On \(k\)-regular embeddings of spaces in Euclidean space

by

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Abstract. If \(k \leq n\) are positive integers, a continuous map \(f: X \to \mathbb{R}^n\) is \(k\)-regular if whenever \(x_1, \ldots, x_k\) are distinct points of \(X\), then \(f(x_1), \ldots, f(x_k)\) are linearly independent. Such maps are of relevance in the theory of Cébysev approximation. In this paper the question of existence of \(k\)-regular maps from a given \(X\) into \(\mathbb{R}^n\) is considered. After discussing some elementary properties of \(k\)-regularity, an algebraic-topological method is introduced to obtain negative results. This method yields the fact that there does not exist a \(3\)-regular map of the real projective plane into \(\mathbb{R}^3\), and this result is best possible. Finally, it is shown how to construct explicit \(2\) and \(3\)-regular maps on real projective spaces which, in terms of homogeneous coordinates, are given by quadratic functions.

1. Introduction. If \(k \leq n\) are positive integers, a continuous map \(f\) of a space \(X\) into Euclidean \(n\)-space \(\mathbb{R}^n\) is \(k\)-regular if whenever \(x_1, \ldots, x_k\) are distinct points of \(X\), then \(f(x_1), \ldots, f(x_k)\) are linearly independent. Closely related to this is the concept of an affinely \(k\)-regular map \(f: X \to \mathbb{R}^n\), where it is required that whenever \(x_0, \ldots, x_k\) are distinct points of \(X\), then \(f(x_0), \ldots, f(x_k)\) are affinely independent (i.e., they are the vertices of a non-degenerate \(k\)-simplex in \(\mathbb{R}^n\)). The latter concept has been considered in [2], [1], and [9]. Clearly, a \(k\)-regular map is affinely \((k-1)\)-regular, and \(f: X \to \mathbb{R}^n\) is affinely \((k-1)\)-regular if and only if the map \(g: X \to \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n\) given by \(g(x) = (1, f(x))\) is \(k\)-regular.

\(k\)-regular maps are of relevance in the theory of Cébysev approximation. A set of \(n\) real-valued continuous functions on \(X\) is called a \(k\)-Cébysev set of length \(n\) if these functions are the components of a \(k\)-regular map of \(X\) into \(\mathbb{R}^n\). The reader is referred to [10], pp. 237–242 for the significance of this concept.

The present paper is concerned with existence and non-existence of \(k\)-regular maps. The following results are obtained:

**Theorem 2.1.** \(X\) admits a 2-regular map into \(\mathbb{R}^n\) if and only if \(X\) admits an affinely 1-regular map into \(\mathbb{R}^{n+1}\). (Thus if \(X\) is compact, existence of a 2-regular map of \(X\) into \(\mathbb{R}^n\) is equivalent to \(X\) being topologically embeddable in \(\mathbb{R}^{n+1}\)).

**Theorem 2.2.** If \(X\) admits a \(k\)-regular map into \(\mathbb{R}^n\), then each \(0 \leq l \leq k-1\), and \(S\) any subset of \(X\) with \(l\) points, \(X - S\) admits a \((k-l)\)-regular map into \(\mathbb{R}^{n-l}\).
2.2. is a slight sharpening of a result of Borsuk [2], p. 355 (proved there for the affinely k-regular case).

In Section 3 we note that existence of a k-regular map of X into R^n implies existence of an equivariant map (with respect to permutation of factors) of the kth configuration space of X into V_k(R^n), the Stiefel manifold of k-frames in R^n. Algebraic topology can be used to prove non-existence of such equivariant maps, and hence non-existence of k-regular maps. As an example, we use this technique to prove that the real projective plane P^2 does not admit a 3-regular map into R^3 (P^2 does admit a 3-regular map into R^4). Since P^2 embeds in R^3 and the complement of any point in P^2 embeds in R^3, this shows that the converse of 2.2 is false when k = 3.

In Section 4, linear algebra is used to produce quadratic 2 and 3-regular maps on real projective spaces.

2. Some properties of k-regularity.

**Theorem 2.1.** X admits a 2-regular map into R^k if and only if X admits an affinely 1-regular map into R^{k+1}.

**Proof.** If f : X → R^{k+1} is affinely 1-regular, then g : X → R^k given by g(x) = (f(x), f(x^2)) is 2-regular, as noted in Section 1.

