Theorem 12.1. Let $p : E \to S^n$ and $q : E' \to S^n$ be cell-like mappings of compact ANR's onto $S^n$. Then for each $\varepsilon > 0$ there exist mappings $h : E \to E'$ and $g : E' \to E$ such that $d(qh, p) < \varepsilon$, $d(pg, q) < \varepsilon$ and the composites $hg$ and $gh$ are homotopic to the identity.

Theorem 12.1 follows also from [1] and [15] in the case when $E = E' = S^n$, $n \neq 4$.

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The Bing-Borsuk conjecture is stronger than the Poincaré conjecture

by

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Abstract. It is shown that the existence of a fake 3-cell implies the existence of a 3-dimensional homogeneous compact ANR-space which is not a manifold.

We say that the space $X$ is homogeneous, if for every pair of points $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. We are concerned with the following conjecture:

CONJECTURE 1 (Bing, Borsuk [H]). Every $n$-dimensional homogeneous compact ANR-space is an $n$-dimensional manifold.

In dimensions 1 and 2 this conjecture was proved by Bing and Borsuk in [4]. Here we prove that in dimension 3 Conjecture 1 is stronger than the Poincaré conjecture.

CONJECTURE 2 (Poincaré). Every homotopy 3-sphere is homeomorphic to a 3-sphere.

By a homotopy 3-sphere we mean a closed 3-dimensional manifold which has a homotopy type of 3-sphere. We shall use the term fake 3-cell for a compact contractible 3-manifold which is not homeomorphic to a 3-cell. It is known ([6], p. 26) that (2) is equivalent to the statement that there are no fake 3-cells. Our main goal may be formulated as follows:

THEOREM 3. If there exists a fake 3-cell $F$, then there exists a 3-dimensional homogeneous compact ANR-space $K$ which is not a manifold.

The proof of Theorem 3 consists of several parts: first we shall construct the space $K$ (assuming the existence of the fake 3-cell), then we shall prove that $K$ is an ANR, that $K$ is homogeneous, that $K$ is not a manifold, and finally that dim $K = 3$. All the time we shall assume the existence of a fixed fake 3-cell $F$ with a given triangulation (by [3] $F$ can be triangulated) and with a fixed orientation. Moreover, we can assume that there exists a homotopy 3-sphere $H$ such that $F$ is obtained from $H$ by removing from it a single open 3-simplex, in particular that the boundary $\partial F$ is equal to the boundary of a 3-simplex [see [6], p. 26].

$\ast$
I. The construction of $K$. We shall construct a sequence of 3-manifolds $K_i$ and maps $a_i$: $K_i \to K_{i+1}$ such that $K = \lim \{K_i, a_{ij}\}$. We define $K_i$ inductively. For $K_0$, we take the boundary of a 4-simplex, and we fix it on an orientation. Let us suppose we have defined $K_{i-1}$ as a manifold with a given triangulation and a fixed orientation. Then for each closed 3-simplex $\sigma$ of $K_{i-1}$, we perform the following construction: we take the second barycentric subdivision of $\sigma$, we choose one of it is closed 3-simplices $\sigma'$ such that $\sigma' \cap \text{Int} \sigma$ and we remove the interior of $\sigma'$. Now we take a copy $F_{\sigma}$ of a fake 3-cell $F$ and we construct the space $(\sigma' \cap \text{Int}\sigma) \cup F_{\sigma}$ where $F_{\sigma}$ and $\sigma'$ is a simplicial homeomorphism which reverses the orientation (both $\partial \sigma' \cap K_{i-1}$ and $\partial F_{\sigma}$ have a triangulation of a boundary of a 3-simplex, and have an orientation induced by the orientations of $K_{i-1}$ and $F_{\sigma}$, respectively). Performing this construction for all 3-simplices of $K_{i-1}$, we get a new manifold $K_i$ with an orientation determined by the orientation of $K_{i-1}$ (i.e. the orientations induced on $\sigma' \cap \text{Int} \sigma$ by the orientations of $K_i$ and $K_{i-1}$ are equal for each $\sigma$). The triangulation of $K_i$ is determined by the second barycentric subdivision on each set of the form $\sigma' \cap \text{Int} \sigma$, and it is a fixed triangulation of $F$ on each set $F_{\sigma}$. $K_i$ is in fact homeomorphic to a connected sum $K_{i-1} \# H_i \# H_3 \# \ldots \# H_n$, where each $H_i$ is a copy of the homotopy 3-sphere $H$ (see [6], p. 24). We define $a_{i-1,i}: K_i \to K_{i-1}$ as follows: for each 3-simplex $\sigma$ of $K_{i-1}$ we take $a_{i-1,i}(\sigma' \cap \text{Int} \sigma) = \text{Id} \cap \text{Int} \sigma$ and on each $F_{\sigma}$ we let $a_{i-1,i}(F_{\sigma}) = \sigma'$ be any simplicial map which restricted to $\partial \sigma = \partial F_{\sigma}$, is equal to the identity map. Finally, we take

