Hereditary $\mathfrak{m}$-separability and the hereditary $\mathfrak{m}$-Lindelöf property in product spaces and function spaces

by

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Abstract. It is shown that under some assumptions on the product $X \times Y$ we have the alternative: either X is hereditarily $\mathfrak{m}$-separable or Y is hereditarily $\mathfrak{m}$-Lindelöf. It is also shown that for every completely regular X the power $X^{\mathfrak{m}}$ is hereditarily $\mathfrak{m}$-separable (hereditarily $\mathfrak{m}$-Lindelöf) if and only if the space $C(X, Y)$ is hereditarily $\mathfrak{m}$-Lindelöf (hereditarily $\mathfrak{m}$-separable) for every space Y with $w(Y) \leqslant \mathfrak{m}$.

1. Introduction. In Section 2, it is shown that if $X \times Y$ is hereditarily $\mathfrak{m}^+$-sequentially compact, then either X is hereditarily $\mathfrak{m}$-separable or Y is hereditarily $\mathfrak{m}$-Lindelöf. In particular then, if $X \times Y$ is hereditarily $\aleph_1$-sequentially compact, then either X is hereditarily separable or Y is hereditarily Lindelöf. With the aid of the generalized continuum hypothesis (GCH), if $X \times Y$ is $\mathfrak{m}$-separable and hereditarily normal, then either X is hereditarily $\mathfrak{m}$-separable or Y is hereditarily $\mathfrak{m}$-Lindelöf. In [4], Michael displays a countable collection $\{X_i \mid i \in \mathcal{N}\}$ of spaces such that, for each n, $\prod_{i=1}^{n} X_i$ is Lindelöf but $\prod \{X_i \mid i \in \mathcal{N}\}$ is not Lindelöf. Suppose that $\{X_a \mid a \in A\}$ is a collection of spaces such that (1) $|A| \leqslant \mathfrak{m}$ and (2) if B is a finite subcollection of A, then $\prod \{X_b \mid b \in B\}$ is hereditarily $\mathfrak{m}$-Lindelöf (hereditarily $\mathfrak{m}$-separable, respectively). Then $\prod \{X_a \mid a \in A\}$ is hereditarily $\mathfrak{m}$-Lindelöf (hereditarily $\mathfrak{m}$-separable, respectively).

In [5], Rudin and Klee show that if X and Y are second countable, then $C(X, Y)$, the space of continuous functions from X into Y with the pointwise topology, is hereditarily separable and hereditarily Lindelöf. In Section 3, we show that if X is completely regular, then $C(X, Y)$ is hereditarily $\mathfrak{m}$-separable (hereditarily $\mathfrak{m}$-Lindelöf, respectively) for every space Y with weight less than or equal to $\mathfrak{m}$ if and only if $X^{\mathfrak{m}}$ is hereditarily $\mathfrak{m}$-Lindelöf (hereditarily $\mathfrak{m}$-separable).

In Section 4, spaces S and R are obtained so that, for each n, S^n is hereditarily Lindelöf but S is not hereditarily separable and, for each n, R^n is hereditarily separable but R is not hereditarily Lindelöf.

Before proceeding to the body of this paper, let us recall some definitions and establish some conventions:

Throughout this paper, $\mathfrak{m}$ will denote an infinite cardinal.

Our notion of $\mathfrak{m}$-Lindelöf is different from the usual convention. We will say that X is *$\mathfrak{m}$-Lindelöf* if each open cover $\mathcal{U}$ of X contains a subcover $\mathcal{U}'$ of X with $|\mathcal{U}'|\leqslant\mathfrak{m}$. Following conventions established in [3], we will write $\mathrm{hl}(X)\leqslant\mathfrak{m}$ if X is hereditarily $\mathfrak{m}$-Lindelöf.

X is $\mathfrak{m}$-separable if X contains a dense subset H with $|H|\leqslant\mathfrak{m}$. If X is $\mathfrak{m}$-separable, we will write $d(X)\leqslant\mathfrak{m}$ (d stands for density). If X is hereditarily $\mathfrak{m}$-separable we will write $\mathrm{hd}(X)\leqslant\mathfrak{m}$.

