Lemma 12. Since $\deg f_i^{i+1} = 1$ for each i, PR_j is an essential map from C onto S^1, thus PR_j is weakly confluent. If $g|M$ were weakly confluent, then $PR_j \circ g|M$ would be weakly confluent. Therefore $g(M)$ is C and $g|M$ is not weakly confluent, implying that C is not in class W. This completes the proof.

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Extending a partial equivalence to a congruence and relative embeddings in universal algebras

by

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Abstract. i) The partial equivalences which extend to congruences on arbitrary finitary universal algebras are characterized (as in [5] but with an additional particularization) freeing SC of [3] from the requirement that the equivalence have an initial generating domain, and yielding ii) the characterization of "admissible" subsets for semigroups developed in [1] as well as iii) a characterization of the partial algebras relatively embeddable in the full agebras of an equational class, which specialized to iv) a characterization of partial S-sets relatively embeddable in full ones, leads to v) that exactly for the subsemigroups T right unitary in S can T-sets be relatively embedded in S-sets.

Let ϱ be a *partial equivalence* ([1], p. 43), i.e. a symmetric transitive relation, on a finitary universal algebra A: we ask when the classes of ϱ are (in their totality) those of a single congruence on A. It is clear that if this is so for any congruence, then it will be so for the congruence θ generated by (i.e. the smallest congruence containing) ϱ; hence we investigate when strengthening ϱ to θ does not enlarge its classes.

The generation of θ from ϱ may be effected in stages. First, one extends ϱ to the smallest containing equivalence on A. Initially ϱ may only be defined on a proper subset D of A: it suffices to make it reflexive by augmenting it with the diagonal on the complement D' of D in A; in this process it loses neither its symmetry nor its transitivity and so becomes an equivalence on A. Its individual classes do not become enlarged; the new classes are just the singletons of the complement D'.

The next stage is to strengthen the relation to one having the substitution property for each of the operations which define the algebraic structure of A. This means that whenever an argument is replaced by an element mod ϱ-related to it, the value of the operation is to change (at most) to an element mod ϱ-related to the value. We must thus strengthen the equivalence to include the relation which holds between (possibly inequivalent) operation values for equivalent arguments — and then iteratively for arguments related in the so strengthened way. This strengthened relation turns out to be still reflexive and symmetric but may fail to be transitive; however its transitive closure is the desired congruence; and since the passage to this closure would of itself not enlarge the original classes of ϱ (which was assumed transitive), we see that the critical condition for one of these classes not to be enlarged is that whenever it contains some value of an operation, it also contains the values for all arguments mod ϱ-related to its arguments.

A more manageable form can be given this criterion by presenting a more explicit form for the weakest relation satisfying the substitution property and stronger than a given equivalence relation. To this end, observe that the functions with respect to which a relation satisfies the substitution property — the values of the function related whenever one of its arguments is replaced by a related element — are closed for composition. It follows that if a relation has this property for the defining algebraic operations, it will have it also for their finite composites: i.e., for all the functions induced on A by the "words" or "terms" built with its operations. Conversely, the relation defined as holding between the two values of any such composite when one of its arguments is changed to an equivalent element, is easily shown to satisfy the substitution property. It is therefore with respect to this relation that the ϱ-classes must be closed, in order to be classes of the generated congruence (1).

We can be still a little more explicit by noting that we need not include in the relation the composites evaluated at arguments in the subalgebra of A generated by D, apart from those in D itself: indeed, these arguments are themselves values of composites for arguments in D, and so the part of the relation obtainable with them as arguments would already be contributed by the composites with arguments from D resulting from substituting for them their forms as composites in elements of D.

To summarize this conveniently, let us call the selfmap of A obtained from a composite of operations by fixing all but one of the arguments, a *translation* with *coefficients* these fixed arguments; and let us say that a partial equivalence ϱ has the *strong substitution property* or is *strong* for a translation if, whenever the latter sends an element of its domain D of ϱ into D, it also sends the whole ϱ-class of this element into the ϱ-class of the image. Then we have

i) A partial equivalence on an algebra A has the same classes as some congruence on A, if and only if it is strong for the translations with coefficients from its domain D or the complement of the subalgebra generated by D — hence if D generates A, strong for just the translations with coefficients from D.

In the latter case the criterion is independent of the containing algebra A generated by D; and in any event (since it is couched in terms of translations) depends on the particular operations chosen to define the algebraic structure only through the system of composites they induce on A.

The result extends Theorem SC of [3] from the case there treated of D generating A so that no operation ever takes a value in D for any argument not in D: for that case the criterion need only be applied with the "elementary" translations, i.e. those resulting from (uncomposed) operations by fixing arguments in D.

(1) This much duplicates Theorem 5 of [5].

