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A note on transfinite sequences

by

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Abstract. The purpose of this paper is to show that transfinite sequences can be used to characterize topologies, various mappings and certain topological properties. The investigation leads to some set theoretic problems and a translation of the Axiom of Choice is obtained in terms of the existence of a certain type of well ordered neighborhood base at each point of any arbitrary topological space. Some of the properties characterized are: paracompactness, the Lindelöf property, compactness, the linearly Lindelöf property and the Hausdorff property. Furthermore, some of these characterizations demonstrate the interaction of the transfinite sequences with the uniform structure of the space.

Applications of these results by other authors to various problems are indicated and the paper concludes with characterizations of topologies and mappings (continuous, open, pseudo-open, closed, quotient) and a treatment of spaces whose topologies are determined by ordinary convergence of transfinite sequences as opposed to the clustering of transfinite sequences used elsewhere in the paper.

Introduction. The theory of convergence is fundamental to the study of topology and analysis. Classically, continuous functions were ones that preserved convergent sequences and metric spaces were compact if each sequence had a convergent subsequence. As metric spaces were generalized it was found that sequences were inadequate for characterizing topological properties. Suitable generalizations were sought; the most obvious being a transfinite sequence. However, preliminary work with transfinite sequences led investigators to the widely accepted conclusion that transfinite sequences are inadequate for describing topological properties.

To date the most fruitful generalizations have been nets and filters. Whereas nets lack two pleasant properties of sequences: 1) they are not well ordered and 2) the domain of a subnet need not be a subset of the domain of the original net, filters are better described as a generalization of a neighborhood base than a sequence. The purpose of this paper is to show that transfinite sequences, if properly applied, can be used to characterize topologies and certain topological properties. In addition, the investigation leads to some set theoretic problems and we obtain a translation of the Axiom of Choice (AC) in terms of the existence of a certain type of neighborhood base in an arbitrary space.

Definitions and known results. A non-void set is said to be directed by the binary relation $\leqslant$ provided that 1) if m, n and $p \in D$ with $m \leqslant n$ and $n \leqslant p$ then $m \leqslant p$, 2) $m \leqslant m$ for each $m \in D$ and 3) if $m, n \in D$ then there is a $p \in D$ with $m \leqslant p$ and $n \leqslant p$. A *net* is a function ψ: $D \to X$ from a directed set D into a space X. In the event D is well ordered (i.e., for each $m, n \in D$ either $m = n$ or $n < m$ or $m < n$ and each non-void subset of D has a first element) then ψ is said to be a *transfinite sequence*. For each $\alpha \in D$ let $x_\alpha = \psi(\alpha)$. We will often identify a net or transfinite sequence with its range $\psi(D) = \{x_\alpha\}_{\alpha \in D}$. If the set D is understood we will simply use $\{x_\alpha\}$.

If $R \subset D$ is such that $R \neq \emptyset$ and whenever $\alpha \in R$ and $\beta \in D$ with $\alpha \leqslant \beta$ then $\beta \in R$, we say that R is *residual* in D. If $C \subset D$ such that whenever $\beta \in D$ there is an $\alpha \in C$ with $\beta \leqslant \alpha$ we say C is *cofinal* in D. We also say $\{\alpha_\alpha\}$ is *frequently* in $B \subset X$ if there is a cofinal $C \subset D$ with $\{x_\gamma\}_{\gamma \in C} \subset B$ and *eventually* in B if there is a residual $R \subset D$ with $\{x_\theta\}_{\theta \in R} \subset B$. $\{x_\alpha\}$ is said to *converge* to $p \in X$ if it is eventually in each neighborhood of p and to *cluster* to p if it is frequently in each neighborhood of p. $\{x_\alpha\}$ is said to *cluster to a set* F if it is frequently in each open set containing F. If φ: $A \to X$ is the transfinite sequence $\{y_\beta\}$ and S is cofinal in A, the restriction of φ to S, which we will denote by φ_S: $S \to X$, will be called a *subsequence* of φ. We will also find it convenient to denote φ_s by $\{x_\sigma\}_{\sigma \in S}$ or simply $\{x_\sigma\}$.

A *subnet* is a more complex object than a subsequence. A net $\varphi: E \to X$ is a subnet of the net ψ: $D \to X$ if there exists a function f: $E \to D$ such that 1) $\varphi = \psi \circ f$ or in other words $\varphi(\alpha) = \psi(f(\alpha))$ for each $\alpha \in E$ and 2) for each $\alpha \in D$ there is a $\beta \in E$ such that if $\gamma \geqslant \beta$ then $f(\gamma) \geqslant \alpha$.

In what follows, if a uniformity is not mentioned in the statement of a theorem we will assume the space to be arbitrary, otherwise we will assume the space to be completely regular and T_1 (i.e., uniformizable). A uniformity μ for a space X is a filter of covering with respect to $<^*$ (star refinement) as defined by Tukey [24] whose development is equivalent to the one given by Bourbaki [3]; confer Isbell [14] p. 12. The pair (X, μ) is called a *uniform space*.

A transfinite sequence (resp. net) $\{x_\alpha\}_{\alpha \in A}$ is said to be *cofinally Cauchy* if for each $\mathcal{U} \in \mu$ there is a cofinal $C \subset A$ with $\{x_\gamma\}_{\gamma \in C} \subset U$ for some $U \in \mathcal{U}$. $\{x_\alpha\}$ is *Cauchy* if for each $\mathcal{U} \in \mu$ there is a residual $R \subset A$ with $\{x_\theta\}_{\theta \in R} \subset U$ for some $U \in \mathcal{U}$.

A family of coverings ν in which every member has a star refinement in ν is said to be a *normal family*. Since every collection λ of coverings contains a largest normal family μ we say the members of μ are *normal* in λ. A covering $\mathcal{U}$ is said to be *normal* with respect to λ if it is normal in $\varkappa$ where $\varkappa$ is the collection of all open coverings refined by members of λ. Given a completely regular topology τ, there exists a finest uniformity u for τ. It consists of all coverings that are normal with respect to the family of all open coverings. The open members of u are simply called *normal coverings* and u is called the *universal uniformity*.

