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Generalized quantifiers in models of set theory

by

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Abstract. Generalized quantifiers as described by Keisler in [3] are considered. A complete characterization of such quantifiers in models of ZFC and ZF set theory is given. It is also shown that the generalized quantifiers introduced by Krivine and McAloon in [4] can be characterized as certain subsets of those considered by Keisler (Lemma 3.5 and Theorem 3.6).

This paper is concerned with the kind of generalized quantifiers considered by Keisler [3] and Krivine and McAloon [4]. One adjoins to first-order language a new quantifier Qx which is interpreted as "there exist many x" and which obeys certain natural schemata. Keisler considered quantifiers satisfying the additional schema

$$Qx\exists y\varphi(x,y) \to [\exists yQx\varphi(x,y) \vee Qy\exists x\varphi(x,y)].$$

Here we call such quantifiers regular.

After giving the basic definitions in Section 1, in Section 2 we give a characterization of regular quantifiers which are definable in models of ZF and ZFC. In Section 3 we make some general observations about quantifiers. The main one is that in a model of ZF any quantifier explicit in the sense of [4] generates in a natural way an explicit regular quantifier.

1. Preliminaries. Let L be a countable first-order language. By $L(Q)$ we mean the language obtained by adjoining to L a new quantifier symbol Q. As in [4] we will interpret the formula "$Qx\varphi(x)$" to mean "there exist many x satisfying φ". Also if M is a model of L then by L_M and $L_M(Q)$ we mean the languages L or $L(Q)$ respectively with constants for all elements of $|M|$ adjoined.

DEFINITION 1.1. Let M, N be two models of L and let $M \prec N$. Let φ be a formula of L with one free variable and with parameters from $|M|$. We say φ is *preserved in N* if

$$N \models \varphi(a) \to a \in |M|.$$

φ is *enlarged in N* if there exist in N new elements satisfying φ.

DEFINITION 1.2. Let M be a model of L and $\mathcal{Q}$ be a family of subsets of $|M|$. A pair $\mathcal{M} = \langle M, \mathcal{Q}\rangle$ is called a *weak model* of $L(Q)$. The notion of satisfaction is defined by the usual induction on length of formulas with the additional clause:

$$\mathscr{M} \models \mathrm{Q}x\varphi(x) \;.\!\equiv\!.\; \{a \in |M|:\ \mathscr{M} \models \varphi(a)\} \in \mathscr{Q}\,.$$

We think of $\mathscr{Q}$ as the family of "big" subsets of $|M|$.

DEFINITION 1.3. Let $\mathscr{M} = \langle M, \mathscr{Q}\rangle$ be a weak model of $L(Q)$. We say $\mathscr{Q}$ is a *generalized quantifier* (or simply a *quantifier*) in M if the following axioms are satisfied in $\mathscr{M}$:

(Q.1) $\forall x(\varphi \to \psi) \to (\mathrm{Q}x\varphi \to \mathrm{Q}x\psi)$,

(Q.2) $\mathrm{Q}x(\varphi \vee \psi) \to (\mathrm{Q}x\varphi \vee \mathrm{Q}x\psi)$,

(Q.3) $\mathrm{Q}x(x = x)$,

(Q.4) $\forall y \neg \mathrm{Q}x(x = y)$.

If $\mathscr{M}$ satisfies also

(Q.5) $\mathrm{Q}y\exists x\varphi \to (\exists x\mathrm{Q}y\varphi \vee \mathrm{Q}x\exists y\varphi)$,

then we say $\mathscr{Q}$ is a *regular quantifier* in M. Axioms (Q.1)–(Q.5) are slightly different from those given by Keisler in [3] but they are equivalent as was shown in [3].

