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The equality of dimensions

by

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Dedicated to Professor A. Komatu
on his 70th birthday

Abstract. Let X be a paracompact σ-space where each closed set has a closure-preserving quasi-neighborhood base and an anti-closure-preserving quasi-neighborhood base. Then $\dim X = \operatorname{Ind} X$.

0. Introduction. In this paper all spaces are assumed to be Hausdorff topological spaces, maps to be continuous onto, and images to be those under maps. The closed image of a metric space is shortly said to be a Lašnev space in this paper (cf. [4], or [5]). The aim of this paper is to introduce a concept of L-spaces and to prove, for each L-space X, the equality $\dim X = \operatorname{Ind} X$, where $\dim X$ denotes the covering dimension of X and $\operatorname{Ind} X$ the large inductive dimension of X. Since the class of L-spaces is, as is shown below, an intermediate class between that of Lašnev spaces and that of M_1-spaces due to Ceder [2], then the equality generalizes the corresponding equality for Lašnev spaces which was established by Leĭbo [6], Theorem 1. Restricting the class of L-spaces, we get the class of D-spaces where even the decomposition theorem is valid. The concept of D-spaces stems from Dugundji's canonical covers [3] which have been considered in connection only with extendability of maps. The concept happens to be effective to dimension theory quite unexpectedly. As for undefined terminology refer to Nagami [8] and Kodama–Nagami [4].

1. L-spaces.

1.1. Definition. Let X be a space and F a closed set of X. A subset of X is said to be a *neighborhood* of F if its interior contains F. Let $\mathscr{U} = \{U_\alpha\colon \alpha \in A\}$ be a collection of neighborhoods of F. $\mathscr{U}$ is said to be a *quasi-neighborhood base* of F if for each neighborhood U of F contains some U_α. If moreover each U_α is open, then $\mathscr{U}$ is said to be a *neighborhood base* of F. If $\{X - U_\alpha\colon \alpha \in A\}$ is closure-preserving in $X - F$, then $\mathscr{U}$ is said to be *anti-closure-preserving*. If $\mathscr{U}$ is closure-preserving as well as anti-closure-preserving, then $\mathscr{U}$ is said to be *closure-preserving in both sides*. An open cover of $X - F$ is said to be an *anti-cover* of F. An anti-cover $\mathscr{V}$ is said to be *approaching* (to F in X) if for each neighborhood U of F, $\mathrm{Cl}(\mathscr{V}(X-U))$ does not meet F, where $\mathscr{V}(X-U)$ denotes the star of $X-U$ with respect to $\mathscr{V}$.

1.2. DEFINITION. A space X is said to be an *L-space* if it is a paracompact σ-space satisfying the following two conditions.

(1) Each closed set has a closure-preserving quasi-neighborhood base.

(2) Each closed set has an anti-closure-preserving quasi-neighborhood base.

1.3. THEOREM. *For a paracompact σ-space X the following three conditions are equivalent.*

(1) *X is an L-space.*

(2) *Each closed set F of X has a neighborhood base which is closure-preserving in both sides.*

(3) *Each closed set F of X has an approaching anti-cover.*

Proof. (1)→(3): Let $\mathscr{U}$ be a closure-preserving quasi-neighborhood base of F and $\mathscr{V}$ be an anti-closure-preserving quasi-neighborhood base of F. For each point x of $X-F$ set

$$W(x) = X-(\bigcup \{\overline{U}\colon x\notin \overline{U},\ U\in\mathscr{U}\})\cup(\bigcup\{\overline{X-V}\colon x\notin \overline{X-V},\ V\in\mathscr{V}\}).$$