If X is a completely regular T_1 space there is a uniformity β for X that has a basis consisting of all finite normal coverings. The completion of (X, β) is βX, the Stone-Čech compactification of X. Shirota [21] showed that every completely regular T_1 space has a uniformity that has a basis consisting of all countable normal coverings which he called the e-uniformity and established what Isbell refers to [14] as the first deep theorem of uniform spaces.

Theorem (Shirota). *For a completely regular space X the following are equivalent:*

1) *X is e-complete (complete with respect to the e-uniformity),*

2) *X is realcompact (X is a Q-space, see Hewitt [7]), and*

3) *X is homeomorphic to a closed subspace of a product of real lines.*

Completeness with respect to a uniformity is defined as the property that each Cauchy net converges. In [9] we defined a uniform space to be *cofinally complete* if each cofinally Cauchy net clusters and proved the following:

Theorem. *For a completely regular T_1 space X we have:*

1) *X is paracompact if and only if (X, u) is cofinally complete,*

2) *X is Lindelöf if and only if (X, e) is cofinally complete and*

3) *X is compact if and only if (X, β) is cofinally complete.*

Characterizations of topological properties. First, we can obtain an improvement of the above theorem in terms of transfinite sequences as follows:

Theorem 1. *For a completely regular T_1 space X we have:*

1) *X is paracompact if and only if each transfinite sequence that is cofinally Cauchy with respect to the u-uniformity clusters.*

2) *X is Lindelöf if and only if each transfinite sequence that is cofinally Cauchy with respect to the e-uniformity clusters.*

3) *X is compact if and only if each transfinite sequence that is cofinally Cauchy with respect to the β-uniformity clusters.*

Proof of 1). Let X be paracompact and $\{x_\alpha\}$ be a cofinally Cauchy transfinite sequence with respect to the u-uniformity. Suppose $\{x_\alpha\}$ does not cluster. Then for each $x \in X$ there is an open $U(x)$ containing x with $\{x_\alpha\}$ eventually in $X - U(x)$. Put $\mathcal{U} = \{U(x) |\ x \in X\}$. Since X is paracompact, $\mathcal{U}$ is normal as shown in [23] and hence $\mathcal{U} \in u$, so $\{x_\alpha\}$ is frequently in some $U(y)$. But then $\{x_\alpha\}$ cannot be eventually in $X - U(y)$ which is a contradiction. Therefore $\{x_\alpha\}$ clusters.

Conversely, assume each transfinite sequence that is cofinally Cauchy with respect to the u-uniformity clusters. By a theorem of Mack [17], it suffices to show that each well ordered open covering has a locally finite open refinement. Let $\{U_\alpha\}$ be a well ordered open covering of X (i.e., $\alpha < \beta$ implies $U_\alpha \subset U_\beta$) and put $F_\alpha = X - U_\alpha$ for each α. Then $\bigcap F_\alpha = \emptyset$. For each α let $<_\alpha$ be a well ordering of F_α and let $E = \{(F_\alpha, x) |\ x \in F_\alpha\}$. Define the well ordering $<$ on E by $(F_\alpha, x) < (F_\beta, y)$ if and only if $\alpha < \beta$ or $\alpha = \beta$ and $x <_\alpha y$. For each (F_α, x) in E put $\psi(F_\alpha, x) = x$. The assignment ψ: $E \to X$ is a transfinite sequence that cannot cluster to any $p \in X$ for otherwise $p \in F_\alpha$ for each α which contradicts $\bigcap F_\alpha = \emptyset$. Denote ψ by $\{y_\beta\}$.

For each $\mathcal{U} \in u$ there exists ([14] p. 7) a uniformly continuous f: $X \to M$ where M is a metric space such that for each ball $B(m, \varepsilon)$ of radius $\varepsilon < 1$ in M, $f^{-1}(B(m, \varepsilon)) \subset U$

for some $U \in \mathcal{U}$. $\{f(y_\beta)\}$ cannot cluster to any $m \in M$ for otherwise there is a cofinal $C \subset E$ with $\{f(y_\gamma)\}_{\gamma \in C} \subset B(m, \varepsilon)$ for $\varepsilon < 1$. Then

$$\{y_\gamma\} \subset f^{-1}(\{f(y_\gamma)\}) \subset f^{-1}(B(m, \varepsilon)) \subset U$$

for some $U \in \mathcal{U}$ so $\{y_\beta\}$ is cofinally Cauchy and therefore clusters which is a contradiction.

Suppose $p \in \bigcap \overline{f(F_\alpha)}$. Let U be an open set containing p and $\gamma \in E$ where γ represents some (F_β, z). Pick $\sigma > \beta$. Since $U \cap f(F_\sigma) \neq \emptyset$ there is an $x \in F_\sigma$ with $f(x) \in U$. If $\delta = (F_\sigma, x)$ then $y_\delta = x$ so $f(y_\delta) \in U$. But $\gamma < \delta$ so $\{f(y_\beta)\}$ is frequently in each neighborhood of p which is a contradiction so $\bigcap \overline{f(F_\alpha)} = \emptyset$.

For each α put $V_\alpha = M - \overline{f(F_\alpha)}$ and let $\mathcal{V} = \{V_\alpha\}$. Since M is metric and therefore paracompact, there is a locally finite open refinement $\mathcal{W}$ of $\mathcal{V}$ and hence $f^{-1}(\mathcal{W}) = \{f^{-1}(W) |\ W \in \mathcal{W}\}$ is locally finite in X. If $W \in \mathcal{W}$ there is a $V_\alpha \in \mathcal{V}$ with $W \subset V_\alpha = M - \overline{f(F_\alpha)}$ so that

$$f^{-1}(W) \subset X - f^{-1}(\overline{f(F_\alpha)}) \subset X - F_\alpha = U_\alpha .$$

Therefore $f^{-1}(\mathcal{W})$ refines $\{U_\alpha\}$ so that X is paracompact.