A specialization in a different direction, which can also dispense with the complement of the generated subalgebra, is to the characterization of those partial equivalences on a semigroup, which may be completed to left congruences, as given in Theorem 10.4 (ii) ([1], p. 179). This may be done somewhat more generally for S-sets: namely for sets A equipped with a "multi-unary" structure of selfmaps f_s, indexed by a semigroup S and composing concordantly to its multiplication: $f_s \circ f_t = f_{st}$. The set of f_s thus constitutes the totality of composites of operations, and since they are all unary, also of translations.

ii) A partial equivalence ϱ on an S-set A consists of classes of some congruence, if and only if $a\varrho b$ and $f_s(a)$ in its domain implies $f_s(a)\varrho f_s(b)$: i.e. the image of a class $f_s(C)$ can meet a class C' only by being contained in it.

Another type of application is to the characterization of those partial algebras which can be embedded in full algebras of an equational class as "relative" subalgebras: i.e., so that new operation values never occur among the original elements; equivalently, so that the fully extended operations induce on the given domain exactly the given operations (rather than extensions). This may be derived from the above by taking for A the algebra generated freely in the equational class by the unstructured set of the given partial algebra P, and for ϱ the partial equivalence generated from the evaluations defined in P, modulo which P should obtain its structure: the quotient of A modulo any congruence inducing just ϱ will then furnish an algebra of the class in which P is embedded as desired. Before proceeding with the details we enlarge the setting to treat a more general situation at the same time.

Partial algebras of some type are usually defined as sets on which the specific algebraic operations of the type need only be partially defined, the composites of such operations then becoming partially defined as the composites of partial functions; but we may generalize by construing these composites as partial functions in their own right and so allowing them to be defined elsewhere as well: i.e. also where their values do not result from composing their defined constituent partial operations (of course we will require that these values do so result in the full algebra into which we seek an embedding. Alternatively, we might take the composites as new operations and impose the composition equations as identities) — and now require of the embedding that it effect no extension internal to P of these composite operations, as well as of the basic ones.

In order for P to be so embeddable in a full algebra of an equational class, it is necessary that each of its identites hold in P *strongly*: i.e., if one side is evaluable then so is the other and the two values are the same. Moreover, if some composite operation can be evaluated in P, then any other composite should be evaluable at its value as one of the arguments whenever the combined composite, which results from substituting — qua function — it at this argument-place of the other, is evaluable at the original arguments. (Evaluability of the combination and equality of the values would follow per definitionem from evaluability of the constituents). Conversely, the former condition is equivalent to: In the absolutely free algebra on P as unstructured set of generators, the totality of evaluable elements (i.e. words in the generators which represent composite operations evaluable in P at these arguments) constitutes a union of classes, on each of which the value in P is constant, for the congruence modulo which it becomes the algebra free in the equational class on P as (unstructured) generators; while the latter ensures that the partial equivalence, with the evaluable elements as domain, of having the same value in P, is strong for the translations with coefficients from P: whence also modulo the congruence of identities, in the algebra free in the equational class.

iii) Necessary and sufficient for a partial algebra (in the above general sense) P to be embeddable as a relative subalgebra of a full algebra in an equational class, is that all the identities of the class hold strongly in P, and that the evaluability in P be undisturbed by replacing in a composed operation an evaluable composed sub-operation by its value.

For S-sets, which we construe as multi-unary algebras with operations the f_s, this yields

iv) Necessary and sufficient for a partial S-set to be a relative subset of a full one is: whenever $f_t(a)$ is defined, then $f_{st}(a)$ is if and only if $f_s(f_t(a))$ is and the two are equal.

Here we are taking the partial composed operation $f_s \circ f_t$ on P as the partial (uncomposed) f_{st} (as we may be virtue of the "if" part) whence all identities — $f_s \circ f_t \circ \ldots = f_{s'} \circ f_{t'} \circ \ldots$ whenever $st \ldots = s't' \ldots$ in S — hold strongly: i.e. both sides are the same partial selfmap.

It may be instructive to compare the above condition for embeddability with the definition of "partial operand" as given towards the bottom of p. 254 in [1]: the latter is stronger just by requiring in addition that $f_{st}(a)$ be defined only when $f_t(a)$ is. This makes its one-point completion a full S-set in which it is embedded as a relative subset (as noted top p. 255); conditions for the one-point completion to belong to an equational class in the context of general algebras can be found in [3] p. 14.

A final application is to the embedding of a T — in an S-set where T is a subsemigroup of S. We thus have a partial S-set in which $f_t(a)$ is defined for every a when $t \in T$ but not otherwise. Our condition requires $f_{st}(a)$ defined — i.e. $st \in T$ — only if $f_s(f_t(a))$ defined — whence $s \in T$: thus T be a right unitary semigroup; conversely, for such a subsemigroup the condition is verified:

v) The partial S-set determined by the full action of a subsemigroup T on a set can be embedded as a relative subset: i.e. so that no action of the elements not in T is induced on the subset, if and only if T is right unitary in S.

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(1) This might be the place to record that Section 2 of [3] suffers from some infelicities: namely, the condition of Theorem IS is essentially no more general than that of the corollary and should have been formulated simply for congruences which remain such when the relevant polynomials are taken as operations. Since the extending congruence will in general enlarge classes, that theorem transcends the frame we have set here. Also, the sketchy discussion of the polynomial concept appearing there is being elaborated in a more complete and more satisfying fashion in connection with the write-up of another investigation.

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