To begin with put $E_0 = E$, $A_\emptyset = C_p(X)$ and assume that the construction is done for $k = n$.

By the condition (i) of Lemma B, Sec. 3 there exists i_{k+1} such that the set $G = \{\xi \in E_k\colon f_\xi \in A_{i_1 \ldots i_{k+1}}\}$ is stationary; the set $H = G \setminus (L(V_{\lambda_1}) \cup \ldots \cup L(V_{\lambda_k}))$ is also stationary and for every $\xi \in H$ the set $V_\xi \setminus (V_{\lambda_1} \cup \ldots \cup V_{\lambda_k}) = W_\xi$ is a neighbourhood of the point p_ξ. Since $p_\xi \in \overline{\bigcup_{\alpha<\xi} X_\alpha}$ for $\xi \in H$, there exists $s(\xi) < \xi$ with $W_\xi \cap X_{s(\xi)} \neq \emptyset$. The function $s\colon H \to \Omega$ is a regressive function with the stationary domain H and therefore there exist a stationary set $I \subset H$ and an ordinal $\alpha \in \Omega$ such that $I \subset s^{-1}(\alpha)$. Thus $W_\xi \cap X_\alpha \neq \emptyset$ for every $\xi \in I$ and, since $|X_\alpha| < \mathfrak{m}$, there exist a stationary set $E_{k+1} \subset I$ and a point $a_{k+1} \in X_\alpha$ such that $a_{k+1} \in W_\xi$ whenever $\xi \in E_{k+1}$. Finally, let λ_{k+1} be an arbitrary ordinal from E_{k+1}. We have $f_\xi(a_{k+1}) = 1$ for $\xi \in E_{k+1}$, as $W_\xi \subset V_\xi$ and also $f_{\lambda_i}(a_{k+1}) = 0$ for $i \leqslant k$, as $a_{k+1} \notin V_{\lambda_1} \cup \ldots \cup V_{\lambda_k}$. This completes the construction.

It is now easily verified that the conditions (c_1), (c_2), (c_3) are fullfield if we put $f_i = f_{\lambda_i}$ (indeed, if $i \geqslant k$ then $\lambda_i \in E_i \subset E_k$ and therefore $f_{\lambda_i}(a_k) = 1$).

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Kan fibrations in the category of simplicial spaces

by

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Abstract. The notion of Kan fibration of simplicial spaces is defined as a direct generalization of Kan fibration of simplicial sets. The covering homotopy property for these fibrations is proved.

Introduction. The category of simplicial spaces (see § 1) has two homotopy theories, which might be called, respectively, topological and algebraic. The topological homotopy theory is based on the simplicial space I_*, whose space of n-simplices is the unit interval I for each $n \geqslant 0$, and whose face and degeneracy operators are all identity maps. Thus, in this theory, two morphisms of simplicial spaces, $f_i\colon X \to Y$, $i = 0, 1$, are homotopic if there is a morphism $P\colon I_* \times X \to Y$, such that, for each $n \geqslant 0$, $F_n|\{i\} \times X = f_{in}$ for $i = 0, 1$. On the other hand, the algebraic theory is the natural extension of the usual homotopy theory in the category of simplicial sets, and is based on the simplicial unit interval, $\Delta[1]$, regarded as a discrete simplicial space.

It is proved in [3], 11.9 and 11.10, that geometric realization of simplicial spaces preserves both kinds of homotopies. Thus, in using techniques in homotopy theory which obtain results about spaces by first working with simplicial spaces and then realizing (techniques which have been much in vogue in recent years, largely in connection with infinite loop spaces), it is possible to work with either of the simplicial homotopy theories, whichever is the more convenient. The use of what we have called the topological theory, has been fairly widespread, for example in [3]. However, by analogy with the category of simplicial sets, it seems to the author that the algebraic homotopy theory should prove much richer.

The purpose of the present paper is to extend some of the basic notions which have been developed for simplicial sets, to simplicial spaces, in particular the notion of Kan fibration. The corresponding notion of fibration with respect to the topological simplicial homotopy theory has been developed in [3], § 12, and is there called a simplicial Hurewicz fibration. In § 1 we give some preliminary topological considerations about function spaces. In § 2 we define our notion of Kan fibration via the notion of a lifting function. This is a direct and rather obvious generalization of the usual notion of Kan fibration in the category of simplicial sets. In § 3 we provide examples by showing that there is a "topological singular complex" functor, adjoint to geometric realization, which sends Hurewicz fibrations to Kan fibrations in our sense. This parallels the discrete case in which the usual, discrete, singular complex functor sends Serre fibrations to Kan fibrations in the category of simplicial sets. In § 4 we prove our main theorem, which is that our Kan fibrations have the covering homotopy property with respect to algebraic homotopies of simplicial spaces. We have tried, as far as possible, to model our proof on the elegant method of [1], chapter IV, § 2. However, the method used there, of comparing certain classes of morphisms obtained from basic morphisms by a prescribed list of operations, does not carry over in a straightforward manner, primarily because the listed operations are not strong enough to pick up anything other than a discrete topology. The method we use, in fact, is something of a hybrid between the function space method of [4] (see Corollary 7.12 to Theorem 7.8), and the method of [1].

In this paper we are concerned, by proving the covering homotopy theorem, to show that our notion of Kan fibration is the correct notion of fibration of simplicial spaces for the algebraic simplicial homotopy theory. Elsewhere, we hope to discuss these ideas in relation to geometric realization.

§ 1. Function spaces of simplicial spaces. Let $\mathscr{C}$ denote the category of compactly generated, weak Hausdorff spaces, with continuous maps as morphisms ([7]). It is well known that this category has a product, namely the cartesian product with the compactly generated topology associated with the product topology. Also, if $X, Y \in \mathscr{C}$, then the function space, $Y^X = \mathrm{Hom}_{\mathscr{C}}(X, Y)$ with the compactly generated topology associated to the compact open topology, is again in $\mathscr{C}$. With these topologies understood, we have a law of exponential correspondence in $\mathscr{C}$, namely, the natural maps:

$$\mathrm{Hom}_{\mathscr{C}}(X\times Y, Z) \rightleftarrows \mathrm{Hom}_{\mathscr{C}}(X, \mathrm{Hom}_{\mathscr{C}}(Y, Z))$$

are inverse homeomorphisms ([7], Theorem 5.6). This implies that a function, $f\colon X \to Z^Y$, is continuous if and only if the composite $X\times Y \xrightarrow{f\times 1} Z^Y\times Y \xrightarrow{E} Z$ is continuous, where E is the evaluation map.