Proof of 2). Assume X is Lindelöf and suppose $\{x_\alpha\}_{\alpha \in A}$ is a cofinally Cauchy transfinite sequence with respect to the e-uniformity that does not cluster. For each $x \in X$ there is an open $U(x)$ containing x such that $\{x_\alpha\}$ is eventually in $X - U(x)$. Let $\mathcal{U} = \{U(x) |\ x \in X\}$. Since X is Lindelöf $\mathcal{U}$ has a countable subcovering $\{U(x_i)\}$ and since X is paracompact, $\{U(x_i)\}$ is normal and therefore belongs to the e-uniformity. But then $\{x_\alpha\}$ is frequently in some $U(x_j)$ which contradicts $\{x_\alpha\}$ eventually being in $X - U(x_j)$.

Conversely assume that each transfinite sequence that is cofinally Cauchy with respect to the e-uniformity clusters. Then each transfinite sequence that is cofinally Cauchy with respect to the u-uniformity clusters so by the proof of 1) X is paracompact and therefore countably metacompact. We will first show that a transfinite sequence $\{y_\beta\}_{\beta \in B}$ with no countable subsequence clusters. Let $\mathcal{U} \in e$. Then there is a $\{U_i\} \in e$ that refines $\mathcal{U}$. Put $D_i = \{\beta |\ y_\beta \in U_i\}$ and let $D = \bigcup D_i$. Suppose D_i is not cofinal in B for each i. Since $\{y_\beta\}$ has no countable subsequence, D is not cofinal in B so there is a $\delta \in B$ with $y_\delta \notin U_i$ for each i which is a contradiction. Consequently some D_j is cofinal in B so that $\{y_\beta\}$ is cofinally Cauchy with respect to the e-uniformity and hence clusters.

Next we show that a countably metacompact space in which each transfinite sequence with no countable subsequence clusters, must be Lindelöf. Suppose X is not Lindelöf. Let η be the least cardinal such that some open covering $\mathcal{U} = \{U_\alpha |\ \alpha < \eta\}$ has no countable subcovering. For each α put $V_\alpha = \bigcup \{U_\beta |\ \beta \leqslant \alpha\}$ and let $F_\alpha = X - V_\alpha$. It is easily shown that each $F_\alpha \neq \emptyset$ so pick $x_\alpha \in F_\alpha$ for each α. If $\{x_\alpha\}$ has no countable subsequence it clusters which is impossible so assume a countable cofinal $\{\alpha_i\} \subset \eta$. $\bigcap F_{\alpha_i} = \emptyset$ since $\{V_\alpha\}$ covers X. We can apply a theorem of F. Ishikawa [15] which says: In a countably metacompact space, if $\{F_{\alpha_i}\}$ is a decreasing countable sequence of closed sets with an empty intersection, there is a decreasing countable sequence $\{G_i\}$ of open sets with an empty intersection such that $F_{\alpha_i} \subset G_i$ for each i. Let $H_i = X - G_i$. Then $\{H_i\}$ is an ascending closed covering of X and

$$H_i = X - G_i \subset X - F_{\alpha_i} = V_{\alpha_i} = \bigcup \{U_\beta |\ \beta \leqslant \alpha_i\} .$$

Now $\mathcal{U}_i = \{U_\beta |\ \beta \leqslant \alpha_i\}$ has cardinality $< \eta$ so there is a countable subcollection of $\mathcal{U}_i$ that covers H_i. But since $\{H_i\}$ covers X, $\mathcal{U}$ has a countable subcovering and therefore X is Lindelöf.

Proof of 3). Suppose X is compact and $\{x_\alpha\}_{\alpha \in A}$ is a transfinite sequence. For each α put $M_\alpha = \{x_\beta |\ \beta > \alpha\}$ and let $U_\alpha = X - \overline{M}_\alpha$. If $\{x_\alpha\}$ does not cluster $\{U_\alpha\}$ covers X and therefore has a finite subcover $\{U_{\alpha_i}\}$. Let $\delta \in A$ such that $\alpha_i < \delta$ for each i. Then $x_\delta \notin U_{\alpha_i}$ for each i which is a contradiction. Therefore every transfinite sequence in X clusters.

Conversely assume each transfinite sequence that is cofinally Cauchy with respect to the β-uniformity clusters and suppose X is not compact. Let η be the least cardinal such that there is an open covering $\{U_\alpha |\ \alpha < \eta\}$ having no finite subcovering and pick $x_\alpha \in X - U_\alpha$ for each α. It is easily shown that all transfinite sequences in X are cofinally Cauchy with respect to the β-uniformity so $\{x_\alpha\}$ clusters to some $p \in X$. But then $p \in \bigcap (X - U_\alpha)$ which implies $\{U_\alpha\}$ does not cover X which is a contradiction. ∎

A transfinite sequence $\{x_\alpha\}$ will be called *cofinally Δ Cauchy* if for each open covering $\mathcal{U}$ of X there is a $p \in X$ such that $\{x_\alpha\}$ is frequently in $\mathrm{Star}(p, \mathcal{U})$. Mansfield [18] calls a space almost 2-fully normal if for each open covering $\mathcal{U}$ there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that if $p \in V \in \mathcal{V}$ and $q \in W \in \mathcal{V}$ with $V \cap W \neq \emptyset$ then there is a $U \in \mathcal{U}$ containing both p and q.

COROLLARY. 1) *An almost 2-fully normal T_1 space is paracompact if and only if each cofinally Δ Cauchy transfinite sequence clusters.*

2) *A regular, countably metacompact space is Lindelöf if and only if each transfinite sequence with no countable subsequence clusters.*

3) *A space is compact if and only if each transfinite sequence clusters.*

Proof of 1). This follows from part 1) of Theorem 1 and the method of proof in Proposition 3 of [9].

Proof of 2). The sufficiency was proved in part 3) of Theorem 1 and the necessity in the case where the space is T_1 follows from the fact that each transfinite sequence with no countable subsequence is cofinally Cauchy with respect to the e-uniformity. In the event the space is not T_1, a direct proof is obtainable.