DEFINITION 1.4. Let $\mathscr{Q}$ and $\mathscr{Q}'$ be two families of subsets of the same model M. Denote $\langle M, \mathscr{Q}\rangle$ by $\mathscr{M}$ and $\langle M, \mathscr{Q}'\rangle$ by $\mathscr{M}'$. We say $\mathscr{Q}$ and $\mathscr{Q}'$ are *equivalent* over M if for every sentence φ of $L_M(Q)$

$$\mathscr{M} \models \varphi \leftrightarrow \mathscr{M}' \models \varphi\,.$$

Consider the following expression:

$$(*)\qquad \mathscr{Q} = \{A \subset |M|:\ \exists\varphi \in L_M(Q)[\mathscr{M} \models \mathrm{Q}x\varphi(x)\ \&\ A = \{a \in |M|:\ \mathscr{M} \models \varphi(a)\}]\}\,.$$

Notice that for every family $\mathscr{Q}'$ of subsets of $|M|$ we can find a unique family $\mathscr{Q}$ satisfying $(*)$ such that $\mathscr{Q}$ and $\mathscr{Q}'$ are equivalent over $|M|$. Hence sometimes we will assume that a family $\mathscr{Q}$ satisfies $(*)$.

DEFINITION 1.5. Let M be a model of L and $\mathscr{Q}$ be a quantifier in M. We say $\mathscr{Q}$ is a definable quantifier in M if all elements of $\mathscr{Q}$ are definable in M by formulas of L_M. We say $\mathscr{Q}$ is an explicit quantifier if for every formula $\varphi \in L_M(Q)$ we can find a formula $\psi \in L_M$ which is equivalent to φ.

One can easily check that if $\mathscr{Q}$ is an explicit quantifier in M then $\mathscr{Q} \cap \mathrm{Def}(M)$ is a definable quantifier in M and it is equivalent to $\mathscr{Q}$ over M. Besides if $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are two quantifiers in M and they are equivalent over M then $\mathscr{Q}_1 \cap \mathrm{Def}(M) = \mathscr{Q}_2 \cap \mathrm{Def}(M)$ hence definable quantifiers are $\subset$-smallest in their equivalence classes and they satisfy $(*)$.

2. Regular quantifiers in ZF, ZFC, A_2 **and** A_2^-. Until further notice by "quantifier" we mean "regular quantifier satisfying $(*)$."

Now we start to characterize generalized quantifiers in models for ZFC and ZF

set theories. We define cardinal numbers in ZF as in Jech's book ([2], p. 152); i.e., a cardinal $\varkappa$ will be a set having following properties:

1. $\forall x\forall y[x = \varkappa \to (y \in \varkappa \leftrightarrow \bar{\bar{x}} = \bar{\bar{y}}\ \&\ \varrho(x) = \varrho(y))]$,
2. $\forall x\forall y[x \in \varkappa\ \&\ \bar{\bar{x}} = \bar{\bar{y}} \to \varrho(y) \geqslant \varrho(x)]$.

An infinite initial ordinal will be an aleph. An aleph $\aleph$ is singular if there exists a function f such that $\mathrm{dom}(f)$ is an aleph $<\aleph$ and for all $\gamma \in \mathrm{dom}(f)$ $f(\gamma)$ is also an aleph $<\aleph$. We say aleph $\aleph$ is regular if it is not singular.

If a cardinal $\varkappa$ can be well ordered, i.e., if

$$\exists x\exists y(x \in \varkappa\ \&\ y \in On\ \&\ \bar{\bar{x}} = \bar{\bar{y}}),$$

then there exists an aleph $\aleph$ such that $\bar{\bar{\aleph}} = \varkappa$. In this case we will identify $\varkappa$ with the corresponding aleph.

Let M be a model of ZF and $A \subset |M|$. We say A is a *set* in M if there exists an a in M such that

$$A = \{b \in |M|:\ M \models b \in a\}\,.$$

We say A is a *class* in M if there exists a formula $\varphi(x)$ of L_M such that

$$A = \{b \in |M|:\ M \models \varphi(b)\}\,.$$

DEFINITION 2.1. Let M be a model of ZF and $\varkappa$ be an infinite cardinal in M.

(a) $\mathscr{Q}_\varkappa$ denotes the family of definable subsets of $|M|$ which are either sets of power $\geqslant\varkappa$ in M or classes containing a subset of power $\geqslant\varkappa$.

(b) $\mathscr{Q}_V$ denotes the family of all definable subsets of $|M|$ which are not sets in $|M|$.

The following lemma, stated in a slightly different form, can be found in Keisler's paper ([3], p. 34).

