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On k-regular embeddings of spaces in Euclidean space

by

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Abstract. If $k \leqslant n$ are positive integers, a continuous map $f\colon X \to R^n$ is *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent. Such maps are of relevance in the theory of Čebyšev approximation. In this paper the question of existence of k-regular maps from a given X into R^n is considered. After discussing some elementary properties of k-regularity, an algebraic-topological method is introduced to obtain negative results. This method yields the fact that there does not exist a 3-regular map of the real projective plane into R^5, and this result is best possible. Finally, it is shown how to construct explicit 2- and 3-regular maps on real projective spaces which, in terms of homogeneous coordinates, are given by quadratic functions.

1. Introduction. If $k \leqslant n$ are positive integers, a continuous map f of a space X into Euclidean n-space $\boldsymbol{R}^n$ is *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent. Closely related to this is the concept of an *affinely k-regular map* $f\colon X \to \boldsymbol{R}^n$, where it is required that whenever $x_0, \ldots, x_k$ are distinct points of X, then $f(x_0), \ldots, f(x_k)$ are affinely independent (i.e. they are the vertices of a non-degenerate k-simplex in $\boldsymbol{R}^n$). The latter concept has been considered in [2], [1], and [9]. Clearly, a k-regular map is affinely $(k-1)$-regular, and $f\colon X \to \boldsymbol{R}^n$ is affinely $(k-1)$-regular if and only if the map $g\colon X \to \boldsymbol{R}^{n+1} = \boldsymbol{R} \times \boldsymbol{R}^n$ given by $g(x) = \big(1, f(x)\big)$ is k-regular.

k-regular maps are of relevance in the theory of Čebyšev approximation. A set of n real-valued continuous functions on X is called a *k-Čebyšev set of length n* if these functions are the components of a k-regular map of X into $\boldsymbol{R}^n$. The reader is referred to [10], pp. 237–242 for the significance of this concept.

The present paper is concerned with existence and non-existence of k-regular maps. The following results are obtained:

THEOREM 2.1. *X admits a 2-regular map into $\boldsymbol{R}^n$ if and only if X admits an affinely 1-regular map into $\boldsymbol{R}^{n-1}$. (Thus if X is compact, existence of a 2-regular map of X into $\boldsymbol{R}^n$ is equivalent to X being topologically embeddable in $\boldsymbol{R}^{n-1}$).*

THEOREM 2.2. *If X admits a k-regular map into $\boldsymbol{R}^n$, then each $0 \leqslant i \leqslant k-1$, and S any subset of X with i points, $X-S$ admits a $(k-i)$-regular map into $\boldsymbol{R}^{n-i}$.*

2.2. is a slight sharpening of a result of Borsuk [2], p. 355 (proved there for the affinely k-regular case).

In Section 3 we note that existence of a k-regular map of X into $\boldsymbol{R}^n$ implies existence of an equivariant map (with respect to permutation of factors) of the kth configuration space of X into $V_k(\boldsymbol{R}^n)$, the Stiefel manifold of k-frames in $\boldsymbol{R}^n$. Algebraic topology can be used to prove non-existence of such equivariant maps, and hence non-existence of k-regular maps. As an example, we use this technique to prove that the real projective plane P^2 does not admit a 3-regular map into $\boldsymbol{R}^5$. (P^2 does admit a 3-regular map into $\boldsymbol{R}^6$). Since P^2 embeds in $\boldsymbol{R}^4$ and the complement of any point in P^2 embeds in $\boldsymbol{R}^3$, this shows that the converse of 2.2 is false when $k = 3$.

In Section 4, linear algebra is used to produce quadratic 2 and 3-regular maps on real projective spaces.

2. Some properties of k-regularity.

THEOREM 2.1. *X admits a 2-regular map into $\boldsymbol{R}^n$ if and only if X admits an affinely 1-regular map into $\boldsymbol{R}^{n-1}$.*

Proof. If $f\colon X \to \boldsymbol{R}^{n-1}$ is affinely 1-regular, then $g\colon X \to \boldsymbol{R}^n$ given by $g(x) = (1, f(x))$ is 2-regular, as noted in Section 1.