Then $W(x)$ is an open neighborhood of x with $W(x)\cap F=\emptyset$. Set

$$\mathscr{W}=\{W(x)\colon x\in X-F\}.$$

To prove that $\mathscr{W}$ is approaching to F let W be an arbitrary neighborhood of F. Let V be an element of $\mathscr{V}$ with $W\supset X-\overline{X-V}$. Let U be an element of $\mathscr{U}$ with $X-\overline{X-V}\supset\overline{U}$. If $x\in X-(\overline{X-V}\cup F)$, then $W(x)\subset X-\overline{X-V}$ and $W(x)\cap(X-W)=\emptyset$. Thus the inequality $W(y)\cap(X-W)\neq\emptyset$, $y\in X-F$, implies $y\in\overline{X-V}$ and hence $W(y)\cap\overline{U}=\emptyset$. Therefore $\mathscr{W}(X-W)\cap\overline{U}=\emptyset$, which implies that $\mathscr{W}(X-W)\cap\mathrm{Int}\overline{U}=\emptyset$. Since $\mathrm{Int}\overline{U}\supset F$ and $\mathrm{Cl}(\mathscr{W}(X-W))\cap\mathrm{Int}\overline{U}=\emptyset$, then $\mathrm{Cl}(\mathscr{W}(X-W))\cap F=\emptyset$.

(3)→(2): Let $\mathscr{W}$ be an approaching anti-cover of F. Since X is hereditarily paracompact, $\mathscr{W}$ is refined by an open cover $\mathscr{U}=\{U_\alpha\colon \alpha\in A\}$ of $X-F$ which is locally finite in $X-F$. $\mathscr{U}$ is again approaching to F. Let Λ be the collection of all subsets B of A such that $V_B=(\bigcup\{U_\alpha\colon \alpha\in B\})\cup F$ are open neighborhoods of F. Then $\{V_B\colon B\in\Lambda\}$ is a neighborhood base of F which is closure-preserving in both sides.

The implication (2)→(1) is evident and the proof is finished.

1.4. THEOREM. *The closed image of an L-space is an L-space.*

Proof. Let $f\colon X\to Y$ be a closed map, X and L-space, and F a closed set of Y. Let $\{U_\alpha\colon \alpha\in A\}$ be a closure-preserving quasi-neighborhood base of $f^{-1}(F)$. Then $\{f(\overline{U}_\alpha)\colon \alpha\in A\}$ is a closure-preserving quasi-neighborhood base of F. Let $\{V_\lambda\colon \lambda\in\Lambda\}$ be an anti-closure-preserving quasi-neighborhood base of $f^{-1}(F)$. Then $\{Y-f(\overline{X-V_\lambda})\colon \lambda\in\Lambda\}$ is an anti-closure-preserving neighborhood base of F. Y is a paracompact σ-space as the closed image of a paracompact σ-space. That completes the proof.

The following is essentially proved in Leĭbo [6]. We present a proof for the reader's convenience.

1.5. LEMMA. *A metric space (X,d) is an L-space.*

Proof. By the preceding theorem it suffices to prove that each closed set F of X has an approaching anti-cover. For each point $x\in X-F$ set $r(x)=d(x,F)$. Let $W(x)$ be the spherical region of radius $\frac{1}{3}r(x)$ with the center x. Set $\mathscr{W}=\{W(x)\colon x\in X-F\}$. To see that $\mathscr{W}$ is approaching to F let U be an open neighborhood of F. Assume that $W(x)\cap(X-U)\neq\emptyset$. Then for each $y\in W(x)$, we have $d(y,X-U)<\frac{2}{3}r(x)$ and $d(y,F)\geqslant d(x,F)-d(x,y)>r(x)-\frac{1}{3}r(x)=\frac{2}{3}r(x)$. Hence $d(y,X-U)<d(y,F)$. Set $V=\{z\in X\colon d(z,X-U)>d(z,F)\}$. Then V is an open neighborhood of F and the last inequality assures that $\mathscr{W}(X-U)\cap V=\emptyset$. That completes the proof.

By this lemma we get at once the following.

1.6. THEOREM. *A Lašnev space is an L-space.*

By Borges–Lutzer [1], Remark 2.7, a paracompact σ-space in which each closed set has a σ-closure-preserving neighborhood base in an M_1-space. Thus the following is a direct consequence of Theorem 1.3.

1.7. THEOREM. *An L-space is an M_1-space.*

1.8. THEOREM. *A closed subset F and an open subset U of an L-space X are L-spaces.*

Proof. Let H be a relatively closed subset of F. Since H is closed in X, there exists an anti-cover $\mathscr{U}$ of H which is approaching to H in X. Then $\mathscr{U}|F$ is clearly approaching to H in F.