As usual, we denote by Δ^* the simplicial category (see [4], p. 4; also [1], p. 23, where the category is called Δ). Let $\mathscr{C}^\Delta$ denote the category of simplicial objects in $\mathscr{C}$ (simplicial spaces) with simplicial maps as morphisms. Define a functor, $\mathrm{Hom}_{\mathscr{C}^\Delta}(\_, \_)\colon \mathscr{C}^\Delta\times\mathscr{C}^\Delta \to \mathscr{C}$, as follows. As a set, $\mathrm{Hom}_{\mathscr{C}^\Delta}(X, Y)$ consits of all morphisms: $X\to Y$ in $\mathscr{C}^\Delta$. The topology on $\mathrm{Hom}_{\mathscr{C}^\Delta}(X, Y)$ is the subspace topology arising from the natural inclusion:

$$\mathrm{Hom}_{\mathscr{C}^\Delta}(X, Y) \subseteq \prod_{n\geq 0} \mathrm{Hom}_{\mathscr{C}}(X_n, Y_n)$$

given by $f \mapsto \prod_{n\geq 0} f_n$. Here, $\prod$ denotes the product in $\mathscr{C}$. As an immediate consequence of this, we have:

1.1. PROPOSITION. *A function $f\colon Z \to \mathrm{Hom}_{\mathscr{C}^\Delta}(X, Y)$ is continuous if and only if the composites: $X_n\times Z \xrightarrow{1\times f} X_n \times \mathrm{Hom}_{\mathscr{C}^\Delta}(X, Y) \xrightarrow{E_n} Y_n$, are continuous for each $n\geq 0$, where E_n is the evaluation map, $E_n(x, f) = f_n(x)$.*

Let $\Delta[n]$ denote the standard simplicial n-simplex ([1], p. 25; [4], p. 14), regarded as a simplicial space with the discrete topology. We define function spaces in $\mathscr{C}^\Delta$ as follows. If X and Y are simplicial spaces, Y^X is the simplicial space given by:

$$(Y^X)_n = \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times X, Y)$$

and, if $\theta\colon [n]\to[m]$ is a morphism in Δ^*, $\theta^*\colon (Y^X)_m \to (Y^X)_n$ is the map induced in the obvious way.

1.2. PROPOSITION. *For $X, Y, Z \in \mathscr{C}^\Delta$, there is a natural exponential homeomorphism:*

$$\mathrm{Hom}_{\mathscr{C}^\Delta}(X\times Y, Z) \rightleftarrows \mathrm{Hom}_{\mathscr{C}^\Delta}(X, Z^Y)\,.$$

Proof. Define $\theta\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(X\times Y, Z) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(X, Z^Y)$ by: $\theta(f)_n(x)(\lambda, y) = f_m(\lambda^*(x), y)$, for $x\in X_n$, $y\in Y_m$ and $\lambda\in\Delta[n]_m$. Applications of 1.1 show that θ is well defined and continuous.

Define $\varphi\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(X, Z^Y) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(X\times Y, Z)$ by: $\varphi(g)_n(x, y) = g_n(x)_n(i_n, y)$, where $g\colon X\to Z^Y$ in $\mathscr{C}^\Delta$, $x\in X_n$, $y\in Y_n$ and $i_n\in\Delta[n]_n$ is the fundamental simplex. Again, applications of 1.1 show that φ is well defined and continuous.

It is clear that $\varphi\theta = 1$. That $\theta\varphi = 1$ follows as below:

$$\theta\varphi(f)_n(x)(\lambda, y) = \varphi(f)_m(\lambda^*(x), y) = f_m(\lambda^*(x))_m(i_m, y)\,.$$

Now, since $f\colon X\to Z^Y$ is a simplicial map, $f_m(\lambda^*(x)) = \lambda^*(f_n(x))$. Thus, since $\lambda^*\colon (Z^Y)_n \to (Z^Y)_m$ is given by, $\lambda^*(h)(\mu, y) = h(\lambda\mu, y)$ for $h\in(Z^Y)_n$, $(\mu, y)\in\Delta[m]\times Y$, we have

$$\theta\varphi(f)_n(x)(\lambda, y) = f_n(x)(\lambda, y)$$

and hence $\theta\varphi = 1$, as required.

1.3. COROLLARY. *Let $X, Y, Z\in\mathscr{C}^\Delta$ and $f\colon X\to Z^Y$ be a simplicial function. Then f is continuous if and only if the composite: $X\times Y \xrightarrow{f\times 1} Z^Y\times Y \xrightarrow{\varepsilon} Z$ is continuous, where ε is the evaluation map given by: $\varepsilon_n(g, y) = g_n(i_n, y)$ for $g\in (Z^Y)_n$ and $y\in Y_n$.*

Proof. $\varepsilon(f\times 1) = \varphi(f)$, where φ is the homeomorphism defined in the proof of 1.2.

1.4. COROLLARY. *For any $X\in\mathscr{C}^\Delta$, evaluation at the fundamental simplex is a homeomorphism, $\varepsilon_n\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n], X) \to X_n$.*

Proof. The identity on $X^{\Delta[0]}$ is continuous, hence, by 1.3, so is $\varepsilon\colon X^{\Delta[0]}\times \Delta[0] \to X$; i.e. ε_n is continuous for each $n\geq 0$.

Let $\delta\colon X \to X^{\Delta[0]}$ be the map given by: $\delta(x)(\lambda) = \lambda^*(x)$ for $x \in X$, $\lambda \in \Delta[n]$. Then, if $c\colon X \times \Delta[0] \to X$ is the projection, $\delta = \theta(c)$, where θ is the homeomorphism defined in the proof of 1.2. Thus, δ is continuous. It is now straightforward to check that ε and δ are inverse homeomorphisms.