Conversely, suppose $g\colon X \to \boldsymbol{R}^n$ is 2-regular. Define $h\colon X \to S^{n-1}$ by $h(x) = g(x)/\|g(x)\|$. h is injective, and the image of h does not contain an antipodal pair since no two distinct points of X are mapped by g into the same line through the origin. In particular h maps X injectively into a proper subset of S^{n-1}, and hence X can be mapped injectively, i.e. affinely 1-regularly, into $\boldsymbol{R}^{n-1}$.

THEOREM 2.2. *If X admits a k-regular map into $\boldsymbol{R}^n$, then for each $0 \leqslant i \leqslant k-1$ and S any subset of X with i points, $X - S$ admits a $(k-i)$-regular map into $\boldsymbol{R}^{n-i}$.*

Proof. Let $f\colon X \to \boldsymbol{R}^n$ be k-regular. Let x be any point of X, V the orthogonal complement of $f(x)$ in $\boldsymbol{R}^n$, and $\pi\colon \boldsymbol{R}^n \to V$ orthogonal projection. If $x_1, \ldots, x_{k-1}$ are distinct points of $X - \{x\}$, then $f(x), f(x_1), \ldots, f(x_{k-1})$ are linearly independent, and hence $\pi f(x_1), \ldots, \pi f(x_{k-1})$ are linearly independent in V. Thus πf, followed by a linear isomorphism of V onto $\boldsymbol{R}^{n-1}$, yields a $(k-1)$-regular map of $X - \{x\}$ into $\boldsymbol{R}^{n-1}$. The general result follows by iteration.

THEOREM 2.3. *If X embeds in the n-sphere S^n, then there exists a 3-regular embedding of X in $\boldsymbol{R}^{n+2}$.*

Proof. The standard embedding of S^n in $\boldsymbol{R}^{n+1}$ is affinely 2-regular (no line in $\boldsymbol{R}^{n+1}$ meets S^n is more than 2 points), and hence by Section 1, S^n embeds in $\boldsymbol{R}^{n+2}$ in a 3-regular fashion.

3. Equivariant maps. Let $F_k(X)$ denote the kth configuration space of X, i.e. the subspace of the k-fold cartesian product of X consisting of k-tuples of distinct points of X. Let $V_k(\boldsymbol{R}^n)$ denote the space of linearly independent k-frames in $\boldsymbol{R}^n$. The symmetric group on k letters, $\mathscr{S}_k$, acts freely on both $F_k(X)$ and $V_k(\boldsymbol{R}^n)$ by permutation of factors. (It is more convenient for us to use $V_k(\boldsymbol{R}^n)$ rather than the Stiefel manifold of orthonormal k-frames in $\boldsymbol{R}^n$. The two are $\mathscr{S}_k$-equivariantly homotopy equivalent.)

THEOREM 3.1. *If X admits a k-regular map into $\boldsymbol{R}^n$, then there exists an $\mathscr{S}_k$-equivariant map $g\colon F_k(X) \to V_k(\boldsymbol{R}^n)$.*

Proof. If $f\colon X \to \boldsymbol{R}^n$ is k-regular, define $g\colon F_k(X) \to V_k(\boldsymbol{R}^n)$ by $g(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$.

Let $\mathscr{A}_k$ denote the alternating group on k letters. If X is Hausdorff, we have a double covering $F_k(X)/\mathscr{A}_k \to F_k(X)/\mathscr{S}_k$, where $F_k(X)/\mathscr{A}_k$, $F_k(X)/\mathscr{S}_k$ denote the orbit spaces of $F_k(X)$ with respect to the actions of $\mathscr{A}_k$, $\mathscr{S}_k$, respectively. Similarly we have a double covering $V_k(\boldsymbol{R}^n)/\mathscr{A}_k \to V_k(\boldsymbol{R}^n)/\mathscr{S}_k$. An $\mathscr{S}_k$-equivariant map $g\colon F_k(X) \to V_k(\boldsymbol{R}^n)$ induces a map of double coverings of the former to the latter. Thus, combining with 3.1 we obtain:

THEOREM 3.2. *If X admits a k-regular map into $\boldsymbol{R}^n$, then there exists a map of double coverings*

$$\begin{tikzcd} F_k(X)/\mathscr{A}_k \arrow[r] \arrow[d] & V_k(\boldsymbol{R}^n)/\mathscr{A}_k \arrow[d] \\ F_k(X)/\mathscr{S}_k \arrow[r] & V_k(\boldsymbol{R}^n)/\mathscr{S}_k \end{tikzcd}$$

If $\mathscr{S}_k$ (resp. $\mathscr{A}_k$) acts on Y, for each point $y \in Y$ write $y_{\mathscr{S}}$ (resp. $y_{\mathscr{A}}$) for the point in $Y/\mathscr{S}_k$ (resp. $Y/\mathscr{A}_k$) determined by y. Let $T\colon V_k(\boldsymbol{R}^n)/\mathscr{A}_k \to V_k(\boldsymbol{R}^n)/\mathscr{A}_k$ be the involution which interchanges the two points of each fibre in the double covering $V_k(\boldsymbol{R}^n)/\mathscr{A}_k \to V_k(\boldsymbol{R}^n)/\mathscr{S}_k$. T is given by $T(v_1, \ldots, v_k)_{\mathscr{A}} = (v_2, v_1, \ldots, v_k)_{\mathscr{A}}$. Let λ denote the real line bundle over $V_k(\boldsymbol{R}^n)/\mathscr{S}_k$ associated with the above double covering. For each positive integer m, the total space of $m\lambda$, the Whitney sum of m copies of λ, is $E(m\lambda) = V_k(\boldsymbol{R}^n)/\mathscr{A}_k \times_2 R^m$ = quotient space of $V_k(\boldsymbol{R}^n)/\mathscr{A}_k \times \boldsymbol{R}^m$ obtained by identifying $(v_{\mathscr{A}}, x) \sim (Tv_{\mathscr{A}}, -x)$ for all $v \in V_k(\boldsymbol{R}^n)$ and $x \in R^m$. Write $(v_{\mathscr{A}}, x)_2$ for the point in $E(m\lambda)$ determined by $(v_{\mathscr{A}}, x) \in V_k(\boldsymbol{R}^n)/\mathscr{A}_k \times \boldsymbol{R}^m$.

LEMMA 3.3. *Let $k \geqslant 3$ be odd, and λ the line bundle over $V_k(\boldsymbol{R}^n)/\mathscr{S}_k$ as above. Then $n\lambda$ is isomorphic to $\lambda \oplus \alpha \oplus \gamma$, where α is a $(k-1)$-plane bundle with $\lambda \oplus \alpha$ orientable.*

Proof. Let β denote the k-plane subbundle of $n\lambda$ given as follows: $E(\beta) = \{((v_1, \ldots, v_k)_{\mathscr{A}}, x)_2 |\ x$ is a linear combination of $v_1, \ldots, v_k\}$. If $v = (v_1, \ldots, v_k) \in V_k(\boldsymbol{R}^n)$, the ordered basis $(v_{\mathscr{A}}, v_1)_2, \ldots, (v_{\mathscr{A}}, v_k)_2$ of the fibre over $v_{\mathscr{S}}$ in β gives a well-defined orientation of this fibre since an even permutation does not change the orientation, and the above orientation coincides with that given by $(v_{\mathscr{A}}, -v_2)_2$, $(v_{\mathscr{A}}, -v_1)_2$, $(v_{\mathscr{A}}, -v_3)_2, \ldots, (v_{\mathscr{A}}, -v_k)_2$ since k is odd. Hence β is orientable.

For v as above, write $\sum v = v_1 + \ldots + v_k$, and let λ' denote the line subbundle of β whose fibre over $v_{\mathscr{S}}$ is spanned by $(v_{\mathscr{A}}, \sum v)_2$. We have an isomorphism of line bundles $\lambda \to \lambda'$ given by $(v_{\mathscr{A}}, t)_2 \mapsto (v_{\mathscr{A}}, t \sum v)_2$. Thus $\beta \cong \lambda' \oplus \alpha$ where α is a complementary subbundle to λ' in β. Take γ to be any complement to β in $n\lambda$.