Let $\{U_\alpha\colon \alpha\in A\}$ be a locally finite (in U) open cover of U such that $\overline{U}_\alpha\subset U$ for each $\alpha\in A$. Let H be a relatively closed subset of U. Since each $\overline{U}_\alpha$ is an L-space by the above argument, there exists a neighborhood base $\mathscr{U}_\alpha=\{U_{\alpha\lambda}\colon \lambda\in\Lambda_\alpha\}$ of $\overline{U}_\alpha\cap H$ in the relative space $\overline{U}_\alpha$ which is closure-preserving in both sides. Set

$$V_\xi=\bigcup\{U_{\alpha\lambda_\alpha}\colon \alpha\in A\},\qquad \xi=(\lambda_\alpha)\in\prod\{\Lambda_\alpha\colon \alpha\in A\},$$
$$\mathscr{U}=\{V_\xi\colon \xi\in\prod\Lambda_\alpha\}.$$

Then $\mathscr{U}$ is, as can easily be seen, a quasi-neighborhood base of H in U which is closure-preserving in both sides. That completes the proof.

The author does not know whether each subset of an L-space is an L-space.

2. Examples.

2.1. EXAMPLE (Michael [7]). An L-space which is not a Lašnev space.

Let βN be the Stone–Čech compactification of the natural numbers N. Let p be an arbitrary point of $\beta N-N$. Set $X=N\cup\{p\}$. Then X is clearly an L-space.

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Since p is not the limit point of any subsequence of N with any order, X is not a Fréchet space and hence not a Lašnev space by Lašnev [5], Theorem 1.

2.2. EXAMPLE (San-ou [10], Example 4.1). An M_1-space which is not an L-space.

Let X be the box product of countably infinite number of the rationals. Let p be the point of X whose coordinates are 0 and Ξ_p the subspace of X consisting of points all but a finite number of whose coordinates are 0. Then Ξ_p is an M_1-space by San-ou [10], Theorem 3.1. The argument in [10], Example 4.1, shows that $\{p\}$ cannot have an approaching anti-cover.

2.3. EXAMPLE (Okuyama–Yasui [9]). The product of L-spaces which is not an L-space.

Let $N \cup \{p\}$ be the space in Example 2.1 and I the unit interval. By [9], Theorem 3, if each point of $(N \cup \{p\}\times I$ would have an approaching anti-cover, then $N \cup \{p\}$ should be first-countable. Thus $(N \cup \{p\}) \times I$ cannot be an L-space.

2.4. Remark (T. Nogura). Let X and Y be non-discrete spaces. Let X' and Y' be their respective derived sets. Set $t(x, X) = \min\{|M|: x \in \overline{M}-M, M \subset X\}$. Let $\chi(x, X)$ be the character of x in X. If each point of $X\times Y$ has an approaching anti-cover, then

$$\begin{aligned}\inf\{t(x, X)\colon x \in X'\} &= \sup\{t(x, X)\colon x \in X'\}\\ = \inf\{\chi(x, X)\colon x \in X'\} &= \sup\{\chi(x, X)\colon x \in X'\}\\ = \inf\{t(y, Y)\colon y \in Y'\} &= \sup\{t(y, Y)\colon y \in Y'\}\\ = \inf\{\chi(y, Y)\colon y \in Y'\} &= \sup\{\chi(y, Y)\colon y \in Y'\}.\end{aligned}$$

Pick points $p \in X'$ and $q \in Y$. Let M be a set of X with $p \in \overline{M}-M$. To show that $\chi(q, Y) \leqslant |M|$, let V be an arbitrary open neighborhood of q. Let $\mathscr{U}$ be an approaching anti-cover of (p, q). Since $W = X\times Y - \mathrm{Cl}(\mathscr{U}(X\times(Y-V)))$ is a neighborhood of (p, q) and $(p, q) \in \overline{M}\times\{q\}$, then there exists a point $x \in M$ with $(x, q) \in W$. Let U_x be an element of $\mathscr{U}$ with $(x, q) \in U_x$. Then $U_x \subset X\times V$. Let V_x be the image of U_x under the projection of $X\times Y$ to Y. Then $V_x \subset V$, which shows that $\{V_x\colon x \in M\}$ forms a neighborhood base of q. This argument proves the essential part of the assertion.