§ 2. Lifting functions and Kan fibrations in $\mathscr{C}^{\Delta}$. Let $p\colon E \to B$ and $q\colon X \to Y$ be morphisms in $\mathscr{C}^{\Delta}$. Let $p_*\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(\_, E) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(\_, B)$ and $q^*\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(Y, \_) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(X, \_)$ be the induced maps, and $\Gamma_q(p)$ be the pullback in $\mathscr{C}$ in the diagram:

$$\begin{array}{ccc} \Gamma_q(p) & \xrightarrow{v} & \mathrm{Hom}_{\mathscr{C}^\Delta}(X, E) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{Hom}_{\mathscr{C}^\Delta}(Y, B) & \xrightarrow{q^*} & \mathrm{Hom}_{\mathscr{C}^\Delta}(X, B) \end{array}$$

Clearly there is a unique map $\pi = \pi_q(p)\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(Y, E) \to \Gamma_q(p)$ such that $u\pi = p_*$ and $v\pi = q^*$. We say that q admits a *lifting function* with respect to p, if there is a map $\lambda\colon \Gamma_q(p) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(Y, E)$ such that $\pi\lambda = 1$. Below we give some basic properties of lifting functions.

2.1. PROPOSITION. (i) *Let*

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow{u'} & Y' \end{array}$$

be a pushout diagram in $\mathscr{C}^{\Delta}$. Suppose q admits a lifting function with respect to a morphism, p. Then q' admits a lifting function with respect to p.

(ii) *Let*

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & X \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \xi} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & X' \end{array}$$

be a retraction diagram in $\mathscr{C}^{\Delta}$; that is, $v'u' = 1$ and $vu = 1$. Then, if η admits a lifting function with respect to p, so does ξ.

(iii) *Let $\{q_\alpha\colon X_\alpha \to Y_\alpha\}$ be a family of morphisms in $\mathscr{C}^{\Delta}$, each of which admits a lifting function with respect to p. Then $\coprod_\alpha q_\alpha\colon \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$ admits a lifting function with respect to p.*

(iv) *Let $\xi\colon X \to Y$ and $\eta\colon Y \to Z$ be morphisms in $\mathscr{C}^{\Delta}$, each of which admits a lifting function with respect to p. Then $\eta\xi\colon X \to Z$ admits a lifting function with respect to p.*

(v) *More generally, if $q_n\colon X_n \to X_{n+1}$, $n \geqslant 1$, is a direct system of morphisms in $\mathscr{C}^{\Delta}$, each of which admits a lifting function with respect to p, then the inclusion, $q\colon X_1 \to \varinjlim_n X_n$, admits a lifting function with respect to p.*

(vi) *Each of the claimed lifting functions in* (i) *to* (v) *is natural in the following sense. Given a morphism between a situation of the prescribed type and another one of the same type such that the maps induced by the morphism preserve the given lifting functions, then the maps induced by the morphism preserve the constructed lifting functions.*

Proof. (i) Let λ be the lifting function for q. Let $(f, F) \in \Gamma_{q'}(p)$, so that $pf = q'F$. Define $\lambda'\colon \Gamma_{q'}(p) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(Y', E)$ by: $\lambda'(f, F)\colon Y' \to E$ is the unique morphisms satisfying, $\lambda'(f, F)q' = f$ and $\lambda'(f, F)u' = \lambda(fu, Fu')$. An application of 1.1 shows that λ' is continuous, and it is clearly a lifting function for q'.

(ii) Let λ_η be the lifting function for η. Suppose $(f, F) \in \Gamma_\xi(p)$, so that $pf = F$. Define $\lambda_\xi\colon \Gamma_\xi(p) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(X', E)$ by: $\lambda_\xi(f, F) = u'^*\lambda_\eta(fv, Fv')$. Again, by 1.1, λ_ξ is continuous, and is clearly the required lifting function.

(iii) A lifting function for $\coprod_\alpha q_\alpha$ is given by $\prod_\alpha \lambda_\alpha$, where λ_α is a lifting function for q_α.

(iv) Let λ_ξ, λ_η be lifting functions for ξ and η, respectively. Define $\lambda_{\eta\xi}\colon \Gamma_{\eta\xi}(p) \to \mathrm{Hom}_{\mathscr{C}^\Delta}(Z, E)$ by: $\lambda_{\eta\alpha}(f, F) = \lambda_\eta(\lambda_\xi(f, F\eta), F)$. Again, it is easy to check that this gives the required lifting function.

(v) Let $j_n = q_{n-1}q_{n-2} \ldots q_1\colon X_1 \to X_n$. Thus, $q = \varinjlim_n j_n$. By induction from (iv), there are lifting functions, λ_n, for j_n with respect to p. Further, it is clear from the construction of (iv), that we have commutative diagrams:

$$\begin{array}{ccc} \Gamma_{j_n}(p) & \xrightarrow{(q_{n-1})^*} & \Gamma_{j_{n-1}}(p) \\ {\scriptstyle \lambda_n}\downarrow & & \downarrow{\scriptstyle \lambda_{n-1}} \\ \mathrm{Hom}_{\mathscr{C}^\Delta}(X_n, E) & \xrightarrow{(q_{n-1})^*} & \mathrm{Hom}_{\mathscr{C}^\Delta}(X_{n-1}, E) \end{array}$$

Let $i_\alpha\colon X_\alpha \to \varinjlim_n X_n$ be the inclusion (so that $q = i_1$). Define a lifting function, λ, for q by: $\lambda(f, F) = \varprojlim_n \lambda_n(f, Fi_n)$. Again, by an application of 1.1, it is easy to see that λ is the required lifting function.

(vi) Each of the above constructions is obviously natural in the stated sense.

Let $\delta_k\colon [n-1] \to [n]$ be the monomorphism in Δ^* whose image is $[n]-\{k\}$, and define $\Delta^k[n]$ to be the subcomplex of $\Delta[n]$ whose m-simplices are those morphisms in Δ^*, $\lambda\colon [m] \to [n]$, whose image does not contain the image of δ_k. Let $i_k\colon \Delta^k[n] \to \Delta[n]$ be the inclusion. We shall say that $p\colon E \to B$ is a *Kan fibration in $\mathscr{C}^{\Delta}$* if, for each n and k, $0 \leqslant k \leqslant n$, i_k admits a lifting function with respect to p. A simplicial space X is a *Kan object in $\mathscr{C}^{\Delta}$* if and only if the constant morphism, $c\colon X \to \Delta[0]$, is a Kan fibration. The following are trivial to verify.