If Y admits a basis of cardinality less than or equal to $\mathfrak{m}$, then we write $w(X)\leqslant\mathfrak{m}$.

We will say that X is hereditarily $\mathfrak{m}$-compact if each subset H of X with $|H|\geqslant\mathfrak{m}$ contains a limit point of itself.

$\mathfrak{m}^+$ is the first cardinal greater than $\mathfrak{m}$ and $\Omega(\mathfrak{m})$ will denote the first ordinal of cardinality $\mathfrak{m}$ or, equivalently, the set of ordinals of cardinality less than $\mathfrak{m}$. $C(X, Y)$ will denote the space of continuous functions from X into Y with the pointwise topology. E^n will denote the Euclidean n-space.

We make extensive use of the following results which were obtained in [3]:

LEMMA 1. $\mathrm{hd}(X)\leqslant\mathfrak{m}$ *if and only if it is true that if* $f\colon \Omega(\mathfrak{m}^+)\to X$ *is one-to-one, then there is a member* a *of* $\Omega(\mathfrak{m}^+)$ *such that* $f(a)$ *is a limit point of* $\{f(b)|\ b<a\}$.

LEMMA 2. $\mathrm{hl}(X)\leqslant\mathfrak{m}$ *if and only if it is true that if* $f\colon \Omega(\mathfrak{m}^+)\to X$ *is one-to-one, then there is a member* a *of* $\Omega(\mathfrak{m}^+)$ *such that* $f(a)$ *is a limit point of* $\{f(b)|\ b>a\}$.

2. Products.

THEOREM 1. *If* $X\times Y$ *is hereditarily* $\mathfrak{m}^+$*-compact, then either* X *is hereditarily* $\mathfrak{m}$*-separable or* Y *is hereditarily* $\mathfrak{m}$*-Lindelöf.*

Proof. Suppose the theorem is false. Then according to Lemmas 1 and 2, there are one-to-one functions $f\colon \Omega(\mathfrak{m}^+)\to X$ and $g\colon \Omega(\mathfrak{m}^+)\to Y$ such that if $a\in\Omega(\mathfrak{m}^+)$ then $f(a)$ is not a limit point of $\{f(b)|\ b<a\}$ and $g(a)$ is not a limit point of $\{g(b)|\ b>a\}$. Let $L=\{(f(a), g(a))|\ a\in\Omega(\mathfrak{m}^+)\}$. Since $X\times Y$ is hereditarily $\mathfrak{m}^+$-compact, there is a point b of $\Omega(\mathfrak{m}^+)$ such that $(f(b), g(b))$ is a limit point of L. Since $f(b)$ is not a limit point of $\{f(a)|\ a<b\}$ and $g(b)$ is not a limit point of $\{g(a)|\ a>b\}$, there are open sets U and V in X and Y respectively such that

$$f(b)\in U\subset\big(X-\{f(a)|\ a<b\}\big) \quad\text{and}\quad g(b)\in V\subset\big(Y-\{g(a)|\ a>b\}\big).$$

But then $U\times V$ is an open subset of $X\times Y$ containing $(f(b), g(b))$ but no other point of L. This is a contradiction from which the theorem follows.

Note that if S denotes the Sorgenfrey line, then $S\times S$ serves as an example to show that the converse to Theorem 1 is not true. The examples in Section 4 show that Theorem 1 is the best available result.

THEOREM 2 (GCH). *If* $X\times Y$ *is* $\mathfrak{m}$*-separable and hereditarily normal, then either* X *is hereditarily* $\mathfrak{m}$*-separable or* Y *is hereditarily* $\mathfrak{m}$*-Lindelöf.*