3 Auxiliary lemmas.

3.1. LEMMA (Nagami [8], Theorem 11.12). *Let X be a hereditarily paracompact space. Let H and K be disjoint closed sets of X. If the binary cover $\{X-H, X-K\}$ is refined by a σ-locally finite open cover $\mathscr{U}$ such that $\mathscr{U} < \{X-H, X-K\}$ and* $\mathrm{Ind}(\overline{U}-U) \leqslant n$ *for each element U of $\mathscr{U}$, then H and K can be separated by a closed set P with* $\mathrm{Ind}P \leqslant n$ *and* $P \subset \bigcup\{\overline{U}-U\colon U \in \mathscr{U}\}$.

3.2. LEMMA. *Let X be a paracompact σ-space with a σ-locally finite closed network $\mathscr{F}$. If for each element F of $\mathscr{F}$ and each open neighborhood U of F there exists an open neighborhood V of F with $F \subset V \subset U$ such that* $\mathrm{Ind}(\overline{V}-V) \leqslant n-1$, *then* $\mathrm{Ind}\, X \leqslant n$.

Proof. Let H and K be disjoint closed sets of X. Set $\mathscr{U} = \{X-H, X-K\}$ and $\mathscr{F}_1 = \{F \in \mathscr{F}\colon F < \mathscr{U}\}$. For each $F \in \mathscr{F}_1$ there exists an open neighborhood $U(F)$ of F such that $\{U(F)\colon F \in \mathscr{F}_1\}$ is σ-locally finite and refines $\mathscr{U}$. For each $F \in \mathscr{F}_1$ let $V(F)$ be an open neighborhood of F such that $F \subset V(F) \subset \overline{V(F)} \subset U(F)$ and $\mathrm{Ind}(\overline{V(F)}-V(F)) \leqslant n-1$. Since $\mathscr{F}_1$ covers X, $\{V(F)\colon F \in \mathscr{F}_1\}$ covers X. Since $\{U(F)\colon F \in \mathscr{F}_1\}$ is σ-locally finite, $\{V(F)\colon F \in \mathscr{F}_1\}$ is also σ-locally finite. Thus the criterion of the preceding lemma is satisfied and H and K is separated by a closed set P with $\mathrm{Ind}P \leqslant n-1$. That completes the proof.

3.3. LEMMA. *Let X be a paracompact σ-space and $\mathscr{F} = \bigcup_{i=1}^{\infty} \mathscr{F}_i$ a closed network with each $\mathscr{F}_i$ discrete. Set $P_i = \bigcup\{F\colon F \in \mathscr{F}_i\}$. Let $\mathscr{U}_i = \{U_{i\alpha}\colon \alpha \in A_i\}$ be an approaching anti-cover of P_i which is locally finite in $X-P_i$. Let $H_{ij\alpha}$, $j = 1, 2, \ldots$, be closed sets of X with $U_{i\alpha} = \bigcup_{j=1}^{\infty} H_{ij\alpha}$. If each pair $H_{ij\alpha} \subset U_{i\alpha}$, $\alpha \in A_i$, $i, j = 1, 2, \ldots$, admits an open set $V_{ij\alpha}$ such that $H_{ij\alpha} \subset V_{ij\alpha} \subset \overline{V}_{ij\alpha} \subset U_{i\alpha}$ and* $\mathrm{Ind}(\overline{V}_{ij\alpha}-V_{ij\alpha}) \leqslant n-1$, *then* $\mathrm{Ind}\, X \leqslant n$.

Proof. To apply the preceding lemma let F be an element of $\mathscr{F}$ and U an open neighborhood of F. Assume $F \in \mathscr{F}_i$. Let $\mathscr{F}_i = \{F_\lambda\colon \lambda \in \Lambda\}$ and $F = F_\mu$. Let $\{G_\lambda\colon \lambda \in \Lambda\}$ be a discrete open collection such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$ and $G_\mu \subset U$. Set $G = \bigcup\{G_\lambda\colon \lambda \in \Lambda\}$. Set $\mathscr{V}_j = \{V_{ij\alpha}\colon \alpha \in A_i\}$ and $\mathscr{V} = \bigcup_{j=1}^{\infty} \mathscr{V}_j$. Then $\mathscr{V}$ is an open cover of $X-P_i$. Since each $\mathscr{V}_j$ is locally finite in $X-P_i$, $\mathscr{V}$ is σ-locally finite in $X-P_i$. In the subspace $X-P_i$ consider the relatively open cover