2.2. BASIC PROPERTIES. (i) The composite of Kan fibrations in $\mathscr{C}^{\Delta}$ is a Kan fibration in $\mathscr{C}^{\Delta}$. Hence, if $p\colon E \to B$ is a Kan fibration in $\mathscr{C}^{\Delta}$ and B is a Kan object in $\mathscr{C}^{\Delta}$, then E is a Kan object in $\mathscr{C}^{\Delta}$.

(ii) If

$$\begin{array}{ccc} E' & \to & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \to & B \end{array}$$

is a pullback diagram in $\mathscr{C}^{\Delta}$, and p is a Kan fibration in $\mathscr{C}^{\Delta}$, then so is p'.

(iii) Let E and B be discrete simplicial spaces. Then $p\colon E \to B$ is a Kan fibration in $\mathscr{C}^{\Delta}$ if and only if it is a Kan fibration in the category of simplicial sets (see [1], p. 65; [4], 25).

§ 3. An example: The topological singular complex functor. Let Δ^n be the standard topological n-simplex. For a space X, define SX by: $S_n X = \mathrm{Hom}_{\mathscr{C}}(\Delta^n, X) = X^{\Delta^n}$, with face and degeneracy operators induced from the usual operations on Δ^n. Clearly, S extends to a functor, $S\colon \mathscr{C} \to \mathscr{C}^{\Delta}$. Let $|\ |\colon \mathscr{C}^{\Delta} \to \mathscr{C}$ denote the geometric realization functor. We have an adjunction isomorphism:

$$\mathrm{Hom}_{\mathscr{C}^{\Delta}}(A, SX) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_{\mathscr{C}}(|A|, X)$$

given, as usual, by:

$$\Phi(f)(u, a) = f(a)(u) \text{ for } a \in A_n,\ u \in \Delta^n \text{ and } f\colon A \to SX,$$

$$\Psi(g)(a)(u) = g(u, a) \text{ for } a \in A_n,\ u \in \Delta^n \text{ and } g\colon |A| \to X.$$

By the exponential correspondence theorem in $\mathscr{C}$ (see § 1), we see that Φ is continuous, and, by an application of 1.1, Ψ is also seen to be continuous. Thus, Φ and Ψ are inverse homeomorphisms. Let $\eta_A\colon A \to S|A|$ and $\varepsilon_X\colon |SX| \to X$ be the unit and counit morphisms defined by the above adjunction.

3.1. Proposition. *For any $X \in \mathscr{C}$, $\varepsilon_X\colon |SX| \to X$ is a homotopy equivalence.*

Proof. Let $X_* \in \mathscr{C}^{\Delta}$ be the simplicial space whose space of n-simplices is X for each $n \geqslant 0$, and whose face and degeneracy maps are all identity maps. It is easy to see that the inclusion of the space of 0-simplices, $X \subseteq |X_*|$, is a homeomorphism. Thus we have the unit map, $\eta_{X_*}\colon X_* \to SX$. Define $\nu_n\colon S_n X \to X$ by: $\nu_n(f) = f(1/(n+1), \ldots, 1/(n+1))$. Then we have, $\nu_n \eta_{X_*} = 1_X$ and, for $f \in S_n X$, $\eta_{X_*}\nu_n(f)$ is the singular n-simplex given by: $\eta_{X_*}\nu_n(f)(u) = f(1/(n+1), \ldots, 1/(n+1))$ for all $u \in \Delta^n$. Let $u(t) \in \Delta^n$ be the point, $u(t) = tu + (1-t)(1/(n+1), \ldots, 1/(n+1))$ for $0 \leqslant t \leqslant 1$. Then $h\colon I \times S_n X \to S_n X$ given by $h_t(f)(u) = f(u(t))$, is a homotopy between $\eta_{X_*}\nu_n$ and $1_{S_n X}$. Thus η_{X_*} is a simplicial map which is a homotopy equivalence in each dimension. Now, it is clear that X_* and SX are proper in the sense of [3], Definition 11.2, p. 102, and hence, by [5], Theorem A.4(ii), appendix, $|\eta_{X_*}|\colon X = |X_*| \to |SX|$ is a homotopy equivalence. Finally, by adjointness, the composite:

$$|X_*| \xrightarrow{|\eta_{X_*}|} \big||S|X_*|\big| \xrightarrow{\varepsilon_{|X_*|}} |X_*|$$

is the identity map. Hence, $\varepsilon_X = \varepsilon_{|X_*|}$ is a homotopy equivalence.

3.2. Proposition. *Let $p\colon E \to B$ be a map in $\mathscr{C}$. Then p is a Hurewicz fibration if and only if $Sp\colon SE \to SB$ is a Kan fibration in $\mathscr{C}^{\Delta}$. Hence, S maps $\mathscr{C}$ into the full subcategory of $\mathscr{C}^{\Delta}$ whose objects are the Kan objects.*

Proof. Let $\Pi(p) = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$, and $\zeta\colon E^I \to \Pi(p)$ be the map, $\zeta(\omega) = (\omega(0), p\omega)$. Recall, [6], Theorem 8, p. 92, that p is a Hurewicz fibration if and only if p admits a lifting function; that is, if and only if there is a map $\lambda\colon \Pi(p) \to E^I$ such that $\zeta\lambda = 1$. Given such a λ, define, for each $n \geqslant 1$, $\lambda_n\colon \Pi(p^{I^n}) \to E^{I^{n+1}}$ by: $\lambda_n(u, v)(t_1, \ldots, t_{n+1}) = \lambda(u(t_1, \ldots, t_n), \alpha)(t_{n+1})$, where $\alpha \in B^I$ is the path, $\alpha(t) = v(t)(t_1, \ldots, t_n)$, $v \in (B^{I^n})^I$ and $u \in E^{I^n}$. Then λ_n is a lifting function for $p^{I^n}\colon E^{I^n} \to B^{I^n}$. Thus, it follows that p is a Hurewicz fibration if and only if p^{I^n} is for each $n \geqslant 0$, if and only if p^{I^n} admits a lifting function for each $n \geqslant 0$.