Conversely, suppose g : X → R^{k+1} is 2-regular. Define h : X → S^{k-1} by h(x) = g(x)/‖g(x)‖. h is injective, and the image of h does not contain an antipodal pair since no two distinct points of X are mapped by g into the same line through the origin. In particular h maps X injectively into a proper subset of S^{k-1}, and hence X can be mapped injectively, i.e., affinely 1-regularly, into R^{k+1}.

**Theorem 2.2.** If X admits a k-regular map into R^n, then for each 0 < k < k-1 and S any subset of X with k points, X − S admits a (k-1)-regular map into R^{n-k}.

**Proof.** Let f : X → R^n be k-regular. Let x be any point of X, V the orthogonal complement of f(x) in R^n, and π : R^n → V orthogonal projection. If x_1, ..., x_k are distinct points of X − {x}, then f(x_1), ..., f(x_k) are linearly independent, and hence π(f(x_1), ..., f(x_k)) are linearly independent in V. Thus πf, followed by a linear isomorphism of V onto R^{k-1}, yields a (k-1)-regular map of X − {x} into R^{k-1}. The general result follows by iteration.

**Theorem 2.3.** If X embeds in the n-sphere S^n, then there exists a 3-regular embedding of X in R^{n+2}.

**Proof.** The standard embedding of S^n in R^{n+1} is affinely 2-regular (no line in R^{n+1} meets S^n more than 2 points), and hence by Section 1, S^n embeds in R^{n+2} in a 3-regular fashion.

3. Equivariant maps. Let F_k(X) denote the kth configuration space of X, i.e., the subspace of the k-fold cartesian product of X consisting of k-tuples of distinct points of X. Let V_k(R^n) denote the space of linearly independent k-frames in R^n. The symmetric group on k letters, S_k, acts freely on both F_k(X) and V_k(R^n) by permutation of factors. (It is more convenient for us to use V_k(R^n) rather than the Stiefel manifold of orthonormal k-frames in R^n. The two are S_k-equivariantly homotopy equivalent.)

**Theorem 3.1.** If X admits a k-regular map into R^n, then there exists an S_k-equivariant map g : F_k(X) → V_k(R^n).

**Proof.** If f : X → R^n is k-regular, define g : F_k(X) → V_k(R^n) by g(x_1, ..., x_k) = (f(x_1), ..., f(x_k)).

Let S_k denote the alternating group on k letters. If X is Hausdorff, we have a double covering F_k(X) × S_k → F_k(X) × S_k, where F_k(X) × S_k, F_k(X) × S_k denote the orbit spaces of F_k(X) with respect to the actions of S_k, S_k, respectively. Similarly we have a double covering V_k(R^n) × S_k → V_k(R^n) × S_k. An S_k-equivariant map g : F_k(X) → V_k(R^n) induces a map of double coverings of the former to the latter. Thus, combining with 3.1 we obtain:

**Theorem 3.2.** If X admits a k-regular map into R^n, then there exists a map of double coverings F_k(X) × S_k → V_k(R^n) × S_k.

If S_k × (resp. S_k) acts on Y, for each point y ∈ Y we write y_♭ (resp. y_♯) for the point in Y × S_k (resp. Y × S_k) determined by y. Let T : V_k(R^n) × S_k → V_k(R^n) × S_k be the involution which interchanges the two points of each fibre in the double covering V_k(R^n) × S_k → V_k(R^n) × S_k. T is given by (y_♭, x) → (y_♯, −x) for all (y_♭, x) ∈ V_k(R^n) and x ∈ R^n. Write (y_♭, x) for the point in E(m) determined by (y_♭, x) ∈ V_k(R^n) × R^n.

**Lemma 3.3.** Let k ≥ 3 be odd, and λ the line bundle over V_k(R^n) × S_k as above. Then n/k is isomorphic to λ × η, where η is a (k-1)-plane bundle with η × S_k orientable.

**Proof.** Let β denote the k-plane subbundle of n/k given as follows: E(β) = {(v_1, ..., v_k, x) | x is a linear combination of v_1, ..., v_k}. If v = (v_1, ..., v_k) ∈ E(β), the ordered basis (v_1, v_2, ..., v_k, v_1) of the fibre over v_k in β gives a well-defined orientation of this fibre since an even permutation does not change the orientation, and the above orientation coincides with that given by (v_k, −v_2, ..., v_1, v_k) since k is odd. Hence β is orientable.