Proof of 3). The proof is based on the fact that if the space is uniformizable, all transfinite sequences are cofinally Cauchy with respect to the β-uniformity. If the space is not uniformizable, the necessity was proven in part 3) of Theorem 1 and the sufficiency is a straightforward argument based on the descending chain characterization of compactness, a statement of which is given in [16] p. 163.

The property that each transfinite sequence with no countable subsequence clusters is equivalent (as will be shown) to a property investigated by Alexandroff and Urysohn [1] called "final compactness in the sense of complete accumulation points". A space is $[\alpha, \beta]$-compact in the sense of complete accumulation points, where α and β denote cardinals with $\alpha \leqslant \beta$, if every $M \subset X$ with $|M|$ a regular cardinal in $[\alpha, \beta]$ has a point of complete accumulation; i.e., a point p such that if U is an open set containing p, $|U \cap M| = |M|$. A space is finally compact in the sense of complete accumulation points if it is $[\alpha, \beta]$-compact in the sense of complete accumulation points for all cardinals $\beta > \alpha$.

THEOREM (Alexandroff and Urysohn). *A space is $[\alpha, \beta]$-compact in the sense of complete accumulation points if and only if every open covering $\mathcal{U}$ of X, with $|\mathcal{U}|$ a regular cardinal in $[\alpha, \beta]$, has a subcovering $\mathcal{U}'$ with $|\mathcal{U}'| < |\mathcal{U}|$.*

We define a space to be *linearly Lindelöf* if for each well ordered ascending open covering ($\{U_\alpha|\ \alpha < \varkappa\}$ such that $\alpha < \beta$ implies $U_\alpha \subset U_\beta$) has a countable subcover.

THEOREM 2. *The following properties are equivalent in a space X:*

1) *linearly Lindelöf,*

2) *final compactness in the sense of complete accumulation points,*

3) *each transfinite sequence with no countable subsequence clusters.*

Proof. (1)→(2) Suppose X is not finally compact in the sense of complete accumulation points. Then there is an $M \subset X$ of uncountable regular cardinality η with no complete accumulation point. Let $M = \{m_\alpha|\ \alpha < \eta\}$ and for each α put $U_\alpha = X - \overline{M}_\alpha$ where $M_\alpha = \{m_\gamma|\ \gamma > \alpha\}$. Then $\{U_\alpha\}$ is a well ordered ascending open covering. Let $\{U_{\alpha_i}\}$ be a countable subcovering of $\{U_\alpha\}$. But then $\{\alpha_i\}$ is cofinal in η contradicting the fact that η is regular.

(2)→(3) Suppose there is a transfinite sequence with no countable subsequence in X that does not cluster. Let $\{x_\alpha|\ \alpha < \eta\}$ be such a transfinite sequence of least cardinality η. Then η is regular. Therefore $\{x_\alpha\}$ has a point of complete accumulation say p. It is easily shown that $\{x_\alpha\}$ clusters to p.

(3)→(1) Suppose X is not linearly Lindelöf. Then there is a well ordered ascending open covering $\{U_\alpha|\ \alpha < \eta\}$ of least cardinality η having no countable subcovering. For each $\alpha < \eta$ put $V_\alpha = \bigcup \{U_\alpha|\ \beta < \alpha\}$ and $F_\alpha = X - V_\alpha$. Then $F_\alpha \neq \emptyset$ for otherwise $\{U_\beta|\ \beta < \alpha\}$ would cover X and therefore have a countable subcovering. For each α pick $x_\alpha \in F_\alpha$. Since $\{U_\alpha\}$ has no countable subcovering, $\{x_\alpha\}$ has no countable subsequence so $\{x_\alpha\}$ clusters to some $p \in X$. Then $p \in \bigcap F_\alpha$ which implies $\{U_\alpha\}$ does not cover X which is a contradiction. ■

A. Miščenko [19] exhibited a space that he named $\overset{*}{R}$ that is completely regular, T_1, finally compact in the sense of complete accumulation points, but not Lindelöf. Miščenko's space is constructed as follows: Let

$$R = \prod_{i=1}^{\infty} [0, \omega_i] \quad \text{where} \quad [0, \omega_i] = \{\alpha|\ \alpha \text{ is an ordinal and } \alpha \leqslant \omega_i\}.$$

Then R is a compact Hausdorff space in the product topology, $\overset{*}{R}$ is a subspace of R defined as follows: let

$$R_k = \prod_{i=1}^{k} [0, \omega_i] \times \prod_{i=k+1}^{\infty} [0, \omega_i) \quad \text{and put} \quad \overset{*}{R} = \bigcup_{k=1}^{\infty} R_k .$$

Miščenko did not know if $\overset{*}{R}$ was normal or not. M. E. Rudin showed the author that $\overset{*}{R}$ is not normal by exhibiting two disjoint closed sets

$$H = \{\{x_i\} \in \overset{*}{R}|\ x_i \neq 0 \text{ for each } i\} \text{ and}$$

$$K = \{\{x_i\} \in \overset{*}{R}|\ \text{there is an } n \text{ with } x_{n+1} = 0 \text{ and for } i \leqslant n,\ x_i = \omega_i\}$$

that cannot be separated by disjoint open sets. By Theorem 2, Miščenko's space is a uniformizable (completely regular, T_1) space that is linearly Lindelöf but not Lindelöf.

In the preliminary report [10] the author pointed out that if there existed a normal Hausdorff space that was linearly Lindelöf but not Lindelöf, it would be a Dowker space (this follows from part 2 of the corollary to Theorem 1). If a space X has an uncountable open covering with no subcovering of smaller cardinality let $\lambda(X)$ be the least cardinal such that there is an uncountable open covering $\mathcal{U}$ with $|\mathcal{U}| = \lambda(X)$ having no subcovering $\mathcal{U}'$ with $|\mathcal{U}'| < \lambda(X)$. Otherwise put $\lambda(X) = \omega$. We call $\lambda(X)$ the Lindelöf cardinal of the space.