$$\mathscr{W} = \{G-P_i, \mathscr{U}_i(X-G)\}.$$

Then $\mathscr{W}$ is refined by $\mathscr{U}_i$ and hence so by $\mathscr{V}$. Therefore Lemma 3.1 assures the existence of an open set W of $X-P_i$ such that $X-G \subset W \subset \overline{W}-P_i \subset \mathscr{U}_i(X-G)$ and $\mathrm{Ind}((\overline{W}-P_i)-W) \leqslant n-1$. Since $\mathscr{U}_i$ is approaching to P_i, $\mathrm{Cl}(\mathscr{U}_i(X-G)) \subset X-P_i$. Since $\overline{W} \subset \mathrm{Cl}(\mathscr{U}_i(X-G))$, $\overline{W}-P_i = \overline{W}$. Thus $\mathrm{Ind}(\overline{W}-W) \leqslant n-1$. Set $K = (\overline{W}-W) \cap G_\mu$. Then K is a closed set of X with $\mathrm{Ind}K \leqslant n-1$ separating F_μ and $X-G_\mu$ and hence separating F and $X-U$. Thus $\mathrm{Ind}\, X \leqslant n$ by Lemma 3.2. That completes the proof.

3.4. DEFINITION (Leĭbo [6]). A collection $\mathscr{T} = \{(H_\alpha, K_\alpha)\colon \alpha \in A\}$ of disjoint pairs of closed sets of a space X is said to *determine* Ind X, if there exists a pair (H_α, K_α) in $\mathscr{T}$ such that for each closed set P separating H_α and K_α, $\mathrm{Ind}\, P \geqslant \mathrm{Ind}\, X-1$. Let M be a subset of X. Then $\mathscr{T}$ is said to *determine* $\mathrm{Ind}M$, if $\mathscr{T}|M$ determines $\mathrm{Ind}\, M$.

3.5. LEMMA. *Let X be an L-space. Then there exists a countable collection of disjoint pairs of closed sets of X which determines* Ind *of all closed subsets of X.*

Proof. Let $\mathscr{F} = \bigcup_{i=1}^{\infty} \mathscr{F}_i$ be a closed network of X with each $\mathscr{F}_i$ discrete. Set $P_i = \bigcup \{F\colon F \in \mathscr{F}_1\}$. Let $\mathscr{U}_i$ be an approaching anti-cover of P_i which is locally finite and σ-discrete in $X-P_i$. Let $\mathscr{U}_{ij} = \{U_{ij\alpha}\colon \alpha \in A_{ij}\}$ be an open collection which is discrete in $X-P_i$ such that $\mathscr{U}_i = \bigcup_{j=1}^{\infty} \mathscr{U}_{ij}$. Set $U_{ij} = \bigcup \{U_{ij\alpha}\colon \alpha \in A_{ij}\}$. Let H_{ijk} be closed sets of X such that $U_{ij} = \bigcup_{k=1}^{\infty} H_{ijk}$. Set $\mathscr{T} = \{(H_{ijk}, X-U_{ij})\colon i, j, k = 1, 2, \ldots\}$.

Let us see that $\mathscr{T}$ determines $\operatorname{Ind} X$. Set $U_{ij\alpha} \cap H_{ijk} = H_{ijk\alpha}$, $\alpha \in A_{ij}$. Since $\mathscr{U}_{ij}$ is discrete in $X-P_i$ and H_{ijk} is closed in X, then $H_{ijk\alpha}$ is closed in X. Set $\mathscr{T}' = \{(H_{ijk\alpha}, X-U_{ij\alpha})\colon \alpha \in A_{ij}, k = 1, 2, \ldots\}$. Since $U_{ij\alpha} = \bigcup_{k=1}^{\infty} H_{ijk\alpha}$, $\mathscr{T}'$ determines $\operatorname{Ind} X$ by Lemma 3.3. Thus there exists a pair $(H_{abc\beta}, X-U_{ab\beta})$ such that for each set P separating $H_{abc\beta}$ and $X-U_{ab\beta}$, $\operatorname{Ind} P \geqslant \operatorname{Ind} X-1$. Let Q be an arbitrary set separating H_{abc} and $X-U_{ab}$. Then $Q \cap U_{ab\beta}$ separates $H_{abc\beta}$ and $X-U_{ab\beta}$ and hence $\operatorname{Ind} Q \geqslant \operatorname{Ind}(Q \cap U_{ab\beta}) \geqslant \operatorname{Ind} X-1$. The inequality $\operatorname{Ind} Q \geqslant \operatorname{Ind} X-1$ shows that $\mathscr{T}$ determines $\operatorname{Ind} X$.

