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Banach-Euclidean four-point properties

by

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Abstract. A metric space has the Banach–Euclidean four-point property at a point p provided for each triple of its points q, r, and s, if m_1, m_2, m_3, m_4 are respective midpoints of p and q, q and r, r and s, and s and p then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$ and the quadruple m_1, m_2, m_3, m_4 is congruent to a quadruple of points of the euclidean plane. The main result of the paper is that a complete, convex, externally convex, metric space is a real inner-product space if and only if it has the Banach–Euclidean four-point property at some point.

Let S denote a space which satisfies the axioms of Hilbert's groups I, II, III and V; namely, the axioms of connection, order, congruence, and continuity. Young [6] proved S is euclidean, hyperbolic, or elliptic, respectively, if and only if there is *one* triangle such that the length of the line joining the middle points of two sides is (1) equal to, (2) less than, or (3) greater than the third side, respectively.

Andalafte and Blumenthal [1] extended the notion of (1) above to metric spaces in the following way.

The Young postulate. If p, q, and r are points of a metric space M, and if q. and r' are the midpoints of p and q, and of p and r, respectively, then $q'r' = \frac{1}{2}qr$

They proved a complete, convex, externally convex, metric space with the two-triple property is a Banach space if and only if it satisfies the Young Postulate.

A direct analogue of Young's result is: a complete, convex, externally convex, metric space M is a euclidean (inner-product) space if and only if M contains one triple of points which satisfies the Young condition. This is of course false, for every rotund Banach space satisfies the Young Postulate. To make matters worse, there is a complete, convex, externally convex, metric space which satisfies the Young Postulate at one point, but is not a Banach space, see [4].

We focus our attention on an immediate consequence of the Young postulate.

The quadrilateral midpoint property. If p, q, r, and s are points of a metric space M, and if m_1, m_2, m_3, m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then

$$m_1m_2 = m_3m_4 \quad \text{and} \quad m_1m_4 = m_2m_3 .$$

It is not so immediate that conversely, if a complete, convex, externally convex, metric space has the quadrilateral midpoint property, then it satisfies the Young postulate, see [5].

Loveland and Valentine [4] have shown that the quadrilateral midpoint property need only be valid for a particular point p, together with any other three points. We now have half of the direct analogue to Young's result in the setting of metric spaces. It is clear that if we are going to obtain the other half; namely, euclidean space, we are going to have to add to our hypotheses. We therefore introduce a hybrid four-point property.

The Banach-Euclidean four-point property at a point. A metric space M contains a point p such that for each triple of its points q, r, and s, if m_1, m_2, m_3, m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then

(1) $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$ and

(2) the quadruple m_1, m_2, m_3, m_4 is congruent to a quadruple of points of E_2, the euclidean plane.

THEOREM 1. *A complete, convex, externally convex, metric space M is a real inner-product space if and only if M has the Banach–Euclidean four-point property at some point p.*

Proof. First observe that (1) is the Banach four-point property at a point p; and consequently, M is a rotund Banach space, see [4].

In order to simplify the proof, we will say that if x, y are distinct points of M, then z is a *reflection of the point x in the point y* provided y is between x and z and $xy = yz$. Since M is rotund, the reflection of x in y is uniquely determined.

Suppose m_1, m_3, m_4, t is a quadruple of points of M with t the midpoint of m_1 and m_3. We show m_1, m_3, m_4, t is congruent to a quadruple of points of the euclidean plane E_2.

First suppose p is distinct from m_1, m_3, m_4. Let q be the reflection of p in m_1, s the reflection of p in m_4, and then let r be the reflection of s in m_3. If m_2 is the midpoint of q and r, then by the Banach–Euclidean four-point property at p; $m_1m_2 = m_3m_4$ and $m_2m_3 = m_1m_4$ and m_1, m_2, m_3, m_4 is congruent to a quadruple μ_1, μ_2, μ_3, μ_4 of points E_2. Now μ_1, μ_2, μ_3, μ_4 are vertices of a parallelogram so the diagonals $S(\mu_1, \mu_3)$ and $S(\mu_2, \mu_4)$ bisect each other in a point τ. But m_1, m_2, m_3, m_4 are vertices of a parallelogram in a rotund Banach space, see [1], p. 42 and consequently the segments $S(m_1, m_3)$ and $S(m_2, m_4)$ bisect each other in the point t. Consequently, the quadruple m_1, m_3, m_4, t is congruent to the quadruple μ_1, μ_3, μ_4, τ.

If one of the points m_1, m_3, or m_4 is p, then the preceding argument is invalid. However, we may use a limit type argument to obtain the result. We treat only one case, as the others are quite similar. Suppose for definiteness that $p = m_1$. Let $\{m_{1i}\}$ be a sequence of points on the segment joining m_3 and p with $m_{1i} \neq p$, $(i = 1, 2, 3, ...)$ and $\lim m_{1i} = p$. Duplicating the preceding argument, we see that

each of the quadruples m_{1i}, m_3, m_4, t_i, where t_i is the midpoint of m_3 and m_{1i}, is congruent to a quadruple of points of E_2. Since m_3, m_4 appear in each quadruple, we may assume we have a sequence $\{\mu_{1i}, \mu_3, \mu_4, \tau_i\}$ of quadruples of points of E_2 with μ_{1i}, μ_3, μ_4, τ_i congruent to m_{1i}, m_3, m_4, t_i. Since the sequence $\{\mu_{1i}\}$ is bounded, it contains a convergent subsequence which by relabeling if necessary we may assume to be the original sequence. Let μ_1 denote $\lim \mu_{1i}$. By the continuity of the metric for E_2, it follows that $\lim \tau_i = \tau$, the midpoint of μ_1 and μ_3. Also since $\lim m_{1i} = p$, $\lim t_i$ is the midpoint of p and m_3. It is now clear that the quadruple p, m_3, m_4, t is congruent to the quadruple μ_1, μ_3, μ_4, τ.