$$a_i = a_{i+1,i} \circ a_{i+2,i+1} \circ \ldots \circ a_{i+1,i-1}$$

for $j < i$. By $a_i: K \to K_i$, we shall denote the natural projection of $K = \lim \{K_i, a_{ij}\}$ into $K_i$.

II. Proof that $K \in \text{ANR}$. Let $K_i$ be as in I. Since $\dim K_i = 3$, we can consider $K_i$ as a polyhedron lying in the Euclidean space $E^3$. Let $\{e_1, e_2, \ldots, e_n\}$ be the family of 3-simplices of $K_i$, and let $\{d_1, d_2, \ldots, d_n\}$ be the family of 7-simplices in $E^3$ such that for $1 \leq j \neq k \leq n$, $d_j$ contains $e_j$ as a 3-dimensional face and for $1 \leq j \leq n$. Moreover, we can require that $\dim d_j = \dim e_j$. We put $B_i = d_1 \cup d_2 \cup \ldots \cup d_n$. Of course $B_i \subset K_i$. We notice that $K_{i+1}$ can be realized as a subpolyhedron of $B_i$. In fact, each fake 3-cell $F_{\sigma}$ attached in the construction of $K_{i+1}$ instead of the removed simplex $\sigma'$, can be realized as a subpolyhedron of $\partial \sigma'$ such that $\partial F_{\sigma} = \partial \sigma' \cap F_{\sigma}$, and that each simplex of $F_{\sigma}$ has diameter smaller than $\frac{1}{3} \dim \partial F_{\sigma}$. Moreover, we can claim that $K_{i+1} \supset B_i = B_i$. So we can assume that $B_i \supset B_2 \supset B_3 \supset \ldots$. It is easy to check that $\cap B_i$ is homeomorphic to $K$. For each $i$, polyhedron $K_i$ and $K_{i+1}$ and consequently $B_i$ and $B_{i+1}$ have the same homotopy type, so there exists a retraction $r_i: B_i \to B_i$ (see [10], p. 39). Moreover, we can claim that for each $e_j = B_i$, $r(|e_j|) = e_j$. The last condition implies that $\{r_1, r_2, \ldots, r_i\}$ is a sequence of maps converging to the retraction $r: B_i \to \cap B_i$, which proves that $K \in \text{ANR}$.

III. Some auxiliary remarks. Let $\mathcal{F}$ be the family of 3-cells contained in the interior of a given 3-manifold $M$. By $S(\mathcal{F})$ we shall denote the sum of all interiors of 3-cells $Z \in \mathcal{F}$. In all the cases where we say that $M$ is orientable we assume that we have a fixed orientation on $M$ and the induced orientations on $\partial M$, and on $Z$ and $\partial Z$ for every $Z \in \mathcal{F}$.

We shall say that a family $\mathcal{F}$ of 3-cells in the interior of a given 3-manifold $M$ is good if: 1) for every $Z_1, Z_2 \in \mathcal{F}$, $Z_1 \neq Z_2$, we have $Z_1 \cap Z_2 = \emptyset$, 2) each $Z \in \mathcal{F}$ is a tame 3-cell in $M$, i.e. $(M, Z)$ is homeomorphic to a polyhedral pair, 3) $\mathcal{F}$ is a null-family in $M$, i.e. for every $\varepsilon > 0$ the set $Z \in \mathcal{F}$: $\text{diam} Z > \varepsilon$ is finite, 4) $S(\mathcal{F})$ is dense in $M$. The following proof of the lemma is due to H. Torunczyk:

**Lemma A.** Let $M$ and $N$ be orientable 3-manifolds and let $h: M \to N$ be an orientation-preserving homeomorphism, let $\mathcal{F}$ and $\mathcal{G}$ be two good families of 3-cells contained in the interior of $M$ and $N$ respectively and, for every $(Y, Z) \in \mathcal{F} \times \mathcal{G}$, let $\varphi_Y^Z: \partial Z \to \partial Z$ be an orientation-preserving homeomorphism. Then there exists a bijective function $\psi: \mathcal{F} \to \mathcal{G}$ and a homeomorphism $h'$: $M \setminus \text{S}(\mathcal{F}) \to N \setminus \text{S}(\mathcal{G})$ such that $h' \circ |M| = h \circ |M|$, and $h' \circ \partial Z = \psi^Z \circ \partial Z$ for every $Z \in \mathcal{F}$.