LEMMA 3.4. *Let u denote the first Stiefel–Whitney class of the double covering $V_3(\boldsymbol{R}^5)/\mathscr{A}_3 \to V_3(\boldsymbol{R}^5)/\mathscr{S}_3$. Then $u^3 = 0$.*

Proof. We have $5\lambda \cong \lambda\oplus\alpha\oplus\gamma$ where α and γ are 2-plane bundles over $V_3(\mathbf{R}^5)/\mathscr{S}_3$ as in 3.3. We have $w(\lambda) = 1+u$. By the Whitney product formula [8], p. 37,

$$(1)\qquad (1+u)^4 = w(\alpha)w(\gamma)\,.$$

Since $\lambda\oplus\alpha$ is orientable, $w_1(\lambda\oplus\alpha) = 0$ [8], p. 146 and so it follows from the Whitney product formula that $w_1(\alpha) = u$. Thus (1) yields

$$(2)\qquad 1+u^4 = \big(1+u+w_2(\alpha)\big)\big(1+w_1(\gamma)+w_2(\gamma)\big)\,.$$

Hence the 1, 2 and 3-dimensional components of the right hand side of (2) are 0, which yields

$$(3)\qquad \begin{aligned} u &= w_1(\gamma)\,,\\ uw_1(\gamma)+w_2(\alpha)+w_2(\gamma) &= 0\,,\\ w_1(\gamma)w_2(\alpha)+uw_2(\gamma) &= 0\,. \end{aligned}$$

Multiplying the second equation in (3) by u, together with the first and third equations yields $u^3 = 0$.

THEOREM 3.5. *There does not exist a 3-regular map of P^2 into $\mathbf{R}^5$.*

Proof. Regard P^2 as the space of lines through the origin in $\mathbf{R}^3$. If x and y are distinct points of P^2, let $p(x,y)$ denote the unique line through 0 in $\mathbf{R}^3$ which is perpendicular to the plane of x and y. We have a map of double converings

$$\begin{array}{ccc} F_2(P^2) & \xrightarrow{f} & F_3(P^2)/\mathscr{A}_3 \\ \downarrow & & \downarrow \\ F_2(P^2)/\mathscr{S}_2 & \longrightarrow & F_3(P^2)/\mathscr{S}_3 \end{array}$$

where $f(x,y) = \big(x, y, p(x,y)\big)_{\mathscr{A}}$. Let v denote the first Stiefel–Whitney class of the double covering $F_2(P^2)\to F_2(P^2)/\mathscr{S}_2$. Since $\overline{w}_1(P^2) \neq 0$, it follows from [11], p. 380 (also [5], Theorem 3.7) that $v^3 \neq 0$. Thus by 3.4, there does not exist a map of double coverings

$$\begin{array}{ccc} F_3(P^2)/\mathscr{A}_3 & \longrightarrow & V_3(\mathbf{R}^5)/\mathscr{A}_3 \\ \downarrow & & \downarrow \\ F_3(P^2)/\mathscr{S}_3 & \longrightarrow & V_3(\mathbf{R}^5)/\mathscr{S}_3 \end{array}$$

and so we are done by 3.2.

Note that since P^2 embeds in S^4, there does exist a 3-regular embedding of P^2 into $\mathbf{R}^6$ by 2.3. In fact, in the next section we will show that a quadratic (defined below) 3-regular embedding of P^2 into $\mathbf{R}^6$ exists.

4. Quadratic 2 and 3-regular embeddings of projective spaces. Regard real projective m-space P^m as the quotient space obtained from S^m by identifying antipodal points. Write $[x]$ for the point in P^m determined by $x\in S^m$.

DEFINITION 4.1. A *quadratic map* $f\colon P^m\to\mathbf{R}^n$ is one of the form $f[x] = g(x\otimes x)$ where $g\colon \mathbf{R}^{m+1}\otimes\mathbf{R}^{m+1}\to\mathbf{R}^n$ is a linear map.

Equivalently, f is quadratic if the coordinates of $f[x]$ are homogeneous quadratic polynomials in the coordinates of x.