Choose a fixed homeomorphism, $h_k\colon |\Delta[n]| \to I^n$ such that $h_k(\Delta^k[n]) = I^{n-1} \times \{0\}$. Then h_k induces homeomorphisms, via the adjunction defined above:

$\mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[n], SX) \cong \mathrm{Hom}_{\mathscr{C}}(I^n, X) = X^{I^n}$; $\mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta^k[n], SX) \cong X^{I^{n-1}}$, under which the map $(i_k)^*\colon \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[n], SX) \to \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta^k[n], SX)$ induced by the inclusion, $\Delta^k[n] \subset \Delta[n]$, corresponds to the map $X^{I^n} \to X^{I^{n-1}}$ induced by the inclusion, $I^{n-1} \times \{0\} \subset I^n$. These homeomorphisms induce a homeomorphism of pullbacks, $\Gamma_{i_k}(Sp) \cong \Pi(p^{I^{n-1}})$, where $\Gamma_{i_k}(Sp)$ is defined as in § 2. Further, we have a commutative diagram:

$$\begin{array}{ccc} \Gamma_{i_k}(Sp) & \cong & \Pi(p^{I^{n-1}}) \\ {\scriptstyle \pi_k}\uparrow & & \uparrow{\scriptstyle \zeta_n} \\ \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[n], SE) & \cong & E^{I^n} \end{array}$$

where π_k and ζ_n are the canonical inclusions. Thus, p admits a lifting function if and only if Sp does.

3.3. Remark. As a further example of a Kan fibration in $\mathscr{C}^{\Delta}$, we offer the following. Recall from [2], § 7, that, if G is a topological group, X a left G-space and Y a right G-space, then we may form a simplicial space, $B_*(Y, G, X)$, the so-called 2-sided geometric bar construction. If $*$ denotes a points, and $p_*\colon B_*(Y, G, X) \to B_*(Y, G, *)$ is the simplicial map induced by the projection $X \to \{*\}$, then it is easy to show that p_* is a Kan fibration in $\mathscr{C}^{\Delta}$. In particular, taking $Y = *$, $X = G$ with left G-action given by multiplication, p_* is the universal fibration for G in $\mathscr{C}^{\Delta}$; its geometric realization is a universal principal G-bundle. We should note, however, that, if group is replaced by monoid in the above, then p_* need not be a Kan fibration in $\mathscr{C}^{\Delta}$.

§ 4. The covering homotopy property. In this section we shall prove the following theorem.

4.1. Theorem. *Let $p\colon E \to B$ be a Kan fibration in $\mathscr{C}^{\Delta}$ and K any simplicial space. Then, $p^K\colon E^K \to B^K$ is a Kan fibration in $\mathscr{C}^{\Delta}$.*

As an immediate consequence of this theorem, we deduce that the following covering homotopy theorem holds in $\mathscr{C}^{\Delta}$.

4.2. THEOREM. *Let $p\colon E\to B$ be a Kan fibration in $\mathscr{C}^{\Delta}$, and let $i\colon \Delta[0]\to\Delta[1]$ be the morphism induced by $\delta_0\colon [0]\to[1]$. Then, given any commutative diagram in $\mathscr{C}^{\Delta}$:*

$$\begin{array}{ccc} \Delta[0]\times K & \xrightarrow{f} & E \\ {\scriptstyle i\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta[1]\times K & \xrightarrow{F} & B \end{array}$$

there is a lifting, $\tilde{F}\colon \Delta[1]\times K\to E$, which makes the diagram commute.

To prove Theorem 4.1, we shall first prove the theorem for $K=\Delta[q]$. More than that, we shall show that lifting functions can be chosen for $p^{\Delta[q]}$ which are natural with respect to morphisms $[q']\to[q]$ in Δ^*. This is done in Lemma 4.3 and Proposition 4.4 below, and constitutes the main step in the proof of Theorem 4.1. Finally, we show how Theorem 4.1 follows from this.

4.3. LEMMA. *Let X and Y be simplicial sets (i.e. discrete simplicial spaces), $U\subseteq X$ and $V\subseteq Y$ subcomplexes. Consider the maps:*

$$h\colon \Delta[1]\times U\cup\Delta[0]\times X\to\Delta[1]\times X,$$
$$k\colon \Delta[1]\times V\cup\Delta[0]\times Y\to\Delta[1]\times Y$$

induced by $i\colon\Delta[0]\to\Delta[1]$. Let $\Sigma_n(X,U)$ be the set of non-degenerate n-simplices of X which do not belong to U, and let $\mathscr{S}((X,U);(Y,V))$ denote the set of morphisms, $f\colon (X,U)\to(Y,V)$, in $\mathscr{C}^{\Delta}$ such that $f_n(\Sigma_n(X,U))\subseteq\Sigma_n(Y,V)$ for each $n\geqslant 0$. Then, if $p\colon E\to B$ is a Kan fibration in $\mathscr{C}^{\Delta}$, there are lifting functions with respect to p, λ_X for h and λ_Y for k, such that the following diagram commutes for all $f\in\mathscr{S}((X,U);(Y,V))$:

$$\begin{array}{ccc} \Gamma_h(p) & \xrightarrow{\lambda_X} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[1]\times X, E) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Gamma_k(p) & \xrightarrow{\lambda_Y} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[1]\times Y, E) \end{array}$$

Proof. Consider the inclusions, $q_n\colon \Delta[1]\times\dot{\Delta}[n]\cup\Delta[0]\times\Delta[n]\to\Delta[1]\times\Delta[n]$ where $\dot{\Delta}[n]$ is the boundary of $\Delta[n]$ ([1], p. 29). We first observe that, as in [1], Chapter IV, 2.1.1, p. 61, these inclusions are obtained from the inclusion $\Delta^k[n]\subset\Delta[n]$ via the operations listed in Proposition 2.1 above. It therefore follows from Proposition 2.1, that there are lifting functions for q_n with respect to p for each $n\geqslant 0$.