LEMMA 2.2. *If M is a model of* ZF *then $\mathscr{Q}_V$ is a quantifier in M. Also if $\varkappa$ is a regular aleph in M then $\mathscr{Q}_\varkappa$ is a quantifier in M.*

THEOREM 2.3. *Let M be a well-founded model of* ZFC. *Then the only quantifiers in M are $\mathscr{Q}_V$ and those of the form $\mathscr{Q}_\varkappa$ where $\varkappa$ is a regular cardinal in M.*

Proof. Let $\mathscr{Q}$ be a quantifier in M. Then either (1) there are no elements of $\mathscr{Q}$ which are sets in M or (2) there are sets in $\mathscr{Q}$. We will show that in case (1) $\mathscr{Q} = \mathscr{Q}_V$ and in case (2) there exists $\varkappa \in M$ which is a regular cardinal in M and such that $\mathscr{Q} = \mathscr{Q}_\varkappa$.

First assume (1) holds. Then obviously $\mathscr{Q} \subseteq \mathscr{Q}_V$. To prove the converse inclusion suppose A is any element of $\mathscr{Q}_V$. Let ϱ be the usual rank function. Then $\bar{\rho}(A) = A_1 \subseteq On$ and A_1 is not a set in M. Since A_1 is well ordered we can define by transfinite induction a one-to-one function f from A_1 onto On. Hence the function $g = f \circ (\varrho \restriction A)$ from A onto On is definable in M. Now we prove that $On \in \mathscr{Q}$. Let $\mathscr{M} = \langle M, \mathscr{Q}\rangle$. Since $\mathscr{M} \models \mathrm{Q}x\exists y(y = \varrho(x))$ then by (Q.5)

$$\mathscr{M} \models \exists y\mathrm{Q}x(y = \varrho(x)) \vee \mathrm{Q}y\exists x(y = \varrho(x))$$

1·

and since $\mathcal{Q}$ contains no set $\mathcal{M} \models Qy\exists x(y = \varrho(x))$. This implies $On \in \mathcal{Q}$. Now we can finish the proof that $A \in \mathcal{Q}$. From above we can deduce that $\mathcal{M} \models Qx\exists y(x = g(y))$. Again by (Q.5) this implies

$$\mathcal{M} \models \exists y Qx(x = g(y)) \vee Qy\exists x(x = g(y)).$$

The left-hand formula of this disjunction fails because of (Q.4), hence $\mathcal{M} \models Qy\exists x(x = g(y))$. This means that $A \in \mathcal{Q}$ which gives $\mathcal{Q} = \mathcal{Q}_V$.

Assume now (2). Let $\varkappa$ be the smallest cardinal which is the cardinality of an element of $\mathcal{Q}$. Then also $\varkappa \in \mathcal{Q}$. To prove that $\varkappa$ is regular suppose the converse; i.e., there exists $\gamma<\varkappa$ and a sequence $\{\alpha_\xi\colon \xi<\gamma\}$ of elements of $\varkappa$ such that $\varkappa = \sup\{\alpha_\xi\colon \xi<\gamma\}$. Then $\mathcal{M} \models Qx\exists y(x \in \alpha_y)$ and by (Q.5)

$$\mathcal{M} \models \exists y Qx(x \in \alpha_y) \vee Qy\exists x(x = \alpha_y).$$

The left-hand formula of this disjunction means that $\alpha_\xi \in \mathcal{Q}$ for some $\xi<\gamma$ and the right-hand formula means that $\gamma \in On$. Both contradict the fact that there are no sets of power less than $\varkappa$ in $\mathcal{Q}$. Now we shall prove $\mathcal{Q} = \mathcal{Q}_\varkappa$. $\mathcal{Q} \subseteq \mathcal{Q}_\varkappa$ is immediate. To prove $\mathcal{Q}_\varkappa \subseteq \mathcal{Q}$ suppose $A \in \mathcal{Q}_\varkappa$. This implies that there is a function f from A onto $\varkappa$ in M. Hence $\mathcal{M} \models Qx\exists y(x = f(y))$ and using an argument similar to that used in the first part of the proof we get $\mathcal{M} \models Qy\exists x(x = f(y))$ which implies $A \in \mathcal{Q}$ and hence $\mathcal{Q} = \mathcal{Q}_\varkappa$.