PROPOSITION 1. *A necessary and sufficient condition for a linearly Lindelöf space X to be Lindelöf is that $\lambda(X)$ is a regular cardinal.*

COROLLARY. *If there is a normal Hausdorff space that is linearly Lindelöf but not Lindelöf, the Lindelöf cardinal of the space must be singular.*

The proof of the above proposition is based upon the techniques used in the proofs of Theorems 1 and 2. Since $|\overset{*}{R}|$ is the first singular cardinal, and $\overset{*}{R}$ is linearly Lindelöf but fails to be Lindelöf $\lambda(\overset{*}{R}) = |\overset{*}{R}|$. Therefore, it was pointed out by the author that a modification of $\overset{*}{R}$ would be a good candidate from which to construct a normal Hausdorff space that was linearly Lindelöf but not Lindelöf. Such a space would then be a Dowker space. This is the singular cardinal idea referred to by M. E. Rudin in [20]. In fact, she increased the quantity of open sets of $\overset{*}{R}$ by considering a strong product topology (called a box topology in [20]) and after removing some undesirable points of $\overset{*}{R}$ so topologized, produced a Dowker space.

However, as she pointed out to the author, her space fails badly at being linearly Lindelöf, being the union of uncountably many disjoint open sets. Consequently, we still do not know if there exists a normal Hausdorff space that is linearly Lindelöf but not Lindelöf. Since the mentioning of this question in the preliminary report [10], there have been further results by other authors that shed additional light on the problem. [8] and [25] are systematic treatments of $[\alpha, \beta]$-compactness. Smirnov [22] defined a space X to be $[\alpha, \beta]$-compact in the sense of covering if every open cover $\mathscr{U}$ of X with $|\mathscr{U}| \leqslant \beta$ has a subcover $\mathscr{U}'$ with $|\mathscr{U}'| < \alpha$. A space is said to be *finally compact* (*$[\alpha, \infty]$-compact*) in the sense of coverings if it is $[\alpha, \beta]$-compact in the sense of coverings for all $\beta \geqslant \alpha$.

Following the convention of Hodel and Vaughan [8] we will abbreviate $[\alpha, \beta]$-compactness in the sense of coverings to $[\alpha, \beta]$-compactness and $[\alpha, \beta]$-compactness in the sense of complete accumulation points to $[\alpha, \beta]$-compactness^r where they state that the superscript r is a reminder of the restriction to regular cardinals. With this convention we note that $[\aleph_0, \infty]$-compact is the same as compact, $[\aleph_0, \aleph_0]$-compact is countably compact and $[\aleph_1, \infty]$-compact is the Lindelöf property. The above problem then becomes: Does there exist a normal Hausdorff $[\aleph_1, \infty]$-compact^r space that is not $[\aleph_1, \infty]$-compact? [8] is a study of the more general question of when an $[\alpha, \beta]$-compact^r space is $[\alpha, \beta]$-compact. The method of attack here is to generalize part 2) of the corollary to Theorem 1 and other results by Alexandroff and Urysohn and Miščenko simultaneously. [25] is a study of three other properties that have been reported in the literature as being equivalent to $[\alpha, \beta]$-compactness or $[\alpha, \beta]$-compactness^r. It is shown that none of these claims are correct and an implication diagram is given showing the relationship among these properties and $[\alpha, \beta]$-compactness.

Part 2) of the corollary to Theorem 1 shows that countable metacompactness is a sufficient condition for a linearly Lindelöf space to be Lindelöf and it is both necessary and sufficient if the space is regular (T_3). It is possible to generalize both Proposition 1 and part 2) of the corollary by considering only the boundaries of proper open subsets.

PROPOSITION 2. *A linear Lindelöf space is Lindelöf if and only if the boundary ∂U of each proper open subset U has one of the following properties*:

1) *∂U is regular and countably metacompact,*

2) *$\lambda(\partial U)$ is a regular cardinal.*

Characterizing topologies and mappings. The success of the theory of convergent sequences in metric spaces is due to the existence of a countable well ordered neighborhood base at each point such that the well ordering is identical to the partial ordering of set inclusion. Our next result is that in any space, each point has a well ordered neighborhood base such that the well ordering is compatible with the partial ordering of set inclusion. In other words, if U and V are two neighborhoods in this well ordered neighborhood base such that $U \subset V$, then U must follow V in the well ordering. Let (N) be the following statement: each point in a space X has a well ordered neighborhood base such that the well ordering is compatible with the partial ordering of set inclusion.

THEOREM 3. AC $\leftrightarrow$ N.

The proof relies on the following lemma which is an equivalent form of the Axiom of Choice and was first stated by the author in [12] and proved in [11]. Vaughan has recently included a proof in [25] which he claims to be slightly shorter than the author's so we will omit the proof here. Vaughan uses the lemma in showing that the property which he calls $S[\alpha, \beta]$ introduced by Smirnov [22] is equivalent to $[\alpha, \beta]$-compactness provided the cofinality of β is $\geqslant \alpha$ and in showing that another property which he denotes by $G[\alpha, \beta]$ which was introduced by Gaal [5] is implied by $S[\alpha, \beta]$.

LEMMA. *If $(P, <)$ is a partially ordered set and $\prec$ is any well ordering of P then there is a subset S of P, cofinal with respect to $<$ such that $\prec$ is compatible with $<$ on S; i.e., if $a, b \in S$ with $a<b$ then $a \prec b$.*

Proof of Theorem 3. AC implies the above lemma which in turn implies (N). Conversely, given the statement (N). Let X be a set and let τ be the finite complement topology on X. Let $(B, <)$ be a well ordered neighborhood base for some $p \in X$. For each $y \in X$ distinct from p let $\Phi(y) = A$ such that A is the first neighborhood in B with respect to $<$ with $A \subset X - \{y\}$. Then $\Phi\colon X - \{p\} \to B$ and $\Phi^{-1}(A)$ contains at most finitely many members. For each $A \in B$ with $\Phi^{-1}(A) \neq \emptyset$ we can well order $\Phi^{-1}(A)$ by some well ordering $<_A$ since $\Phi^{-1}(A)$ is finite. Define the well ordering $\prec$ on $X - \{p\}$ as follows: if $x, y \in X - \{p\}$ put $x \prec y$

1) if $\Phi(x) < \Phi(y)$ or

2) if $\Phi(x) = \Phi(y)$ and $x <_A y$ where $A = \Phi(x)$.