We have now shown that each quadruple of points of M which contains a linear triple with one of the linear triple a midpoint of the other two is congruent to a quadruple of points of E_2. The result of Blumenthal [3] shows that M is a real inner-product space.

If we are willing to sacrifice the particular point p, we can use a formally weaker hybrid property and still obtain a characterization of real inner-product spaces.

The Banach-Euclidean feeble four-point property. If p, q, r, and s are points of M with s a midpoint of q and r and if m_1, m_2, m_3, and m_4 are respective midpoints of p and r, r and s, s and q, and q and p, then

(1) $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$ and

(2) the quadruple m_1, m_2, m_3, m_4 is congruent to a quadruple of points of E_2, the euclidean plane.

THEOREM 2. *A complete, convex, externally convex, metric space is a real inner product space if and only if M has the Banach–Euclidean feeble four-point property.*

Proof. It is easily seen that if M has the Banach–Euclidean feeble four-point property, then M satisfies the Young postulate and M has the two-triple property. Thus, M is a real rotund Banach space by the Andalafte and Blumenthal theorem [1]. Consequently the algebraic line determined by two distinct points of M is the only metric line joining those two points. Let p, q, m, and s be points of M, with m a midpoint of q and s. If $t = -p+2m$, then m is the midpoint of p and t. Letting $x = -[\frac{1}{2}(s+t)]+2t$ and $y = -[\frac{1}{2}(s+t)]+2s$, we have $\frac{1}{2}(s+t) = \frac{1}{2}(x+y)$; that is $\frac{1}{2}(s+t)$ is the midpoint of x and y. Choosing $z = -y+2p$ and $w = -x+2q$, we see $z = w$ and q, t, s, p are the respective midpoints z and x, x and $\frac{1}{2}(s+t)$, $\frac{1}{2}(s+t)$ and y, and y and z. Thus, by the Banach–Euclidean feeble four-point property, $pq = st$ and $ps = qt$ and E_2 contains a quadruple p', q', s', t' which is congruent to the quadruple p, q, s, t. Since the segment joining p and t and the segment joining q and s bisect each other in the point m and the segment joining p' and t' and the segment joining q' and s' bisect each other in a point m', the quadruple p, q, m, s is congruent to the quadruple p', q', m', s'. Once again we apply the result of Blumenthal [3] to see that M is a real inner-product space.

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On k-regular embeddings of spaces in Euclidean space

by

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Abstract. If $k \leqslant n$ are positive integers, a continuous map $f\colon X \to R^n$ is *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent. Such maps are of relevance in the theory of Čebyšev approximation. In this paper the question of existence of k-regular maps from a given X into R^n is considered. After discussing some elementary properties of k-regularity, an algebraic-topological method is introduced to obtain negative results. This method yields the fact that there does not exist a 3-regular map of the real projective plane into R^5, and this result is best possible. Finally, it is shown how to construct explicit 2- and 3-regular maps on real projective spaces which, in terms of homogeneous coordinates, are given by quadratic functions.

1. Introduction. If $k \leqslant n$ are positive integers, a continuous map f of a space X into Euclidean n-space $\boldsymbol{R}^n$ is *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent. Closely related to this is the concept of an *affinely k-regular map* $f\colon X \to \boldsymbol{R}^n$, where it is required that whenever $x_0, \ldots, x_k$ are distinct points of X, then $f(x_0), \ldots, f(x_k)$ are affinely independent (i.e. they are the vertices of a non-degenerate k-simplex in $\boldsymbol{R}^n$). The latter concept has been considered in [2], [1], and [9]. Clearly, a k-regular map is affinely $(k-1)$-regular, and $f\colon X \to \boldsymbol{R}^n$ is affinely $(k-1)$-regular if and only if the map $g\colon X \to \boldsymbol{R}^{n+1} = \boldsymbol{R} \times \boldsymbol{R}^n$ given by $g(x) = (1, f(x))$ is k-regular.

k-regular maps are of relevance in the theory of Čebyšev approximation. A set of n real-valued continuous functions on X is called a *k-Čebyšev set of length n* if these functions are the components of a k-regular map of X into $\boldsymbol{R}^n$. The reader is referred to [10], pp. 237–242 for the significance of this concept.

The present paper is concerned with existence and non-existence of k-regular maps. The following results are obtained:

THEOREM 2.1. *X admits a 2-regular map into $\boldsymbol{R}^n$ if and only if X admits an affinely 1-regular map into $\boldsymbol{R}^{n-1}$. (Thus if X is compact, existence of a 2-regular map of X into $\boldsymbol{R}^n$ is equivalent to X being topologically embeddable in $\boldsymbol{R}^{n-1}$).*

THEOREM 2.2. *If X admits a k-regular map into $\boldsymbol{R}^n$, then each $0 \leqslant i \leqslant k-1$, and S any subset of X with i points, $X - S$ admits a $(k-i)$-regular map into $\boldsymbol{R}^{n-i}$.*