Proof. The technique used here was first employed by Jones in [2]. Suppose that $\mathrm{hd}(X)>\mathfrak{m}$ and $\mathrm{hl}(Y)>\mathfrak{m}$. Then by Lemmas 1 and 2, there are one-to-one functions $f\colon \Omega(\mathfrak{m}^+)\to X$ and $g\colon \Omega(\mathfrak{m}^+)\to Y$ such that if $a\in\Omega(\mathfrak{m}^+)$, then $f(a)$ is not a limit point of $\{f(b)|\ b<a\}$ and $g(a)$ is not a limit point of $\{g(b)|\ b>a\}$. As in the proof to Theorem 1, $L=\{(f(a), g(a))|\ a\in\Omega(\mathfrak{m}^+)\}$ is a subset of $X\times Y$ that contains none of its limit points. Let S be a dense subset of $X\times Y$ such that $|S|\leqslant\mathfrak{m}$. Since $X\times Y$ is hereditarily normal, for each subset H of L, there is an open set $U(H)$ containing H such that $\overline{U(H)}\cap(L-H)=\emptyset$. For each $H\subset L$, let $s(H)=U(H)\cap S$. Then s is a one-to-one function from 2^L, the set of subsets of L, into 2^S. This is a contradiction since the generalized continuum hypothesis implies that $|2^L|>|2^S|$.

THEOREM 3. *Suppose that* $\{X_i|\ i\in A\}$ *is a collection of spaces such that* $|A|\leqslant\mathfrak{m}$ *and such that if* B *is a finite subset of* A, *then* $\prod\{X_i|\ i\in B\}$ *is hereditarily* $\mathfrak{m}$*-Lindelöf. Then* $\prod\{X_i|\ i\in A\}$ *is hereditarily* $\mathfrak{m}$*-Lindelöf.*

Proof. If $B\subset A$, π_B will denote the projection map of $\prod\{X_i|\ i\in A\}$ onto $\prod\{X_i|\ i\in B\}$. Let $\mathcal{A}$ denote the collection of finite subsets of A. Let Y denote a subspace of $\prod\{X_i|\ i\in A\}$ and let $\mathcal{U}$ be an open cover of Y. For each $B\in\mathcal{A}$, let $\mathcal{U}(B)$ denote the collection to which V belongs if and only if V is an open subset of $\prod\{X_i|\ i\in B\}$ and $\pi_B^{-1}(V)\cap Y$ is a subset of some member of $\mathcal{U}$. Since, for each $B\in\mathcal{A}$, $\mathrm{hl}(\pi\{X_i|\ i\in B\})\leqslant\mathfrak{m}$, there is a subcollection, $\mathcal{W}(B)$, of $\mathcal{U}(B)$ so that $\bigcup(\mathcal{W}(B))=\bigcup(\mathcal{U}(B))$ and such that $|\mathcal{W}(B)|\leqslant\mathfrak{m}$. Then the collection

$$\mathcal{W}=\{\pi_B^{-1}(W)\cap Y|\ W\in\mathcal{W}(B),\ B\in\mathcal{A}\}$$

is an open cover of Y refining $\mathcal{U}$ so that $|\mathcal{W}|\leqslant\mathfrak{m}$.

THEOREM 3*. *Suppose that* $\{X_i|\ i\in A\}$ *is a collection of spaces such that* $|A|\leqslant\mathfrak{m}$ *and such that if* B *is a finite subset of* A, *then* $\prod\{X_i|\ i\in B\}$ *is hereditarily* $\mathfrak{m}$*-separable. Then* $\prod\{X_i|\ i\in A\}$ *is hereditarily* $\mathfrak{m}$*-separable.*

Proof. Let Y be a subspace of $\prod\{X_i|\ i\in A\}$. We will use the notation established in the proof of Theorem 3. For each $B\in\mathcal{A}$, let M_B be a subset of Y so that $|M_B|\leqslant\mathfrak{m}$ and $\pi_B(M_B)$ is dense in $\pi_B(Y)$. Then $\bigcup\{M_B|\ B\in\mathcal{A}\}$ is a dense subset of Y with cardinality no greater than $\mathfrak{m}$.

3. Function spaces. The author is grateful to the referee for the following lemma which greatly simplifies the Author's original Proofs to Theorems 4, 4*, 5, and 5*.

LEMMA 3. *Let* R, S, T *be three topological spaces and* $\Phi\colon R\times S\to T$ *be a function such that:*

(i) Φ *is continuous with respect to the variable* s,

(ii) *the topology of* R *is the weakest topology for which* Φ *is continuous with respect to* r.