If $u\in\mathbf{R}^n\otimes\mathbf{R}^n$, $u\neq 0$, the rank of u is the smallest integer r such that u is expressible in the form $u = x_1\otimes y_1+\ldots+x_r\otimes y_r$. If $e_1,\ldots,e_n$ is a basis of $\mathbf{R}^n$ and we identify $\mathbf{R}^n\otimes\mathbf{R}^n$ with $M_n(\mathbf{R})$, the space of real $n\times n$ matrices, under the identification $\sum a_{ij}e_i\otimes e_j \mapsto (a_{ij})$, then the above notion of rank coincides with the usual matrix rank.

THEOREM 4.2. *Let $k = 2$ or 3. If $f\colon P^m\to\mathbf{R}^n$ is the quadratic map given by $f[x] = g(x\otimes x)$ where $g\colon \mathbf{R}^{m+1}\otimes\mathbf{R}^{m+1}\to\mathbf{R}^n$, then f is k-regular if the kernel of g contains no non-zero symmetric elements of rank $\leqslant k$.*

Proof. We give the proof for $k = 3$. (The case $k = 2$ is simpler.)

If $[x]$, $[y]$, $[z]$ are distinct points of P^m such that $f[x]$, $f[y]$, $f[z]$ are linearly dependent, then there exist real numbers a, b, c, not all 0, such that

$$0 = af[x]+bf[y]+cf[z] = g(ax\otimes x+by\otimes y+cz\otimes z)\,.$$

Hence, by the hypothesis on the kernel of g,

$$(1)\qquad ax\otimes x+by\otimes y+cz\otimes z = 0\,.$$

Since $[x]$, $[y]$, $[z]$ are distinct, no two of x, y, z are linearly dependent, and hence no two of $x\otimes x$, $y\otimes y$, $z\otimes z$ are linearly dependent. Hence by (1), a, b, c must all be non-zero.

x, y, z cannot be linearly independent, for otherwise $x\otimes x$, $y\otimes y$, $z\otimes z$ would be, contradicting (1). Say $z = sx+ty$, $s, t\in\mathbf{R}$. Then both s and t are non-zero, for otherwise $[z]$ would not be distinct from $[x]$ and $[y]$. Substituting into (1) yields

$$(2)\qquad (a+cs^2)x\otimes x+(b+ct^2)y\otimes y+cst(x\otimes y+y\otimes x) = 0\,.$$

But $x\otimes x$, $y\otimes y$, $x\otimes y+y\otimes x$ are linearly independent, and so $cst = 0$, a contradiction.

Note that if $k\geqslant 4$ and $m\geqslant 1$, a quadratic map on P^m cannot be k-regular, for if x, y are orthogonal points on S^m, then the lines $[x]$, $[y]$, $\left[\frac{x+y}{\sqrt{2}}\right]$, $\left[\frac{x-y}{\sqrt{2}}\right]$ are distinct, but $x\otimes x$, $y\otimes y$, $\left(\frac{x+y}{\sqrt{2}}\right)\otimes\left(\frac{x+y}{\sqrt{2}}\right)$, $\left(\frac{x-y}{\sqrt{2}}\right)\otimes\left(\frac{x-y}{\sqrt{2}}\right)$ are linearly dependent.

DEFINITION 4.3. A linear subspace of $M_n(\mathbf{R})$ is *k-regular* if it contains no non-zero symmetric matrices of rank $\leqslant k$.

COROLLARY 4.4. *Let $k = 2$ or 3. If $M_{m+1}(\mathbf{R})$ contains a k-regular subspace of dimension $(m+1)^2-n$, then P^m admits a quadratic k-regular map into $\mathbf{R}^n$.*

Proof. Such a subspace corresponds to a subspace K of $\boldsymbol{R}^{m+1}\otimes\boldsymbol{R}^{m+1}$ with no non-zero symmetric elements of rank $\leqslant k$, and so the composition

$$\boldsymbol{R}^{m+1}\otimes\boldsymbol{R}^{m+1}\to(\boldsymbol{R}^{m+1}\otimes\boldsymbol{R}^{m+1})/K\cong\boldsymbol{R}^{n}$$

yields the desired quadratic k-regular map by 4.2.