We conclude that X can be well ordered so that (N) implies the Well Ordering Principle and consequently AC. ■

A transfinite sequence $\{x_\alpha | \ \alpha < \eta\}$ is said to cluster to two points p and q *simultaneously* if for each pair of neighborhoods U and V of p and q respectively, there is a cofinal $C \subset \eta$ with $\{x_\alpha | \ \alpha \in C\} \subset U \cap V$.

THEOREM 4. *The following statements are valid in any space X.*

1) *$U \subset X$ is open if and only if no transfinite sequence in $X - U$ clusters to a point of U.*

2) *$F \subset X$ is closed if and only if a transfinite sequence in F can only cluster to a point of F.*

3) *p is a limit point of $M \subset X$ if and only if there is a transfinite sequence in $M - \{p\}$ that clusters to p.*

4) *X is Hausdorff (T_2) if and only if no transfinite sequences can cluster to two distinct points simultaneously.*

5) *X is T_1 if and only if for each pair of distinct points there are two transfinite sequences clustering to the two points respectively but neither clustering to the other point.*

6) *X is T_0 if and only if for each pair of distinct points there is a transfinite sequence that clusters to one of the points but not the other.*

All parts of Theorem 4 rely directly or indirectly on the lemma to Theorem 3. We will only indicate a proof of 1) and 4) as representative. It is possible to formulate characterizations of regularity and normality similar to the characterization of Hausdorff in 4). Also, it is easily shown that the T_1 property is equivalent to the statement that each constant transfinite sequence clusters to exactly one point.

Proof of 1). Assume no transfinite sequence in $X-U$ clusters to a point of U and suppose U is not open. Then there is a $p \in U$ each neighborhood of which meets $X-U$. By Theorem 3 there is a well ordered neighborhood base $\{V_\alpha | \alpha<\eta\}$ such that the well ordering is compatible with the partial ordering of set inclusion. For each $\alpha<\eta$ pick $x_\alpha \in V_\alpha \cap (X-U)$. It is easily shown that $\{x_\alpha\} \subset (X-U)$ clusters to p which is a contradiction.

Proof of 4). Suppose X is not Hausdorff. Then there are distinct points p and q with each neighborhood of p meeting each neighborhood of q. Let $B(p)$ and $B(q)$ be neighborhood bases for p and q respectively and put $P = B(p)\times B(q)$. Define $\leqslant$ on P as follows: if $U, W \in B(p)$ and $V, Z \in B(q)$ with $U\subset W$ and $V\subset Z$ then $(W, Z)\leqslant(U, V)$. By the lemma to Theorem 3, there is a well ordered cofinal subset $(S, \prec)$ of $(P, \leqslant)$ such that $\prec$ is compatible with $\leqslant$ on S. For each $(U, V)\in S$ pick $\psi(U, V) \in U\cap V$. Then the assignment $\psi\colon S\to X$ is a transfinite sequence which can be shown to simultaneously cluster to both p and q.

Let E be a well ordered set and for each $\alpha \in E$ suppose $\varphi_\alpha\colon A_\alpha \to X$ is a transfinite sequence. Consider the A_α to be formally disjoint and put $A = \bigcup A_\alpha$. Define the well ordering $<$ on A as follows: if $a, b\in A$ then $a<b$

1) if $\gamma<\delta$ in E where $a\in A_\gamma$ and $b\in A_\delta$ or

2) if a and b both belong to the same A_γ and $a<b$ in A_γ.

The ordering $<$ on A is usually called the lexicographic ordering of A. Next we define the transfinite sequence $\Sigma\colon A\to X$ called the *sum* of the transfinite sequences $\{\varphi_\alpha\}$ as follows: $\Sigma(a) = \varphi_\gamma(a)$ where $a\in A_\gamma$.

PROPOSITION 3. *Let X be a space, $\varphi\colon E\to X$ a transfinite sequence and for each $\alpha\in E$ let $\varphi_\alpha\colon A_\alpha\to X$ be transfinite sequences and Σ be the sum of the φ_α. Then*

1) *if φ is constant ($\varphi(x) = p$ for all x) then φ clusters,*

2) *if $E = A\cup B$ and φ clusters to p then either φ_A or φ_B exists and clusters to p,*

3) *if for each α, $\varphi_\alpha\subset\{\varphi(\beta)|\ \beta\leqslant\alpha\}$ and φ_α clusters to p then φ clusters to p.*

4) *if for each α, φ_α clusters to $\varphi(\alpha)$ and φ clusters to p then Σ clusters to p.*

The proof of Proposition 3 amounts to a routine verification of the four properties above. Next we define a *transfinite sequence class* (TS class). Let S be a set and $\mathscr{C}$ a class of ordered pairs (φ, p) where φ is a transfinite sequence in S and $p\in S$. We will call $\mathscr{C}$ a TS class on S if $\mathscr{C}$ satisfies the four conditions of Proposition 3, i.e., if "φ clusters to p" can be replaced by "$(\varphi, p)\in\mathscr{C}$" in 1) through 4) of Proposition 3 and "φ does not cluster to p" can be replaced by "$(\varphi, p)\notin\mathscr{C}$."

THEOREM 5. *Let $\mathscr{C}$ be a TS class on a set S and for each $A\subset S$ let A^c be the set of all $a\in S$ with $(\varphi, a)\in\mathscr{C}$ and $\varphi\subset A$. Then ^{c} is a closure operator on S and $(\varphi, a)\in\mathscr{C}$ if and only if φ clusters to a relative to the topology associated with ^{c}.*

Proposition 3 and the above theorem set up a one-to-one correspondence between the various topologies a set can have and the TS classes on the set. It is clear from the definition of clustering that if $\mathscr{C}_1$, and $\mathscr{C}_2$ are two TS classes and τ_1 and τ_2 are the associated topologies, that $\mathscr{C}_1\subset\mathscr{C}_2$ if and only if $\tau_2\subset\tau_1$. Also, one notices that if $\mathscr{C}_1\vee\mathscr{C}_2$ denotes the smallest TS class containing both $\mathscr{C}_1$ and $\mathscr{C}_2$ then $\mathscr{C}_1\vee\mathscr{C}_2$ is the TS class associated with $\tau_1\cap\tau_2$.