To see that $\mathscr{T}$ determines Ind of all closed sets of X let M be an arbitrary closed set of X. As was noticed in Theorem 1.8 $\mathscr{U}_i|M$ is approaching to $P_i \cap M$. Consider the restrictions of $\mathscr{F}$, $\mathscr{F}_i$, $\mathscr{U}_i$, $\mathscr{U}_{ij}$ and $H_{ijk\alpha}$ to M. Then we can know that $\mathscr{T}|M$ determines $\operatorname{Ind} M$ by an argument quite analogous to the case when $M = X$. That completes the proof.

3.6. Lemma (Leĭbo [6]). *Let X be a paracompact σ-space and $\mathscr{H}$ a countable collection of closed sets of X. Then there exist a metric space Y and a contraction, i.e. a one-one map, $f\colon X \to Y$ such that $\dim X \geqslant \dim Y$ and $f(H)$ is closed for each element H of $\mathscr{H}$.*

3.7. Lemma (Leĭbo [6]). *Let X be a space and $\mathscr{T} = \{(H_\alpha, K_\alpha)\colon \alpha \in A\}$ a collection of disjoint pairs of closed sets determining* Ind *of all closed sets of X. Let $f\colon X \to Y$ be a contraction to another space Y such that $f(H_\alpha)$ and $f(K_\alpha)$ are closed for each $\alpha \in A$. Then $\operatorname{Ind} X \leqslant \operatorname{Ind} Y$.*

Proof (by induction). It suffices to consider the case when $\operatorname{Ind} Y$ is finite. Set $\operatorname{Ind} Y = n$. When $n = 0$, $f(H_\alpha)$ and $f(K_\alpha)$ can be separated by the empty set for each α. Hence H_α and K_α can also be separated by the empty set for each α, which implies that $\operatorname{Ind} X \leqslant 0$. Thus the theorem is true for $n = 0$.

Put the induction hypothesis that the theorem is true for $n \leqslant m-1$. Assume that $n = m$. Let P_α be a closed set of Y with $\operatorname{Ind} P_\alpha \leqslant m-1$ separating $f(H_\alpha)$ and $f(K_\alpha)$. Then $f^{-1}(P_\alpha)$ is a closed set of X separating H_α and K_α. Set $\mathscr{T}' = \mathscr{T}|f^{-1}(P_\alpha)$. Since $f^{-1}(P_\alpha)$ is closed, $\mathscr{T}'$ determines Ind of all closed sets of $f^{-1}(P_\alpha)$. Thus $\operatorname{Ind} f^{-1}(P_\alpha) \leqslant \operatorname{Ind} P_\alpha \leqslant m-1$ by induction hypothesis. Since $\mathscr{T}$ determines $\operatorname{Ind} X$, $\operatorname{Ind} X \leqslant m$. The induction is thus completed and the proof is finished.

4. Main theorems.

4.1. Theorem. *Let X be an L-space. Then $\dim X = \operatorname{Ind} X$.*

Proof. By Lemma 3.5 there exists a countable collection

$$\mathscr{T} = \{(H_i, K_i)\colon i = 1, 2, \ldots\}$$

of disjoint pairs of closed sets of X determining Ind of all closed sets of X. By Lemma 3.6 there exist a metric space Y with $\dim X \geqslant \dim Y$ and a contraction $f\colon X \to Y$ such that $f(H_i)$ and $f(K_i)$ are closed sets of Y for $i = 1, 2, \ldots$ Then by Lemma 3.7 $\operatorname{Ind} X \leqslant \operatorname{Ind} Y$. Since $\dim Y = \operatorname{Ind} Y$ (cf. [8], Theorem 12.6), $\dim X \geqslant \dim Y = \operatorname{Ind} Y \geqslant \operatorname{Ind} X$. Since $\dim X \leqslant \operatorname{Ind} X$ (cf. [8], Theorem 10.1), we have $\dim X = \operatorname{Ind} X$. That completes the proof.