If a model M is not well founded the situation is more complicated. There may be sets in $\mathcal{Q}$ but no least cardinal which is the cardinality of a set from $\mathcal{Q}$. The following example illustrates this possibility. J. Hutchinson in [1] gave an example of a model $M \models$ ZFC such that M is of power $\aleph_1$ and there is no least uncountable ordinal in M. Let $\mathcal{Q}$ be a family of all uncountable subsets of $|M|$. Then $\mathcal{Q}$ is a quantifier in M and it is neither of the form $\mathcal{Q}_V$ nor of the form $\mathcal{Q}_\varkappa$ for some $\varkappa$ from M. Unlike the quantifiers $\mathcal{Q}_V$ and $\mathcal{Q}_\varkappa$, $\mathcal{Q}$ is not an explicit quantifier in M.

The following generalization of Theorem 2.3 is proved in similar fashion.

THEOREM 2.4. *Let M be a not necessarily well founded model of* ZFC *and $\mathcal{Q}$ be a quantifier in M. Then either $\mathcal{Q} = \mathcal{Q}_V$ or $\mathcal{Q} = \mathcal{Q}_\varkappa$ for some $\varkappa$ which is a regular cardinal in M or $\mathcal{Q} = \{A \subseteq |M|\colon \bar{A} \geqslant \varkappa$ for some $\varkappa \in K\}$ where K is a final segment of the cardinals of M with no least element.*

The situation becomes much more complicated when we start to consider models for set theory without choice. It is impossible to give so simple a characterization of quantifiers in models of ZF as in case ZFC because cardinal numbers are not linearly ordered. But we will try to give as good a characterization as possible.

First notice that the axiom of choice was not used in the proof of case (1) in the proof of Theorem 2.3. Hence if M is a model of ZF and $\mathcal{Q}$ is a quantifier in M such that there are no sets in $\mathcal{Q}$ then $\mathcal{Q} = \mathcal{Q}_V$. Using the same idea we obtain the following lemma:

LEMMA 2.5. *Let M be a model of* ZF *and $\mathcal{Q}$ be a quantifier in M. Then*

(a) *If there exists an $A \subseteq |M|$, $A \in \mathcal{Q}$ such that A is a class in M and no subsets of A which are in M belong to $\mathcal{Q}$ then $\mathcal{Q}_V \subseteq \mathcal{Q}$.*

(b) *Also, if $On \in \mathcal{Q}$ then $\mathcal{Q}_V \subseteq \mathcal{Q}$.*

This lemma gives us no new information for models of ZFC because obviously if $\mathcal{Q}$ is a quantifier in $M \models$ ZFC then all classes of M belong to $\mathcal{Q}$. However, if the axiom of choice fails in M it may be the case that $\mathcal{Q}_V \not\subseteq \mathcal{Q}$. To see this let us first recall the definition of the class WO^∞ which proceeds by transfinite induction:

1^0 $x \in \mathrm{WO}^0 .\!\equiv\!. x$ can be well ordered,

2^0 $x \in \mathrm{WO}^{\alpha+1} .\!\equiv\!. x = \bigcup\{x_i\colon i \in I\}$ and $I \in \mathrm{WO}^0$ and $x_i \in \mathrm{WO}^\alpha$ for all $i \in I$,

3^0 $x \in \mathrm{WO}^\lambda$ for $\lambda \in \mathrm{Lim} .\!\equiv\!. x \in \mathrm{WO}^\alpha$ for some $\alpha<\lambda$,

4^0 $x \in \mathrm{WO}^\infty .\!\equiv\!. x \in \mathrm{WO}^\alpha$ for some $\alpha \in On$.

Let M be a model of ZF+V $\neq \mathrm{WO}^\infty$. Define the following family of subsets of $|M|$:

$$\mathcal{Q}_{\mathrm{WO}^\infty} = \{A \subseteq |M|\colon A \text{ is a class in } M \text{ and there exists } B \subseteq A \text{ s.t. } B \text{ is a set in } M \text{ and } B \notin (\mathrm{WO}^\infty)^M\}.$$

$\mathcal{Q}_{\mathrm{WO}^\infty}$ is a quantifier in M and $On^M \notin \mathcal{Q}_{\mathrm{WO}^\infty}$ (see [3] p. 32 and [4] p. 253).