Let $w(T)\leqslant\mathfrak{m}$. *Then*:

(a) $(\forall\, n\,\mathrm{hl}(S^n)\leqslant\mathfrak{m}) \Rightarrow \mathrm{hd}(R)\leqslant\mathfrak{m}$,

(b) $(\forall\, n\,\mathrm{hd}(S^n)\leqslant\mathfrak{m}) \Rightarrow \mathrm{hl}(R)\leqslant\mathfrak{m}$.

Proof. for $x\in S^{\aleph_0}$, $x=\{s_i\}$ and $r\in R$, put $\Psi(r,x)=\{\Phi(r,s_i)\}\in T^{\aleph_0}$. The function $\Psi\colon R\times S^{\aleph_0}\to T^{\aleph_0}$ is continuous with respect to the second variable, and the sets of the form $M(x,U)=\{r|\ \Psi(r,x)\in U\}$, where U belongs to the basis $\mathscr{B}$ (of cardinality $\leqslant\mathfrak{m}$) of the space $T^{\aleph_0}$ and $x\in S^{\aleph_0}$, form the basis of R (according to (ii)).

Proof of (a). Let $A\subseteq R$. For $U\in\mathscr{B}$ and $a\in A$ let $U(a)=\{x\in S^{\aleph_0}|\ \Psi(a,x)\in U\}$ and $\mathscr{S}_U=\{U(a)|\ a\in A\}$. The family $\mathscr{S}_U$ is open in $S^{\aleph_0}$, so there exists $A_U\subseteq A$ such that $|A_U|\leqslant\mathfrak{m}$ and $\bigcup\mathscr{S}_U=\bigcup\{U(a)|\ a\in A_U\}$. Let $B=\bigcup\ \{A_U|\ U\in\mathscr{B}\}$. We have $|B|\leqslant\mathfrak{m}$ and B is dense in A.

Proof of (b). Let $\mathscr{A}=\{M(x,U)|\ x\in S^{\aleph_0},\ U\in\mathscr{B}\}$. Let $\mathscr{A}'\subseteq\mathscr{A}$. For $U\in\mathscr{B}$ let $A_U=\{x\in S^{\aleph_0}|\ M(x,U)\in\mathscr{A}'\}$. Since $\mathrm{hd}(S^{\aleph_0})\leqslant\mathfrak{m}$ (according to Theorem 3*), there exists $B_U\subseteq A_U$ such that $|B_U|\leqslant\mathfrak{m}$ and $\overline{B_U}\supseteq A_U$. Let

$$\mathscr{A}''=\{M(x,U)|\ U\in\mathscr{B},\, x\in B_U\}\,.$$

We have $|A''|\leqslant\mathfrak{m}$ and $\bigcup\mathscr{A}''=\bigcup\mathscr{A}'$, which finishes the proof.

THEOREM 4. *If, for each positive integer* n, X^n *is hereditarily* $\mathfrak{m}$*-Lindelöf and* $w(Y)\leqslant\mathfrak{m}$, *then* $C(X,Y)$ *is hereditarily* $\mathfrak{m}$*-separable.*

THEOREM 4*. *If, for each positive integer* n, X^n *is hereditarily* $\mathfrak{m}$*-separable and* $w(Y)\leqslant\mathfrak{m}$ *then* $C(X,Y)$ *is hereditarily* $\mathfrak{m}$*-Lindelöf.*

Proof of Theorems 4 and 4*. Take $R=C(X,Y)$, $S=X$, $T=Y$, and $\Phi(f,x)=f(x)$ for $f\in C(X,Y)$, $x\in X$.

THEOREM 5. *Suppose that* X *is completely regular*, n *is an integer, and* $C(X,E^n)$ *is hereditarily* $\mathfrak{m}$*-separable, then* X^n *is hereditarily* $\mathfrak{m}$*-Lindelöf.*

THEOREM 5*. *Suppose that* X *is completely regular*, n *is an integer, and* $C(X,E^n)$ *is hereditarily* $\mathfrak{m}$*-Lindelöf. Then* X^n *is hereditarily* $\mathfrak{m}$*-separable.*

Proof of Theorems 5 and 5*. Take $R=X^n$, $S=C(X,E^n)$, $T=E^n$ and for $x=(x_1,\ldots,x_n)\in X^n$, $f=(f_1,\ldots,f_n)\in C(X,E^n)$, let $\Phi(x,f)=(f_1(x_1),\ldots,f_n(x_n))\in E^n$.