For u as above, write Σ u = v_1 + ... + v_k and let X denote the line subbundle of β whose fibre over v_k is spanned by (v_1, ..., v_k). We have an isomorphism of line bundles λ → λ_♭ given by (v_♭, x) → (v_♭, x), Σ u_♭. Thus β ≅ Λ × S_k where Λ is a complementary subbundle to λ_♭ in β. Take γ to be any complement to β in n/k.

**Lemma 3.4.** Let u denote the first Stiefel–Whitney class of the double covering V_k(R^n) × S_k → V_k(R^n) × S_k. Then u^2 = 0.
Proof. We have \( S^2 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \) where \( \alpha \) and \( \gamma \) are 2-plane bundles over \( V_3(\mathbb{R})/\mathbb{R}^3 \) as in 3.3. We have \( w(3) = 1 + u \). By the Whitney product formula [8], p. 37,
\[
(1 + u)^4 = w(\alpha)w(\gamma).
\]
Since \( \lambda \otimes \alpha \) is orientable, \( w_4(\lambda \otimes \alpha) = 0 \) [8], p. 146 and so it follows from the Whitney product formula that \( w(4) = u \). Thus (1) yields
\[
1 + u^4 = (1 + u)w(2)(1 + u)w(2).
\]
Hence the 1, 2 and 3-dimensional components of the right hand side of (2) are 0, which yields
\[
\begin{align*}
\nu &= w(1), \\
u w_2(3) + w(2) &= 0, \\
u w_2(3) + w(2) &= 0.
\end{align*}
\]
Multiplying the second equation in (3) by \( u \), together with the first and third equations yields \( u^2 = 0 \).

Theorem 3.5. There does not exist a 3-regular map of \( P^3 \) into \( \mathbb{R}^3 \).

Proof. Regard \( P^3 \) as the space of lines through the origin in \( \mathbb{R}^3 \). If \( x \) and \( y \) are distinct points of \( P^3 \), let \( p(x,y) \) denote the unique line through 0 in \( \mathbb{R}^3 \) which is perpendicular to the plane of \( x \) and \( y \). We have a map of double coverings
\[
\begin{array}{ccc}
F_3(P^3) & \longrightarrow & F_3(P^3)/\mathbb{R}^3 \\
\downarrow & & \downarrow \\
F_3(P^3)/\mathbb{R}^3 & \longrightarrow & F_3(P^3)/\mathbb{R}^3
\end{array}
\]
where \( f(x,y) = (x,y,p(x,y),u) \). Let \( v \) denote the first Stiefel--Whitney class of the double covering \( F_3(P^3) \rightarrow F_3(P^3)/\mathbb{R}^3 \). Since \( w_2(1) \neq 0 \), it follows from [11], p. 360 (also [5], Theorem 3.7) that \( u \neq 0 \). Thus by 3.4, there does not exist a map of double coverings
\[
\begin{array}{ccc}
F_3(P^3)/\mathbb{R}^3 & \longrightarrow & F_3(\mathbb{R})/\mathbb{R}^3 \\
\downarrow & & \downarrow \\
F_3(P^3)/\mathbb{R}^3 & \longrightarrow & F_3(\mathbb{R})/\mathbb{R}^3
\end{array}
\]
and so we are done by 3.2.

4. Quadratic 2 and 3-regular embeddings of projective spaces. Regard real projective \( m \)-space \( P^m \) as the quotient space obtained from \( S^m \) by identifying antipodal points. Write \([x]\) for the point in \( P^m \) determined by \( x \in S^m \).

**Definition 4.1.** A quadratic map \( f: P^m \rightarrow \mathbb{R}^n \) is one of the form \( f[x] = g(x \otimes x) \) where \( g: \mathbb{R}^{m+1} \otimes \mathbb{R}^{m+1} \rightarrow \mathbb{R}^n \) is a linear map.

Equivalently, \( f \) is quadratic if the coordinates of \( f[x] \) are homogeneous quadratic polynomials in the coordinates of \( x \).