One can easily prove using the definition of WO^∞ and condition (Q.5) that if $M \models$ ZF, $\mathcal{Q}$ is a quantifier in M and there is an $x \in (\mathrm{WO}^\infty)^M$ which is in $\mathcal{Q}$ then $On^M \in \mathcal{Q}$. This together with Lemma 2.5 implies the following theorem:

THEOREM 2.6. *Let M be a model of* ZF. *Then*:

(a) *There exists a family $\mathcal{Q}$ of subsets of $|M|$ which does not contain all classes of $|M|$ and which is a quantifier in M if and only if $M \models$ V $\neq \mathrm{WO}^\infty$.*

(b) *Every quantifier which does not contain all classes of M is a subset of $\mathcal{Q}_{\mathrm{WO}^\infty}$.*

(c) *If $\varkappa$ is a cardinal in M and $\varkappa \notin (\mathrm{WO}^\infty)^M$ then $\mathcal{Q}_\varkappa \subseteq \mathcal{Q}_{\mathrm{WO}^\infty}$.*

This theorem is a stronger version of Theorems 5.2 and 5.3 from [3].

Notice that in models of ZFC quantifiers (satisfying (*)) are linearly ordered by $\subseteq$ because of the linear ordering of cardinal numbers. Since every infinite partial ordering can be embedded in the ordering of cardinals of some model of ZF (see [2], p. 151) one can expect that the ordering of quantifiers can also be very complicated. The following example shows that it may not be linear.

Let M be the model of ZF constructed by Cohen in which axiom of choice fails but every set can be linearly ordered (see [2], p. 141). This model contains a Dedekind set U; i.e., infinite set which has no subset of power ω. Define the following family $\mathcal{Q}_u$ of subsets of $|M|$:

$A \in \mathcal{Q}_u .\!\equiv\!. A \subseteq |M|$ and there exists a relation $R \subseteq U \times A$ which is a set in M such that $\mathrm{dom} R$ is infinite and for each a in A there are at most a finite number of u such that uRa.

$\mathcal{Q}_u$ is a quantifier in M and $\omega \notin \mathcal{Q}_u$. Also $U \notin \mathcal{Q}_\omega$ by definition of Dedekind set. Hence $\mathcal{Q}_u$ and $\mathcal{Q}_\omega$ are two quantifiers in M such that $\mathcal{Q}_u \not\subseteq \mathcal{Q}_\omega$ and $\mathcal{Q}_\omega \not\subseteq \mathcal{Q}_u$.

At this point we should mention that Krivine and McAloon also considered quantifiers based on the notion of Dedekind set. They let a set $A \subseteq |M|$ be in $\mathcal{Q}$ if $A \cap U$ was a Dedekind set. Note that $\mathcal{Q}$ defined in this way is not a regular quantifier and thus not a quantifier in the restricted sense of this section.

To say more about quantifiers in models of ZF we have to extend the notion of regular cardinal to cardinals which are not alephs.

DEFINITION 2.7. Let M be a model of ZF.

(a) A subset K of Card^M is *regular* in M if for every family $\{S_i\colon i \in I\}$ which is a set in M, if card $\bigcup \{S_i\colon i \in I\} \geqslant \varkappa$ for some $\varkappa \in K$ then either $\bar{I} \geqslant \lambda$ for some $\lambda \in K$ or there exists an $i \in I$ such that $\bar{S}_i \geqslant \lambda$ for some $\lambda \in K$.

(b) If $\varkappa \in M$ is a cardinal in M then we say $\varkappa$ is *regular* in M if $\{\varkappa\}$ is a regular class in M.

(c) If $\varkappa$ is a cardinal in M then we say $\varkappa$ is $*$-*regular* in M if for any $A \subseteq |M|$ such that A is in M a set of the cardinality less than $\varkappa$ and all elements of A have in M cardinality less than $\varkappa$ then $\overline{\overline{\bigcup A}} < \varkappa$.

The definition of a regular class and a regular cardinal was first formulated by Keisler in [3], p. 32.