To summarize the results obtained so far, we have, letting H denote Hilbert space:

THEOREM 6. *If* X *is completely regular then the following are equivalent for the cardinal* $\mathfrak{m}$:

A. X^n *is hereditarily* $\mathfrak{m}$*-Lindelöf* ($\mathfrak{m}$*-separable, respectively*) *for every* n.

B. $X^{\mathfrak{m}}$ *is hereditarily* $\mathfrak{m}$*-Lindelöf* ($\mathfrak{m}$*-separable, respectively*).

C. $C(X,E^n)$ *is hereditarily* $\mathfrak{m}$*-separable* ($\mathfrak{m}$*-Lindelöf, respectively*) *for every* n.

D. $C(X,H)$ *is hereditarily* $\mathfrak{m}$*-separable* ($\mathfrak{m}$*-Lindelöf, respectively*).

E. $C(X,Y)$ *is hereditarily* $\mathfrak{m}$*-separable* ($\mathfrak{m}$*-Lindelöf, respectively*) *for every space* Y *with* $w(Y)\leqslant\mathfrak{m}$.

4. Two Examples.

EXAMPLE 1. We construct a space S such that S^n is hereditarily separable for every n but S is not hereditarily Lindelöf. Let $\varphi\colon\Omega(\aleph_1)\to E^1$ be a one-to-one function. Let $\mathscr{B}=\{\varphi^{-1}(U)\cap(0,a]|\ a\in\Omega(\aleph_1),\ U$ is open in $E^1\}$. Then $\mathscr{B}$ is a basis for a Hausdorff topology $\mathscr{T}$ on $\Omega(\aleph_1)$. Let $S=(\Omega(\aleph_1),\mathscr{T})$. It follows from Lemmas 1 and 2 that S is hereditarily separable but not hereditarily Lindelöf. For each i, let S_i denote a copy of S. We wish to show that, for each n, $\prod_{i=1}^{n}S_i$ is hereditarily separable. To this end, suppose otherwise. Let N denote the first integer so that $\prod_{i=1}^{N}S_i$ is not hereditarily separable. By Lemma 1, there is a one-to-one function $f\colon\Omega(\aleph_1)\to\prod_{i=1}^{N}S_i$ such that if $a\in\Omega(\aleph_1)$, then $f(a)$ is not a limit point of $\{f(b)|\ b<a\}$. For each $i\leqslant N$, let π_i denote the projection of $\prod_{i=1}^{N}S_i$ onto S_i.

Suppose that there are an integer K and an uncountable subset Γ of $\Omega(\aleph_1)$ such that π_K is constant on $f(\Gamma)$. Let π denote the projection of $\prod_{i=1}^{N}S_i$ onto $S^{N-1}=\prod_{i\neq k}S_i$. Let g be an order preserving map of $\Omega(\aleph_1)$ onto Γ. Then $(\pi\circ f\circ g)$ is a one-to-one function from $\Omega(\aleph_1)$ into S^{N-1}. By our hypothesis, S^{N-1} is hereditarily separable; and so, there is a point $\boldsymbol{a}$ of $\Omega(\aleph_1)$ such that $(\pi\circ f\circ g)(a)$ is a limit point of $(\pi\circ f\circ g)(\{b|\ b<a\})$. Thus since $\pi|f(b)$ is a homeomorphism, $g(a)$ is a point of Γ, and thus of $\Omega(\aleph_1)$, such that $f(g(a))$ is a limit point of $f((0,g(a))\cap\Gamma)$ and, hence, of $\{f(b)|\ b<g(a)\}$. This is a contradiction, from which it follows that if $i\leqslant N$ and if $a\in\Omega(\aleph_1)$, then the set $\{b\in\Omega|\pi_i(f(b))=\pi_i(f(a))\}$ is countable. This permits us to choose an uncountable subset Γ_1 of $\Omega(\aleph_1)$ such that if $i\leqslant N$, then the restriction of $\pi_i\circ f$ to Γ_1 is one-to-one.