Since a paracompact σ-space with a countable network has the star-finite property, the following is clear from the equality $\operatorname{ind} X = \operatorname{Ind} X$ for such a space X (cf. [8], Corollary 11.13).

4.2. Corollary. *Let X be an L-space with a countable network. Then* $\dim X = \operatorname{Ind} X = \operatorname{ind} X$.

4.3. Remark. As can be seen from the argument presented we do not need the condition that all closed sets of X have approaching anti-covers. If X is merely a paracompact σ-space having a σ-discrete closed network $\mathscr{F}$ such that each element F of $\mathscr{F}$ has an approaching anti-cover, yet $\dim X = \operatorname{Ind} X$.

4.4. Definition. Let X be a space and F a closed set of X. An anti-cover $\mathscr{U}$ of F is said to be *uniformly approaching* (to F in X) if for each open set V of X, $\operatorname{Cl}(\mathscr{U}(X-V)) \cap V \cap F = \emptyset$. X is said to be a *D-space* if it is a paracompact σ-space and each closed set has a uniformly approaching anti-cover.

4.5. Remark. The following propositions are easily verified. (1) Each D-space is an L-space. (2) (Dugundji [3], Lemma 2.1) Each metric space is a D-space. Actually $\mathscr{W}$ given in Lemma 1.5 is uniformly approaching. (3) The closed image of a D-space is a D-space. Thus each Lašnev space is a D-space. (4) Example 2.1 is a D-space which is not a Lašnev space. (5) Each subset of a D-space is a D-space.

4.6. Lemma. *Let X be a D-space. Then there exists a countable collection of disjoint pairs of closed sets of X which determines* Ind *of all subsets of X.*

Proof. Let us continue to use the notions $\mathscr{F} = \bigcup \mathscr{F}_i$, P_i, $\mathscr{U}_i = \bigcup_{j=1}^{\infty} \mathscr{U}_{ij}$, $\mathscr{U}_{ij} = \{U_{ij\alpha}\colon \alpha \in A_{ij}\}$, U_{ij}, H_{ijk}, $\mathscr{T}$, $H_{ijk\alpha}$ $(\alpha \in A_{ij})$, which are the same as in the proof of Lemma 3.5, except that each $\mathscr{U}_i$ is uniformly approaching to P_i in X in the present case. Let S be an arbitrary subset of X. Then $\mathscr{F}|S$ is a σ-discrete relatively closed network of S and $\mathscr{U}_i|S$ is (uniformly) approaching to $P_i \cap S$ in S. Thus $\mathscr{T}|S$ determines $\operatorname{Ind} S$. That completes the proof.

4.7. Theorem. *For a D-space X the following three conditions are equivalent.* (1) $\dim X \leqslant n$. (2) $\operatorname{Ind} X \leqslant n$. (3) *$X$ is the sum of sets X_i, $i = 0, 1, \ldots, n$, with $\dim X_i \leqslant 0$ for each i.*

Proof (by induction). As is well known (3) implies (2) (cf. [8], p. 76). The implication (2)→(3), say (P_n), is proved by induction. (P_0) is clearly true. Let $n>0$. Put the induction hypothesis that (P_i) is true for $i<n$. By the preceding lemma there exists a collection $\mathscr{T} = \{(H_i, K_i)\colon i = 1, 2, \ldots\}$ of disjoint pairs of closed sets which determines Ind of all subsets of X. Let Q_i be a closed set, with $\operatorname{Ind} Q_i \leqslant n-1$, separating H_i and K_i. Set $Q = \bigcup Q_i$. Then $\operatorname{Ind} Q \leqslant n-1$. Hence Q is the sum of sets X_i, $i = 1, \ldots, n$, with $\dim X_i \leqslant 0$. Set $X_0 = X - Q$. Then $Q_i \cap X_0$ separates $H_i \cap X_0$ and $K_i \cap X_0$. Since $Q_i \cap X_0 = \emptyset$ and $\mathscr{T}$ determines $\operatorname{Ind} X_0$, then $\operatorname{Ind} X_0 \leqslant 0$ and hence $\dim X_0 \leqslant 0$. That completes the proof.

4.8. **Problem.** Is each L-space a D-space?

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