Notice that if $\varkappa$ is an aleph then $\varkappa$ is regular in the sense of this definition iff it is regular in the previous sense. The notion of $*$-regularity can be quite different from the notion of regularity. Using this notion we can define the following quantifier:

Let $M \models$ ZF and $\varkappa$ be $*$-regular cardinal in M. Define the family $\mathcal{Q}^*$ as follows:

$$A \subseteq |M| \text{ is in } \mathcal{Q}^*_\varkappa \text{ iff } A \text{ is not a set of cardinality } < \varkappa \text{ in } M.$$

$\mathcal{Q}^*_\varkappa$ is a quantifier and it may be different from $\mathcal{Q}_\varkappa$. For example, in the model M mentioned above $\mathcal{Q}^*_\omega$ contains Dedekind sets and $\mathcal{Q}_\omega$ does not, so $\mathcal{Q}^*_\omega$ is different from $\mathcal{Q}_\omega$ (and also from $\mathcal{Q}_u$).

For the following lemma only we drop the convention that all quantifiers are regular.

LEMMA 2.8. *Let M be a model of L and $\mathcal{Q}_1$, $\mathcal{Q}_2$ be two generalized quantifiers in M. Then $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is also a generalized quantifier in M. Moreover if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are regular quantifiers then so is $\mathcal{Q}_1 \cup \mathcal{Q}_2$.*

Now we give a characterization of quantifiers in models of ZF.

THEOREM 2.9. *Let M be a model of* ZF. *Then*

(a) *If $\mathcal{Q}$ is a quantifier in M which does not contain sets from M then $\mathcal{Q} = \mathcal{Q}_V$.*

(b) *If $\mathcal{Q}$ is a quantifier in M and there are sets in $\mathcal{Q}$ then $K = \mathcal{Q} \cap \mathrm{Card}^M$ is a regular class in M.*

(c) *If K is a regular class in M then $\mathcal{Q}_\varkappa$ and $\mathcal{Q}_\varkappa \cup \mathcal{Q}_V$ are quantifiers in M. They are different iff $M \models \mathrm{V} \neq \mathrm{WO}^\infty$ and $K \cap (\mathrm{WO}^\infty)^M = \varnothing$.*

(d) *If $\mathcal{Q}$ is a quantifier in M then $\mathcal{Q}$ is of the form either $\mathcal{Q}_K$ or $\mathcal{Q}_K \cup \mathcal{Q}_V$ for some K being a regular class in M.*

Proof. For the proof that $\mathcal{Q}_K$ is a quantifier see Keisler [3], p. 34. We leave the rest of the proof for the reader since the reasoning is similar to the used above.

We now consider the theories: A_2, second-order arithmetic with the axiom of choice, and A_2^-, second-order arithmetic without the axiom of choice. One can easily prove that $\mathcal{Q}_\omega$ and $\mathcal{Q}^*_\omega$ are quantifiers in models for both these theories and $\mathcal{Q}_V$ is a quantifier in models of A_2. We will define them now as follows:

$$\mathcal{Q}_V = \{A \subseteq \mathcal{P}(\omega)^M\colon A \text{ cannot be coded as a subset of } \omega \text{ in } M \text{ by means of the pairing function}\},$$

$$\mathcal{Q}_\omega = \{A \subseteq \mathcal{P}(\omega)^M\colon \exists X \forall y \forall z (y \neq z \Rightarrow X^{(y)} \neq X^{(z)} \text{ and } X^{(y)} \in A)\},$$

$$\mathcal{Q}^*_\omega = \{A \subseteq \mathcal{P}(\omega)^M\colon A \text{ infinite}\}.$$

If M is a model of $\mathrm{A}_2 + \mathrm{V} = \mathrm{L}$ then $\mathcal{Q}_\omega$ and $\mathcal{Q}_V$ are the only quantifiers in M. Also in models of A_2 the axiom of choice implies $\mathcal{Q}_\omega = \mathcal{Q}^*_\omega$. But it is not known yet whether there are any other quantifiers in models of A_2 and A_2^- and even whether $\mathcal{Q}_V$ is a quantifier in models of A_2^-.

3. Some general remarks about quantifiers. We now drop the convention that quantifiers are regular.

The main reason that we are interested in generalized quantifiers is that one can use them to obtain elementary extensions of countable models. This was first noticed by Keisler [3] who proved the following.