If \( u \in \mathbb{R}^m \otimes \mathbb{R}^n \), \( u \neq 0 \), the rank of \( u \) is the smallest integer \( r \) such that \( u \) is expressible in the form \( u = x_1 \otimes y_1 + \cdots + x_r \otimes y_r \). If \( e_1, \ldots, e_m \) is a basis of \( \mathbb{R}^m \) and we identify \( \mathbb{R}^m \otimes \mathbb{R}^n \) with \( M_m(\mathbb{R}) \), the space of real \( n \times n \) matrices, under the identification \( \sum a_{ij} e_i \otimes e_j \mapsto (a_{ij}) \), then the above notion of rank coincides with the usual matrix rank.

**Theorem 4.2.** Let \( k = 2 \) or \( 3 \). If \( f: P^m \rightarrow \mathbb{R}^n \) is the quadratic map given by \( f[x] = g(x \otimes x) \) where \( g: \mathbb{R}^{m+1} \otimes \mathbb{R}^{m+1} \rightarrow \mathbb{R}^n \), then \( f \) is \( k \)-regular if the kernel of \( g \) contains no non-zero symmetric elements of rank \( \leq k \).

**Proof.** We give the proof for \( k = 3 \). (The case \( k = 2 \) is simpler.)

If \( [x], [y], [z] \) are distinct points of \( P^m \) such that \( f[x], f[y], f[z] \) are linearly dependent, then there exist real numbers \( a, b, c \), not all 0, such that
\[
0 = af[x] + bf[y] + cf[z] = (ax \otimes x + by \otimes y + cz \otimes z).
\]
Hence, by the hypothesis on the kernel of \( g \),
\[
(1) \quad ax \otimes x + by \otimes y + cz \otimes z = 0.
\]
Since \( [x], [y], [z] \) are distinct, no two of \( x, y, z \) are linearly dependent, and hence no two of \( x \otimes x, y \otimes y, z \otimes z \) are linearly dependent. Hence by (1), \( a, b, c \) must all be non-zero.

\( x, y, z \) cannot be linearly independent, otherwise \( x \otimes x, y \otimes y, z \otimes z \) would be, contradicting (1). Say \( x = x_1 + y_1 + z_1 \), \( t \in \mathbb{R} \). Then both \( x \) and \( t \) are non-zero, for otherwise \( [x] \) would not be distinct from \([x]\) and \([y]\). Substituting into (1) yields
\[
(2) \quad (a + ct^2)x \otimes x + (b + ct^2)y \otimes y + (c + ct^2)z \otimes z = 0.
\]
But \( x \otimes x, y \otimes y, z \otimes z \) are linearly independent, and so \( ct = 0 \), a contradiction.

Note that if \( k \geq 4 \) and \( m \geq 1 \), a quadratic map on \( P^m \) cannot be \( k \)-regular, for if \( x, y \) are orthogonal points on \( S^m \), then the lines \([x], [y] \), \( \left[ x + y \over \sqrt{2} \right], \left[ x - y \over \sqrt{2} \right] \), are distinct, but \( x \otimes x, y \otimes y, \left[ x + y \over \sqrt{2} \right] \otimes \left[ x + y \over \sqrt{2} \right], \left[ x - y \over \sqrt{2} \right] \otimes \left[ x - y \over \sqrt{2} \right] \) are linearly dependent.

**Definition 4.3.** A linear subspace of \( M_m(\mathbb{R}) \) is \( k \)-regular if it contains no non-zero symmetric matrices of rank \( \leq k \).

**Corollary 4.4.** Let \( k = 2 \) or \( 3 \). If \( M_m(\mathbb{R}) \) contains a \( k \)-regular subspace of dimension \((m+1)^2 - n\), then \( P^m \) admits a quadratic \( k \)-regular map into \( \mathbb{R}^n \).
Proof. Such a subspace corresponds to a subspace \( K \) of \( \mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1} \) with no non-zero symmetric elements of rank \( \leq k \), and so the composition

\[
\mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1} \rightarrow (\mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1})/K \cong \mathbb{R}^{n}
\]

yields the desired quadratic \( k \)-regular map by 4.2.