Let $\mathrm{Sk}^n X$ denote the n-skeleton of X ([1], 3.5, p. 29). As in [1], Fig. 31, p. 62, we have a pushout diagram of simplicial sets:

$$\begin{array}{ccc} \coprod_{\sigma\in\Sigma_n(X,U)} (\Delta[1]\times\dot{\Delta}[n]_\sigma\cup\Delta[0]\times\Delta[n]_\sigma) & \to & \Delta[1]\times(U\cup\mathrm{Sk}^{n-1}X)\cup\Delta[0]\times X \\ {\scriptstyle \coprod_\sigma q_n(\sigma)}\downarrow & & \downarrow{\scriptstyle i_n(X,U)} \\ \coprod_{\sigma\in\Sigma_n(X,U)} (\Delta[1]\times\Delta[n]_\sigma) & \to & \Delta[1]\times(U\cup\mathrm{Sk}^n X)\cup\Delta[0]\times X \end{array}$$

where $\Delta[n]_\sigma=\Delta[n]$, $q_n(\sigma)=q_n$ for each σ, $i_n(X,U)$ is induced by the inclusion $\mathrm{Sk}^{n-1}X\subseteq\mathrm{Sk}^n X$ and the horizontal arrows are induced by the singular simplices, $\tilde{\sigma}\colon\Delta[n]_\sigma\to X$. We also have a similar pushout diagram for (Y,V). Further, if $f\in\mathscr{S}((X,U);(Y,V))$, then f_n maps $\Sigma_n(X,U)$ to $\Sigma_n(Y,V)$, and hence induces a map between the two pushout diagrams, since $f\tilde{\sigma}=\widetilde{f_n(\sigma)}$ for each $\sigma\in\Sigma_n(X,U)$. It now follows from Proposition 2.1(i), (iii) and (vi), that the lifting functions for the q_n induce lifting functions, $\lambda_n^{(X,U)}$ for $i_n(X,U)$ and $\lambda_n^{(Y,V)}$ for $i_n(Y,V)$, with respect to p such that the following diagram commutes for all $f\in\mathscr{S}((X,U);(Y,V))$:

$$\begin{array}{ccc} \Gamma_{i_n(X,U)}(p) & \xrightarrow{\lambda_n^{(X,U)}} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[1]\times(U\cup\mathrm{Sk}^n X)\cup\Delta[0]\times X, E) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Gamma_{i_n(Y,V)}(p) & \xrightarrow{\lambda_n^{(Y,V)}} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[1]\times(V\cup\mathrm{Sk}^n Y)\cup\Delta[0]\times Y, E) \end{array}$$

Now $\Delta[1]\times X=\varinjlim_n(\Delta[1]\times(U\cup\mathrm{Sk}^n X)\cup\Delta[0]\times X)$. It therefore follows from Proposition 2.1(v) and (vi), that there are lifting functions, λ_X for h and λ_Y for k, with respect to p, with the stated naturality property with respect to morphisms in $\mathscr{S}((X,U);(Y,V))$.

4.4. PROPOSITION. *Let $p\colon E\to B$ be a Kan fibration in $\mathscr{C}^{\Delta}$ and $i_k\colon\Delta^k[n]\to\Delta[n]$ be the inclusion. Then, for each $q\geqslant 0$, there are lifting functions, λ_k^q, for*

$$i_k\times 1\colon \Delta^k[n]\times\Delta[q]\to\Delta[n]\times\Delta[q]$$

such that, if $\theta\colon[q']\to[q]$ is any morphism in Δ^*, the following diagram commutes:

$$\begin{array}{ccc} \Gamma_{i_k}(p^{\Delta[q]})=\Gamma_{i_k\times 1}(p) & \xrightarrow{\lambda_k^q} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[n]\times\Delta[q], E) \\ \downarrow{\scriptstyle\theta^*} & & \downarrow{\scriptstyle\theta^*} \\ \Gamma_{i_k}(p^{\Delta[q']})=\Gamma_{i_k\times 1}(p) & \xrightarrow{\lambda_k^{q'}} & \mathrm{Hom}_{\mathscr{C}^{\Delta}}(\Delta[n]\times\Delta[q'], E) \end{array}$$

Proof. Let $X=\Delta[n]\times\Delta[q']$, $U=\Delta^k[n]\times\Delta[q']$, $Y=\Delta[n]\times\Delta[q]$, $V=\Delta^k[n]\times\Delta[q]$, $Z=\Delta[n]\times\Delta[q']\times\Delta[q]$ and $W=\Delta^k[n]\times\Delta[q']\times\Delta[q]$. Define a morphism of pairs, $\alpha_0\colon(X,U)\to(Z,W)$, by $\alpha_0(\lambda,\mu)=(\lambda,\mu,\theta\mu)$ for $(\lambda,\mu)\in\Delta[n]\times\Delta[q']$. We assert that α_0 maps $\Sigma_r(X,U)$ to $\Sigma_r(Z,W)$ for each $r\geqslant 0$. Now $\Sigma_r(X,U)=\Sigma_r(Z,W)=\emptyset$ for $r<n$, since, in this range, $\Delta^k[n]_r=\Delta[n]_r$. So suppose $r\geqslant n$, then:

$$\Sigma_r(X,U)=\{(\lambda,\mu)\,|\,\lambda\colon[r]\to[n],\ \mu\colon[r]\to[q'],\ \mathrm{Im}\,\lambda\supset\mathrm{Im}\,\delta_k \text{ and either } \lambda \text{ or } \mu \text{ is a monomorphism}\},$$

$$\Sigma_r(Z,W)=\{(\lambda,\mu,\nu)\,|\,\lambda\colon[r]\to[n],\ \mu\colon[r]\to[q'],\ \nu\colon[r]\to[q],\ \mathrm{Im}\,\lambda\supset\mathrm{Im}\,\delta_k \text{ and at least one of } \lambda,\mu,\nu \text{ is a monomorphism}\}.$$

If $(\lambda,\mu)\in\Sigma_r(X,U)$, then it is now clear that $\alpha_0(\lambda,\mu)=(\lambda,\mu,\theta\mu)\in\Sigma_r(Z,W)$, which proves our assertion. It now follows from Lemma 4.3 that there are lifting functions with respect to p for $h\colon\Delta[1]\times U\cup\Delta[0]\times\lambda\to\Delta[1]\times X$ and $k\colon\Delta[1]\times W\cup\Delta[0]\times Z\to\Delta[1]\times Z$ which are natural with respect to morphisms $\theta\colon[q']\to[q]$.