THEOREM 6. *Let $f\colon X\to Y$ be a function from a space X into a space Y. Then*

1) *f is continuous if and only if for each transfinite sequence $\{x_\alpha\}$ in X that clusters to some point $p\in X$, $\{f(x_\alpha)\}$ clusters to $f(p)$.*

2) *If f is onto, f is open if and only if for each transfinite sequence $\{y_\beta\}\subset Y$ that clusters to some $p\in Y$ and for each $q\in f^{-1}(p)$ there is a transfinite sequence $\{x_\alpha\}\subset\bigcup\{f^{-1}(y_\beta)\}$ that clusters to q.*

3) *f is closed if and only if whenever a transfinite sequence $\{y_\beta|\ \beta<\eta\}\subset Y$ clusters to some $p\in Y$, each transfinite sequence $\{x_\beta|\ \beta<\eta\}$ with $x_\beta\in f^{-1}(y_\beta)$ clusters to $f^{-1}(p)$.*

4) *If f is onto, f is pseudo-open if and only if for each transfinite sequence $\{y_\beta\}\subset Y$ clustering to $p\in Y$ there is a transfinite sequence $\{x_\alpha\}\subset\bigcup\{f^{-1}(y_\beta)\}$ clustering to q for some $q\in f^{-1}(p)$.*

5) *f is a quotient mapping if and only if for each transfinite sequence $\{y_\beta\}\subset Y$ clustering to some $p\in Y$ there is a transfinite sequence $\{x_\alpha\}\subset\bigcup\{f^{-1}(y_\beta)\}$ that is frequently in each open inverse image of an open set in Y that contains $f^{-1}(p)$.*

All parts of Theorem 6 rely on the lemma to Theorem 3. Part 1) has the simplest proof however, it is still representative of the proofs of 2) through 5) so we will only indicate its proof here.

Proof of 1). Assume f is continuous and suppose $\{x_\alpha\}\subset X$ clusters to p. Let $U\subset Y$ be a neighborhood of $f(p)$ and V a neighborhood of p with $f(V)\subset U$. Then $\{x_\alpha\}$ frequently in V implies $\{f(x_\alpha)\}$ is frequently in $f(V)\subset U$, so $\{f(x_\alpha)\}$ clusters to $f(p)$.

Conversely suppose f is not continuous. Then there is a $p\in X$ and a neighborhood U of $f(p)$ such that $f(V)$ is not contained in U for each neighborhood V of p. By Theorem 3 there exists a well ordered neighborhood base $\{V_\alpha|\ \alpha<\eta\}$ for p such that the well ordering is compatible with the partial ordering of set inclusion. For each α pick $x_\alpha\in V_\alpha$ such that $f(x_\alpha)\notin U$. Then $\{x_\alpha\}$ clusters to p but $\{f(x_\alpha)\}$ does not cluster to $f(p)$. ■

Convergence of transfinite sequences. A. V. Arhangel'skiĭ [2] introduced the concept of a *Frechét space* as one that whenever $p\in \mathrm{Cl}(F)$, there is a countable sequence $\{x_n\}\subset F$ that converges to p. He characterized Frechét spaces as pseudo-open images of first countable spaces. We will call a space *Transfinite Frechét* (TF) if whenever $p\in\mathrm{Cl}(F)$ there is a transfinite sequence $\{x_\alpha\}\subset F$ that converges to p.

S. P. Franklin [4] investigated a slight variation of Arhangel'skiĭ's idea. He called a space *sequential* if each countable sequence converging to a point of U is eventually in U implies U is open. It is easy to see that Frechét spaces are sequential and pseudo-open mappings are quotients by definition. Therefore Franklin arrived at the following: sequential spaces are quotients of first countable spaces. We will call a space *Transfinite sequential* (TS) if each transfinite sequence converging to a point of U is eventually in U implies U is open. Furthermore, our transfinite generalization of a first countable space will be a *chain local base* (CLB) space which is defined as a space in which each point has a local base well ordered by set inclusion.

It is fairly easy to show directly that CLB $\rightarrow$ TF $\rightarrow$ TS but we will not do so here because this implication diagram will follow as a corollary to Theorem 7. An example of a TF space that is not CLB is the space $\overset{*}{X}$ defined as follows. Let $X = \{\alpha \mid \alpha < \omega_1\}$ and let X have the discrete topology. Let $\overset{*}{X}$ be the one point compactification of X. It can be shown that the point at infinity has no chain local base yet $\overset{*}{X}$ is TF.

THEOREM 7. 1) *A space is* TF *if and only if it is the continuous pseudo-open image of a* CLB *space.*

2) *A space is* TS *if and only if it is the quotient of a* CLB *space.*

The proof of 1) is representative of the proof of both parts of Theorem 7, so we will only include it here.

The proof of Theorem 7, was given in [11]. However, it has recently been pointed out to the author that there exists an earlier proof of part 2) in [6]. In any event, Theorem 7, is a straight forward generalization of the earlier work of Arhangel'skiĭ [2] who is responsible for the central idea. The proof of part 1) uses the following two lemmas, the first of which is a slight improvement of Theorem 6 part 4).