**Example 4.5.** Let \( S \) be the subspace of \( M_{m+1}(\mathbb{R}) \) consisting of all matrices \( A = (a_{ij}) \) satisfying \( \sum_{i+j=q} a_{ij} = 0 \) for \( 2 \leq q \leq 2m+2 \). The dimension of \( S \) is \((m+1)^2-(2m+1)\). S is 2-regular, for if \( (a_{ij}) \) is a non-zero symmetric matrix in \( S \) and \( g_0 \) is the smallest integer for which the \( a_{ij}, i+j = g_0 \) are not all 0, then at least 3 of these \( a_{ij} \) are non-zero and so rank \( (a_{ij}) \geq 3 \). Hence, by 4.4, there is a quadratic 2-regular embedding of \( P^m \) in \( \mathbb{R}^{2m+1} \) for all \( m \).

This result is best possible when \( m = p \) is a power of 2, for then \( P^m \) does not embed in \( \mathbb{R}^{2^m} \) ([4], p. 34 or [11]) and hence, by 2.1, does not 2-regularly embed in \( \mathbb{R}^{2^m} \).

**Example 4.6.** Let \( S \) be the space of all skew-symmetric matrices in \( M_3(\mathbb{R}) \). \( S \) is \( k \)-regular for all \( k \), \( S \) is 3-dimensional, and so by 4.4 there exists a quadratic 3-regular embedding of \( P^2 \) in \( \mathbb{R}^6 \). By 3.5, this result is best possible.

**Example 4.7.** Let \( S_0 \) consist of all matrices in \( M_3(\mathbb{R}) \) of the form

\[
\begin{pmatrix}
    a & b & c \\
    -b & a & 0 \\
    c & 0 & -a-b \\
    0 & e & b \\
    0 & 0 & a
\end{pmatrix}
\]

and let \( S \) be the direct sum of \( S_0 \) with the space of all skew-symmetric matrices in \( M_3(\mathbb{R}) \). Since every non-zero matrix in \( S_0 \) has rank 4, \( S \) is 3-regular. \( S \) is 9-dimensional, and so by 4.4 there exists a quadratic 3-regular embedding of \( P^2 \) in \( \mathbb{R}^7 \).

We conjecture that \( P^2 \) does not 3-regularly embed in \( \mathbb{R}^7 \). (This would follow from 2.2 if we knew that the complement of a point in \( P^2 \) does not topologically embed in \( \mathbb{R}^7 \).) It is known [3], Corollary 1 that it does not differentiably embed in \( \mathbb{R}^7 \).

A linear map \( \mathbb{R}^m \otimes \mathbb{R}^n \rightarrow \mathbb{R}^s \) is non-singular if its kernel contains no elements of rank 1. Extensive results on the existence of such maps have been obtained in [7] and [6].

**Theorem 4.8.** Suppose there exists a quadratic 3-regular embedding of \( P^m \) in \( \mathbb{R}^n \), and a non-singular map \( \mathbb{R}^{m+1} \otimes \mathbb{R}^{n+1} \rightarrow \mathbb{R}^s \). Then there exists a quadratic 3-regular embedding of \( P^{m+1} \) in \( \mathbb{R}^{m+n} \).

**Proof.** \( M_{m+1}(\mathbb{R}) \) contains a 3-regular subspace \( S_1 \), of dimension \((m+1)^3-n \). The kernel of the non-singular map yields a \((m+1)^3-q \)-dimensional subspace \( T \) of \( M_{m+1}(\mathbb{R}) \) which contains no elements of rank 1. Let

\[
T = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in S_1, B \in T, C \in M_{m+1}(\mathbb{R}), D \in S_1 \right\}
\]

Then \( S \) is a 3-regular subspace of \( M_{2m+2}(\mathbb{R}) \) of dimension \((2m+2)^3-(2m+q) \), so we are done by 4.4.

4.8 is crude, and can undoubtedly be improved.

**Example 4.9.** Quaternionic multiplication yields a non-singular map \( \mathbb{R}^4 \otimes \mathbb{R}^4 \rightarrow \mathbb{R}^4 \). Thus 4.7 and 4.8 yield a quadratic 3-regular embedding of \( P^7 \) in \( \mathbb{R}^{15} \). Cayley multiplication yields a non-singular map \( \mathbb{R}^8 \otimes \mathbb{R}^8 \rightarrow \mathbb{R}^8 \), and hence there is a quadratic 3-regular embedding of \( P^{15} \) in \( \mathbb{R}^{34} \).

**References**


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