EXAMPLE 4.5. Let S be the subspace of $M_{m+1}(\boldsymbol{R})$ consisting of all matrices $A=(a_{ij})$ satisfying $\sum_{i+j=q}a_{ij}=0$ for $2\leqslant q\leqslant 2m+2$. The dimension of S is $(m+1)^2-(2m+1)$. S is 2-regular, for if (a_{ij}) is a non-zero symmetric matrix in S and q_0 is the smallest integer for which the a_{ij}, $i+j=q_0$, are not all 0, then at least 3 of these a_{ij} are non-zero and so rank $(a_{ij})\geqslant 3$. Hence, by 4.4, there is a quadratic 2-regular embedding of P^m in $\boldsymbol{R}^{2m+1}$ for all m.

This result is best possible when m is a power of 2, for then P^m does not embed in $\boldsymbol{R}^{2m-1}$ ([4], p. 34 or [11]) and hence, by 2.1, does not 2-regularly embed in $\boldsymbol{R}^{2m}$.

EXAMPLE 4.6. Let S be the space of all skew-symmetric matrices in $M_3(\boldsymbol{R})$. S is k-regular for all k, S is 3-dimensional, and so by 4.4 there exists a quadratic 3-regular embedding of P^2 in $\boldsymbol{R}^6$. By 3.5, this result is best possible.

EXAMPLE 4.7. Let S_0 consist of all matrices in $M_4(\boldsymbol{R})$ of the form

$$\begin{bmatrix} a & b & c & 0\\ b & -a & 0 & c\\ c & 0 & -a & -b\\ 0 & c & -b & a\end{bmatrix}$$

and let S be the direct sum of S_0 with the space of all skew-symmetric matrices in $M_4(\boldsymbol{R})$. Since every non-zero matrix in S_0 has rank 4, S is 3-regular. S is 9-dimensional, and so by 4.4 there exists a quadratic 3-regular embedding of P^3 in $\boldsymbol{R}^7$.

We conjecture that P^3 does not 3-regularly embed in $\boldsymbol{R}^6$. (This would follow from 2.2 if we knew that the complement of a point in P^3 does not topologically embed in $\boldsymbol{R}^4$. It is known [3], Corollary 1 that it does not differentiably embed in $\boldsymbol{R}^4$).

A linear map $\boldsymbol{R}^m\otimes\boldsymbol{R}^m\to\boldsymbol{R}^n$ is *non-singular* if its kernel contains no elements of rank 1. Extensive results on the existence of such maps have been obtained in [7] and [6].

THEOREM 4.8. *Suppose there exists a quadratic 3-regular embedding of P^m in $\boldsymbol{R}^n$, and a non-singular map $\boldsymbol{R}^{m+1}\otimes\boldsymbol{R}^{m+1}\to\boldsymbol{R}^q$. Then there exists a quadratic 3-regular embedding of P^{2m+1} in $\boldsymbol{R}^{2n+q}$.*

Proof. $M_{m+1}(\boldsymbol{R})$ contains a 3-regular subspace S_1 of dimension $(m+1)^2-n$. The kernel of the non-singular map yields a $(m+1)^2-q$-dimensional subspace T of $M_{m+1}(\boldsymbol{R})$ which contains no elements of rank 1. Let

$$S=\left\{\begin{bmatrix} A & B\\ C & D\end{bmatrix}\middle|\ A\in S_1,\ B\in T,\ C\in M_{m+1}(\boldsymbol{R}),\ D\in S_1\right\}.$$

Then S is a 3-regular subspace of $M_{2m+2}(\boldsymbol{R})$ of dimension $(2m+2)^2-(2n+q)$, so we are done by 4.4.

4.8 is crude, and can undoubtedly be improved.

EXAMPLE 4.9. Quaternionic multiplication yields a non-singular map $\boldsymbol{R}^4\otimes\boldsymbol{R}^4\to\boldsymbol{R}^4$. Thus 4.7 and 4.8 yield a quadratic 3-regular embedding of P^7 in $\boldsymbol{R}^{18}$. Cayley multiplication yields a non-singular map $\boldsymbol{R}^8\otimes\boldsymbol{R}^8\to\boldsymbol{R}^8$, and hence there is a quadratic 3-regular embedding of P^{15} in $\boldsymbol{R}^{44}$.

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