LEMMA. *A function f: $X \rightarrow Y$ is pseudo-open if and only if for each transfinite sequence $\{y_\beta \mid \beta < \eta\} \subset Y$ with $\eta \leqslant 2^{|Y|}$ clustering to some $p \in Y$, there is a transfinite sequence $\{x_\alpha\} \subset \bigcup \{f^{-1}(y_\beta)\}$ clustering to q for some $q \in f^{-1}(p)$.*

LEMMA. *A function f: $X \rightarrow Y$ where X is a* CLB *space is continuous if and only if whenever $\{x_\alpha\} \subset X$ converges to some $p \in X$, $\{f(x_\alpha)\}$ converges to $f(p)$.*

Proof. We will only indicate a proof of the necessity which represents the main part of the proof. Assume Y is TF and construct X as follows; for each $y \in Y$ and each transfinite sequence $x = \{x_\alpha \mid \alpha < \eta\}$ with $\eta \leqslant 2^{|Y|}$ converging to y let $y(x) = \{x_\alpha\} \cup \{y\}$ where each point of $y(x)$ is considered distinct and $y(x)$ is considered to be well ordered by the ordering induced from x with y considered to be the last point. Topologize $y(x)$ by letting each point of $y(x)$ be discrete with the exception of y and let y have a local base consisting of sets of the form $R(x_\alpha) = \{x_\gamma \in y(x) \mid \alpha < \gamma\}$. Then let Σ be the disjoint topological sum of all the $y(x)$'s.

Let $\sigma(x, \alpha)$ denote the member of Σ which comes from the αth element of the transfinite sequence x and if x converges to p, consider $\sigma(x, p) = p$. Then define f: $\Sigma \rightarrow Y$ by $f(\sigma(x, \alpha)) = x_\alpha$ and $f(\sigma(x, p)) = p$. The intuitive idea is that f maps each $\sigma \in \Sigma$ back onto the element of Y which generated it. Clearly f is onto and Σ is a CLB space. That f is continuous follows from the previous lemma. Next we show f is pseudo-open using the other lemma above.

Let $p \in Y$ and suppose $x = \{x_\alpha\}$ clusters to p. Then $p \in \bar{x}$. Then there is a transfinite sequence $y = \{y_\beta\} \subset x$ that converges to p since Y is TF so $p(y) \subset \Sigma$. Then $p(y) \subset f^{-1}(x)$. Let i: $y \rightarrow y$ be the identity transfinite sequence. Then i converges to p, $i \subset f^{-1}(x)$ and $p \in f^{-1}(p)$. Hence for each transfinite sequence $x \subset Y$ that clusters to some $p \in Y$, there is a transfinite sequence $y \subset f^{-1}(x)$ such that y clusters to q for some $q \in f^{-1}(p)$. Therefore f is pseudo-open. ∎

Another type of characterization of TS spaces and sequential spaces can be obtained in terms of mappings. A function f: $X \rightarrow Y$ is said to be *sequentially continuous* if for each sequence $\{x_n\} \subset X$ with $\{x_n\}$ converging to p in X, $\{f(x_n)\}$ converges to $f(p)$. Similarly, f is called TS *continuous* if for each transfinite sequence $\{x_\alpha\} \subset X$ converging to p, $\{f(x_\alpha)\}$ converges to $f(p)$.

PROPOSITION 4. *Given a space X,*

1) *X is* TS *if and only if each* TS *continuous function on X is continuous,*

2) (R. Chandler) *X is sequential if and only if each sequentially continuous function on X is continuous.*

Chandler [18] also proved the following proposition where a space is defined to be *accumulation complete* if each countable sequence that clusters to a point p has a subsequence that converges to p.

PROPOSITION 5. *A sequential space is Frechét if and only if it is accumulation complete.*

Extension of the accumulation complete property to the transfinite case in the obvious way does not lead to a transfinite generalization of Proposition 5 as the following proposition shows:

PROPOSITION 6. *If a space X is T_1, and whenever a transfinite sequence clusters to a point it has a subsequence that converges to the point then X is discrete.*

Proof. The proof is based on the fact that for each $p \in X$ it is possible to construct a transfinite sequence $\psi \subset X - \{p\}$ the cofinality of whose domain is greater than the cardinality of the space and which repeats each point in $X - \{p\}$ cofinally many times. Then, unless p is isolated, ψ clusters to p. Each subsequence of ψ must repeat some point in $X - \{p\}$ cofinally many times. Since X is T_1 it is possible to pick a neighborhood N of p that does not contain this point and hence the subsequence is frequently outside of N so that it cannot converge to p.

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Banach-Euclidean four-point properties

by

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Abstract. A metric space has the Banach–Euclidean four-point property at a point p provided for each triple of its points q, r, and s, if m_1, m_2, m_3, m_4 are respective midpoints of p and q, q and r, r and s, and s and p then $m_1m_2 = m_3m_4$ and $m_1m_4 = m_2m_3$ and the quadruple m_1, m_2, m_3, m_4 is congruent to a quadruple of points of the euclidean plane. The main result of the paper is that a complete, convex, externally convex, metric space is a real inner-product space if and only if it has the Banach–Euclidean four-point property at some point.

Let S denote a space which satisfies the axioms of Hilbert's groups I, II, III and V; namely, the axioms of connection, order, congruence, and continuity. Young [6] proved S is euclidean, hyperbolic, or elliptic, respectively, if and only if there is *one* triangle such that the length of the line joining the middle points of two sides is (1) equal to, (2) less than, or (3) greater than the third side, respectively.

Andalafte and Blumenthal [1] extended the notion of (1) above to metric spaces in the following way.

The Young postulate. If p, q, and r are points of a metric space M, and if q. and r' are the midpoints of p and q, and of p and r, respectively, then $q'r' = \frac{1}{2}qr$

They proved a complete, convex, externally convex, metric space with the two-triple property is a Banach space if and only if it satisfies the Young Postulate.

A direct analogue of Young's result is: a complete, convex, externally convex, metric space M is a euclidean (inner-product) space if and only if M contains one triple of points which satisfies the Young condition. This is of course false, for every rotund Banach space satisfies the Young Postulate. To make matters worse, there is a complete, convex, externally convex, metric space which satisfies the Young Postulate at one point, but is not a Banach space, see [4].

We focus our attention on an immediate consequence of the Young postulate.

The quadrilateral midpoint property. If p, q, r, and s are points of a metric space M, and if m_1, m_2, m_3, m_4 are respective midpoints of p and q, q and r, r and s, and s and p, then

$$m_1m_2 = m_3m_4 \quad \text{and} \quad m_1m_4 = m_2m_3 .$$