THEOREM 3.1. *Let $\mathcal{M} = \langle M, \mathcal{Q}\rangle$ be a countable model of $L(Q)$ and $\mathcal{Q}$ be a regular quantifier in M. Then there exists a countable model $\mathcal{N} \succ \mathcal{M}$ such that for every formula φ, with one free variable, of the language $L_M(Q)$*

$$\varphi \textit{ is enlarged in } \mathcal{N} \leftrightarrow \mathcal{M} \models \mathrm{Q}x\varphi(x).$$

Another interesting paper about generalized quantifiers is [4], by Krivine and McAloon. They showed that quantifiers which do not necessarily satisfy (Q.5) can also be used to extend models elementarily. They introduced a notion of a countable-like formula.

DEFINITION 3.2. Let M be a model of L and $\mathcal{Q}$ be a generalized quantifier in M. Let φ be a formula of $L_M(Q)$, with one free variable. We say φ is *countable-like* in $\mathcal{M} = \langle M, \mathcal{Q}\rangle$ if for every formula ψ of $L_M(Q)$ with x, y free the following sentence holds in $\mathcal{M}$:

$$(\mathrm{Q}.6) \qquad \mathrm{Q}y \exists x [\varphi(x) \,\&\, \psi(x, y)] \rightarrow \exists x \mathrm{Q}y [\varphi(x) \,\&\, \psi(x, y)].$$

Otherwise we say φ is *not countable-like* in $\mathcal{M}$.

The following lemma explains this notion. For the proof see [4].

LEMMA 3.3. *If φ is countable-like in $\mathcal{M}$ then $\mathcal{M} \models \neg \mathrm{Q}x\varphi(x)$. If $\mathcal{Q}$ is a regular quantifier then $\mathcal{M} \models \neg\mathrm{Q}x\varphi(x)$ implies φ is countable-like in $\mathcal{M}$.*

The following theorem was proved in [4].

THEOREM 3.4. *Let $\mathcal{M} = \langle M, \mathcal{Q}\rangle$ be a countable model of $L(Q)$ and $\mathcal{Q}$ be a generalized quantifier in M. Then there exists a countable $\mathcal{N} \succ \mathcal{M}$ such that:*

(i) *All formulas countable-like in $\mathcal{M}$ are preserved in $\mathcal{N}$;*

(ii) *All formulas not countable-like in $\mathcal{M}$ are enlarged in $\mathcal{N}$.*

(iii) *If $A \subseteq |M|$ is definable in both models then it is definable in $\mathcal{M}$ by a countable-like formula.*

There is a simple way of generating many nonregular quantifiers from a given regular one.

LEMMA 3.5. *Let M be a model of L and $\mathcal{Q}$ be a regular quantifier in M. Let $S_1, \ldots, S_n$ be members of $\mathcal{Q}$ and $L(Q)$-definable in M. Then the family*

$$\mathcal{Q}^* = \{A \subseteq |M|: A \cap S_i \in \mathcal{Q} \text{ for some } i \leqslant n\}$$

is a generalized quantifier in M.

Notice that by the use of this lemma and Lemma 2.8 we can characterize all quantifiers considered in the literature. There might be more complicated quantifiers of course.

The above lemma shows that certain subsets of regular quantifiers are still generalized quantifiers. However, they may not satisfy (Q.5). Also the converse is true since for each model M the family of all infinite subsets of $|M|$ is a regular quantifier. Now the question arises what is the smallest regular quantifier containing given quantifier $\mathcal{Q}^*$? We can answer this question in the case of models of set theory and explicit quantifiers.

THEOREM 3.6. *Let M be a model of* ZF *and $\mathcal{Q}^*$ be an explicit quantifier in M. Then the family*

$$\mathcal{Q} = \{A \subseteq |M|: A \text{ definable and } A \text{ not countable-like in } \mathcal{M}^* = \langle M, \mathcal{Q}^*\rangle\}$$

is an explicit regular quantifier in M. Moreover $\mathcal{Q}$ is the smallest regular quantifier containing $\mathcal{Q}^*$.