Choose an uncountable subset Γ of Γ_1 such that, for each i, the restriction $\pi_i\circ f|\Gamma$ is order preserving on Γ. Let φ^N denote the map taking $S^N\to E^N$ defined by $\varphi^N(x_1,\ldots,x_N)=(\varphi(x_1),\ldots,\varphi(x_N))$ and let g be an order preserving map from $\Omega(\aleph_1)$ onto Γ. Since E^N is hereditarily separable, there is a point $\boldsymbol{a}$ of $\Omega(\aleph_1)$ such that $(\varphi^N\circ f\circ g)(\boldsymbol{a})$ is a limit point (in E^N) of $(\varphi^N\circ f\circ g)(\{b|\ b<\boldsymbol{a}\})$. It follows from the construction of S and the fact that, for each i, $\pi_i\circ f\circ g$ is order-preserving that $f(g(a))$ is a limit point of $f(0,g(a))$. This contradiction completes the proof.

EXAMPLE 2. There is a Hausdorff space R such that, for each n, R^n is hereditarily Lindelöf, but R is not hereditarily separable. Let $\varphi\colon\Omega(\aleph_1)\to E^1$ be a one-to-one function. Let $\mathscr{B}'=\{\varphi^{-1}(U)\cap[a,w_1)|\ a\in\Omega(\aleph_1),\ U$ is open in $E^1\}$. Then $\mathscr{B}'$ is a basis for a topology $\mathscr{T}'$ on $\Omega(\aleph_1)$. Let $R=(\Omega(\aleph_1),\mathscr{T}')$. The argument that, for each n, R^n is hereditarily Lindelöf but R is not hereditarily separable is omitted here since it is analogous to the arguments that S (in Example 1) is not hereditarily Lindelöf but that S^n is hereditarily separable for each n.

Finally, the author wishes to thank J. W. Rogers, Jr. for bringing Juhasz's Lemmas to his attention.

Added in proof. Using the continuum hypothesis, K. Kunnen has recently obtained an example of a perfectly normal space X so that X^{ω} is hereditarily separable but X is not Lindelöf.

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О размерности произведений топологических пространств

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Abstract. The inequality $\dim X\times Y\leqslant \dim X+\dim Y$ is established if (1) X and Y are completely regular spaces and the projection of $X\times Y$ onto Y is closed, and if (2) $X\times Y$ is a normal countably paracompact space and Y is a paracompact p-space. Counterparts for Ind ard also obtained under the additional assumption that $X\times Y$ is normal and the finite sum theorem for Ind holds in X and in Y.

В этой заметке мы получим некоторые условия, достаточные для выполнения неравенств

$$(*)\qquad \dim X\times Y\leqslant \dim X+\dim Y,$$

$$(**)\qquad \operatorname{Ind} X\times Y\leqslant \operatorname{Ind} X+\operatorname{Ind} Y,$$

$$\operatorname{ind} X\times Y\leqslant \operatorname{ind} X+\operatorname{ind} Y.$$

Наиболее сильным из предшествующих результатов, связанных с выполнением (*) является теорема Кодамы [10], существенным усилением которой является теорема 2.4 настоящей работы, решающая одну из проблем Нагаты [13].

Отметим, что неравенство (**) не верно даже для бикомпактов, см. [6]. Положительные результаты, связанные с (**) будут нами получены в дополнительных предположениях, а именно, в предположении выполнения теоремы суммы для размерности Ind в сомножителях.

Будем говорить, что *пространство X удовлетворяет условию* (Σ), если для любого конечного семейства γ его замкнутых подмножеств

$$\operatorname{Ind}(\bigcup \gamma) = \max\{\operatorname{Ind} F\colon F\in\gamma\}\qquad (\text{см. [14]}).$$

Результаты этой заметки были сообщены без доказательств в [7].