Now, from [1], 2.1.3, p. 63, there is a retraction diagram:

$$\begin{array}{ccccc} \Delta^k[n] & \to & \Delta[1]\times\Delta^k[n]\cup\Delta[0]\times\Delta[n] & \to & \Delta^k[n] \\ {\scriptstyle i_k}\downarrow & & \downarrow & & \downarrow{\scriptstyle i_k} \\ \Delta[n] & \xrightarrow{u} & \Delta[1]\times\Delta[n] & \xrightarrow{v} & \Delta[n] \end{array}$$

Hence, taking the product with $\Delta[q']$ and with $\Delta[q']\times\Delta[q]$, we obtain that $i_k\times 1\colon U\to X$ is a retract of h and $i_k\times 1\times 1\colon W\to Z$ is a retract of k. Further, any morphism $\theta\colon [q']\to[q]$ in Δ^*, induces a morphism of retraction diagrams which preserves the lifting functions for h and k. Hence, by Proposition 2.1(ii) and (vi) that there are lifting functions, $\lambda_k^{q'}$ for $i_k\times 1\colon U\to X$ and $\lambda_k^{q',q}$ for $i_k\times 1\times 1\colon W\to Z$, which are natural with respect to morphisms $\theta\colon [q']\to[q]$ in Δ^*.

Let λ_0 be the element of $\Delta[q']_r$ whose image is $\{0\}$. Define $\gamma\colon Y\to Z$ by: $\gamma(\lambda,\nu)=(\lambda,\lambda_0,\nu)$ for $(\lambda,\nu)\in\Delta[n]\times\Delta[q]$. We have a retraction diagram:

$$\begin{array}{ccccc} V & \to & W & \longrightarrow & V \\ {\scriptstyle i_k\times 1}\downarrow & & \downarrow{\scriptstyle i_k\times 1\times 1} & & \downarrow{\scriptstyle i_k\times 1} \\ Y & \xrightarrow{\gamma} & Z & \xrightarrow{\delta} & Y \end{array}$$

where δ is the projection. It follows from Proposition 2.1(ii) that the lifting function $\lambda_k^{q',q}$ defined above, induces a lifting function, $\bar{\lambda}_k^q$, for $i_k\times 1$. Further, from the construction of this lifting function in the proof of Proposition 2.1(ii), it is clear that the following diagram commutes:

$$(*)\qquad \begin{array}{ccc} \Gamma_{i_k\times 1}(p) & \xrightarrow{\bar{\lambda}_k^q} & \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times\Delta[q],E) \\ {\scriptstyle\gamma^*}\uparrow & & \uparrow{\scriptstyle\gamma^*} \\ \Gamma_{i_k\times 1\times 1}(p) & \xrightarrow{\lambda_k^{q',q}} & \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times\Delta[q']\times\Delta[q],E) \\ {\scriptstyle\delta^*}\uparrow & & \uparrow{\scriptstyle\delta^*} \\ \Gamma_{i_k\times 1}(p) & \xrightarrow{\bar{\lambda}_k^q} & \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times\Delta[q],E) \end{array}$$

We shall show that $\bar{\lambda}_k^q=\lambda_k^q$, the lifting function for $i_k\times 1$ constructed previously. From this, and the diagram (*), it follows that $\lambda_k^{q',q}\delta^*=\delta^*\lambda_k^q$. Hence, since $\delta\alpha_\theta=1\times\theta\colon \Delta[n]\times\Delta[q']\to\Delta[n]\times\Delta[q]$, we have, for any $\theta\colon[q']\to[q]$, $\theta^*\lambda_k^q=\alpha_\theta^*\delta^*\lambda_k^q=\alpha_\theta^*\lambda_k^{q',q}\delta^*=\lambda_k^{q'}\alpha_\theta^*\delta^*$, since we have shown that $\lambda_k^{q'}$ and $\lambda_k^{q',q}$ are natural with respect to morphisms $\theta\colon[q']\to[q]$. Thus, we conclude that $\theta^*\lambda_k^q=\lambda_k^{q'}\theta^*$, which will prove the proposition.

It remains to show that $\bar{\lambda}_k^q=\lambda_k^q$. To do this, it is clearly sufficient to show that $\lambda_k^q\gamma^*=\gamma^*\lambda_k^{q',q}$. For, since γ^* is an epimorphism the result follows from the diagram (*). We first observe that, for each $r\geqslant 0$, γ maps $\Sigma_r(Y,V)$ to $\Sigma_r(Z,W)$. It now follows from Lemma 4.3 that there are lifting functions for $k\colon \Delta[1]\times W\cup\Delta[0]\times Z\to\Delta[1]\times Z$ and $1\colon \Delta[1]\times V\cup\Delta[0]\times Y\to\Delta[1]\times Y$ which are natural with respect to γ. Now, as above, $i_k\times 1\times 1\colon W\to Z$ and $i_k\times 1\colon V\to Y$ are natural retracts of k and 1, respectively. Further, γ induces a map of retraction diagrams which preserves the lifting functions for k and 1. Hence, by Proposition 2.1(ii) and (vi), there are lifting functions for $i_k\times 1\times 1$ and $i_k\times 1$ which are natural with respect to γ. But, by construction, these lifting functions are precisely λ_k^q and $\lambda_k^{q',q}$, respectively. This completes the proof of Proposition 4.4.

The proof of Theorem 4.1. Let A, K and X be simplicial spaces. Define an evaluation map, $\varepsilon_q\colon K_q\times\mathrm{Hom}_{\mathscr{C}^\Delta}(A\times K,X)\to\mathrm{Hom}_{\mathscr{C}^\Delta}(A\times\Delta[q],X)$ by

$$\varepsilon_q(k,f)(a,\lambda)=f\big(a,\lambda^*(k)\big).$$

Applying 1.1 we see that ε_q is continuous. Further, it is easy to check that ε_q is natural with respect to morphisms $\theta\colon[q']\to[q]$ in Δ^*.