Proof. We first consider the explicitness of $\mathcal{Q}$. There are two cases.

Case 1. For every A in $\mathcal{Q}$ there exists $A_0 \subseteq A$ in $\mathcal{Q}$ such that A_0 is a set in M. Let $\varphi(x)$ be in L_M. By definition $\varphi(x)$ is not countable-like in $\mathcal{M}^*$ if and only if

$$(\#) \qquad \mathcal{M}^* \models Qy\exists x(\varphi(x)\&\psi(x, y)) \;\&\; \forall x \neg Qy(\varphi(x)\&\psi(x, y))$$

for some formula $\psi(x, y)$ of L_M. By the case assumption we may replace $\psi(x, y)$ by a set z and so $\varphi(x)$ is not countable-like in $\mathcal{M}^*$ if and only if

$$\mathcal{M}^* \models \exists z[Qy\exists x(\varphi(x)\&\langle x, y\rangle \in z) \;\&\; \forall x \neg Qy(\varphi(x)\&\langle x, y\rangle \in z)].$$

This shows that $\mathcal{Q}$ satisfies the condition for explicitness since $\mathcal{Q}^*$ does.

Case 2. Otherwise. Then by Lemma 2.5 every proper class of M is in $\mathcal{Q}$. Let $\varphi(x)$ be a formula of L_M. In this case we see that $\varphi(x)$ is not countable-like if and only if

$$\mathcal{M}^* \models \exists z[Qy\exists x(\varphi(x)\&\langle x, y\rangle \in z) \;\&\; \forall x \neg Qy(\varphi(x)\&\langle x, y\rangle \in z) \vee \\ \vee \neg\exists z\forall x(\varphi(x) \to x \in z).$$

For, with the notation of Case 1, if $\psi(x, y)$ cannot be replaced by a set z as in Case 1 it means that $\exists x(\varphi(x)\&\psi(x, y))$ defines a proper class. Also

$$\mathcal{M}^* \models \forall x \neg Qy(\varphi(x)\&\psi(x, y))$$

together with the case assumption yields

$$M \models \forall x\exists z(\varphi(x)\&\psi(x, y). \leftrightarrow y \in z).$$

Thus, if $\psi(x, y)$ cannot be replaced by a set z, then $\varphi(x)$ defines a proper class. Again it follows that $\mathcal{Q}$ is explicit since $\mathcal{Q}^*$ is.

Next we claim that $\mathcal{Q}$ satisfies the axioms for a regular quantifier. This is straightforward; we treat (Q.5) as an illustration. Thus suppose we have

$$\mathcal{M} = \langle M, \mathcal{Q}\rangle \models Qx\exists u\theta(u, x) \;\&\; \forall u \neg Qx\theta(u, x)$$

where $\theta(u, x)$ is in L_M. Since $\exists u\theta(u, x)$ is not countable-like with respect to $\mathcal{Q}^*$ we have a formula $\psi(x, y)$ such that (#) holds when $\varphi(x)$ is taken to be $\exists u\theta(u, x)$. Now the formula $\exists x(\theta(u, x)\&\psi(x, y))$ which we denote by $\chi(u, y)$ witnesses that $\exists x\theta(u, x)$ is in $\mathcal{Q}$. We have $\mathcal{M}^* \models Qy\exists u\chi(u, y)$, because $\exists u\chi(u, y)$ is equivalent to $\exists x(\varphi(x)\&\psi(x, y))$. Also $\mathcal{M}^* \models \forall u \neg Qy\theta(u, y)$ follows from $\mathcal{M} \models \forall u \neg Qx\theta(u, x)$. This completes the proof that (Q.5) is satisfied and the proof of the theorem.

We want to end this paper with a few problems:

1. Describe all regular quantifiers in models of A_2 and A_2^-.

2. Let M be a model of L and $\mathcal{Q}$ be an explicit quantifier in M. Is the family $\{A \subseteq |M|: A$ definable and A not countable-like in $\mathcal{M} = \langle M, \mathcal{Q}\rangle\}$ a regular quantifier in M?

3. Let M be a model of L and $\mathcal{Q}$ be a generalized quantifier in M. Is there a $\subseteq$-smallest regular quantifier containing $\mathcal{Q}$?

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