Define a lifting function $\lambda_k^K\colon \Gamma_{i_k}(p^K)\to\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K,E)$, as follows. Let $(f,F)\in\Gamma_{i_k}(p^K)\subseteq\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta^k[n]\times K,E)\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K,B)$ and suppose $(\lambda,k)\in\Delta[n]_q\times K_q$, then

$$\lambda_k^K(f,F)(\lambda,k)=\lambda_k^q\big(\varepsilon_q(k,f),\varepsilon_q(k,F)\big)(\lambda,i_q)$$

where λ_k^q is the lifting function of Proposition 4.4, and $i_q\in\Delta[q]_q$ is the fundamental simplex. We must check the following:

(i) λ_k^K is well defined; that is, for each $(f,F)\in\Gamma_{i_k}(p^K)$, $\lambda_k^K(f,F)$ is actually a morphism in $\mathscr{C}^\Delta$.

(ii) λ_k^K is continuous.

(iii) If $\pi_k^K\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K,E)\to\Gamma_{i_k}(p^K)$ is the natural map, then $\pi_k^K\lambda_k^K=1$, so that λ_k^K is indeed a lifting function for i_k with respect to p^K.

(i) $\lambda_k^K(f,F)_q$ is continuous, since it is the composite of the following continuous maps:

$$\begin{aligned} \Delta[n]_q\times K_q &\xrightarrow{\alpha_q(f,F)} \Delta[n]_q\times K_q\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta^k[n]\times K,E)\times K_q\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K,B)\\ &\xrightarrow{1\times\varepsilon_q\times\varepsilon_q} \Delta[n]_q\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta^k[n]\times\Delta[q],E)\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times\Delta[q],B)\\ &\qquad\qquad\supset\ \Delta[n]_q\times\Gamma_{i_k}(p^{\Delta[q]})\xrightarrow{1\times\lambda_k^q}\\ &\Delta[n]_q\times\mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times\Delta[q],E)\xrightarrow{\varepsilon}E_q \end{aligned}$$

where $\alpha_q(f,F)$ is the map, $\alpha_q(f,F)(\lambda,k)=(\lambda,k,f,k,F)$, and ε is evaluation at $i_q\in\Delta[q]_q$.

It remains to show that $\lambda_k^K(f,F)$ is a simplicial map. Suppose $\theta\colon[q']\to[q]$ is a morphism in Δ^* and that $(\lambda,k)\in\Delta[n]_q\times K_q$, then:

$$\begin{aligned} [\lambda_k^K(f,F)\theta^*](\lambda,k) &= \lambda_k^K(f,F)(\lambda\theta,\theta^*(k))\\ &= \lambda_k^{q'}\big(\varepsilon_{q'}(\theta^*(k),f),\varepsilon_{q'}(\theta^*(k),F)\big)(\lambda\theta,i_{q'})\\ &= \lambda_k^{q'}\big(\theta^*\varepsilon_q(k,f),\theta^*\varepsilon_q(k,F)\big)(\lambda\theta,i_{q'}) \end{aligned}$$

since the ε_q are natural with respect to morphisms θ. Now, by the naturality statement of Proposition 4.4, we have:

$$\begin{aligned} \lambda_k^{q'}\big(\theta^*\varepsilon_q(k,f),\theta^*\varepsilon_q(k,F)\big)(\lambda\theta,i_{q'}) &= [\theta^*\lambda_k^q(\varepsilon_q(k,f),\varepsilon_q(k,F))](\lambda\theta,i_{q'})\\ &= \lambda_k^q\big(\varepsilon_q(k,f),\varepsilon_q(k,F)\big)(\lambda\theta,\theta)\\ &= \theta^*[\lambda_k^q\big(\varepsilon_q(k,f),\varepsilon_q(k,F)\big)(\lambda,i_q)] \end{aligned}$$

since $\lambda_k^q(\varepsilon_q(k,f),\varepsilon_q(k,F))$ is a simplicial map. Thus, combining the above, we have that $\lambda_k^K(f,F)\theta^* = \theta^*\lambda_k^K(f,F)$, showing that $\lambda_k^K(f,F)$ is a morphism in $\mathscr{C}^\Delta$.

(ii) That λ_k^K is continuous follows easily by applying Proposition 1.1 and using the fact that λ_k^q and ε_q are continuous.

(iii) Let $(f,F)\in\Gamma_{i_k}(p^K)$. Then, $\pi_k^K\lambda_k^K(f,F) = \big((i_k)^*(\lambda_k^K(f,F)),\, p_*(\lambda_k^K(f,F))\big)$ where $(i_k)^*\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K, E)\to \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta^k[n]\times K, E)$ and

$$p_*\colon \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K, E)\to \mathrm{Hom}_{\mathscr{C}^\Delta}(\Delta[n]\times K, B)$$

are the maps induced by $i_k\colon \Delta^k[n]\to\Delta[n]$ and $p\colon E\to B$, respectively.

Now, for $(\lambda,k)\in\Delta^k[n]_q\times K_q$, we have:

$$(i_k)^*(\lambda_k^K(f,F))(\lambda,k) = \lambda_k^q(\varepsilon_q(k,f),\varepsilon_q(k,F))(\lambda,i_q) = \varepsilon_q(k,f)(\lambda,i_q)$$

since λ_k^q is a lifting function for $i_k\times 1\colon \Delta^k[n]\times\Delta[q]\to\Delta[n]\times\Delta[q]$ with respect to p. But, by definition of ε_q, $\varepsilon_q(k,f)(\lambda,i_q) = f(\lambda,k)$. Hence, $(i_k)^*(\lambda_k^K(f,F)) = f$.

Again, we have

$$p_*\lambda_k^K(f,F)(\lambda,k) = p\big(\lambda_k^q(\varepsilon_q(k,f),\varepsilon_q(k,F))(\lambda,i_q)\big) = \varepsilon_q(k,F)(\lambda$$

since λ_k^q is a lifting function. Hence. $p_*\lambda_k^K(f,F) = F$. This proves (iii) o completes the proof of Theorem 4.1.

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