Proof. Without loss of generality we can assume that $N = M$, $h = |M|$, and $\text{diam} M \leq 1$. Each $\psi^Z$ can be extended to a homeomorphism $\psi^Z_Y: Y \to Z$ for $Z \in \mathcal{G}$ and $Y \in \mathcal{F}$. We set $\psi$ by $\psi = \psi^Z_Y: Y \in \mathcal{F}, Z \in \mathcal{G}$, $H(M)$ be the set of all homeomorphisms of $M$ which are an identity on $\partial M$,

$$\mathcal{F} = \{Z \in \mathcal{F}: \text{diam} Z > 2^{-i} \}, \mathcal{G} = \{Y \in \mathcal{G}: \text{diam} Y > 2^{-i} \}$$

and, for any $f \in H(M)$ and any family $\mathcal{F}$ of subsets of $M$, $f(\mathcal{F}) = \{f(T) : T \in \mathcal{F}\}$. We shall construct inductively homeomorphisms $f_n, g_n \in H(M), n = 1, 2, \ldots$ such that the following conditions are satisfied:

(a) if $Z \in \mathcal{F}$, then there is a $Z \in \mathcal{F}$ such that $f_n(Z) = g_n(Z)$ and $g_n(Z) = f_n(Z)$ for $Z \in \mathcal{F}$, then there is a $Z \in \mathcal{F}$ such that $f_n(Z) = g_n(Z)$ and $g_n(Z) = f_n(Z)$,

(b) $\text{diam} f_n(Z) < 2^{-n}$ for every $Z \in \mathcal{F}$,

(c) $\text{diam} g_n(Z) < 2^{-n}$ for every $Z \in \mathcal{F}$,

(d) $|f_n(Z) - f_n(Y)| = |g_n(Z) - g_n(Y)|$ for every $Y \in \mathcal{F}$,

(e) $|g_n(Z) - g_n(Y)| = |f_n(Z) - f_n(Y)|$ for every $Y \in \mathcal{F}$,

(f) $\text{dist}(f_n(Z), f_n(Y)) < 2^{-n-i}$, $\text{dist}(g_n(Z), g_n(Y)) < 2^{-n-i}$,

(g) $\text{dist}(g_n(Z), g_n(Y)) < 2^{-n-i}$, $\text{dist}(f_n(Z), f_n(Y)) < 2^{-n-i}$.

Then $f = \lim f_n$ and $g = \lim g_n$ are in $H(M)$ (see [2], p. 121). Let $h = g^{-1}f$ is a homeomorphism which satifies $h' \circ |M| = |M| \circ h'$ for every $Z \in \mathcal{F}$. So we can take $h' = h' \circ |M| \circ h'$ for every $Z \in \mathcal{F}$. In the construction of $f_n$ and $g_n$ we shall need the following

**Sublemma.** Let $\mathcal{F}$ be a good family of 3-cells in $M$, let $\mathcal{F}$ be a finite subfamily of $\mathcal{F}$, and let $\psi \in H(M)$. Then for any $\varepsilon > 0$ there exists a $\psi \in H(M)$ such that:

$$|V(T) - \psi(T)| < \varepsilon$$

for any $T \in \mathcal{F}$.
that the 2-skeleton of $\tau$ misses all the elements of $\mathcal{F}$. For every 3-simplex $\sigma$ of $\tau$ it follows from the proof of Theorem 2 in [9] that there is a homeomorphism $h_{\tau}: \sigma \to \pi(\sigma)$ with $h_{\tau}(x) = \pi(x)$ for each $x \in \sigma$ where $\pi: M \to M/G$ is a projection. The $h_{\tau}$'s glued together yield the required homeomorphism $h: M \to M/G$.

Let $K_i$ and $F_i \subset K_i$ be the sets described in Section I for a fixed natural $i$ and for a fixed 3-simplex $\sigma$ of $K_{i-1}$. We denote $(F, \mathcal{F}) = (a_{i-1}^w(F), a_{i-1}^w(\mathcal{F}))$, where $a_{i-1}^w: \text{dim}(K_i, a_{i-1}^w) \to K_i$ is the natural projection. It is easy to see that, if we take any other natural number $i'$ and simplex $\sigma'$, we shall get a pair $(F', \mathcal{F}')$ homeomorphic to $(F, \mathcal{F})$, $a_{i-1}^w(F') = id_{\mathcal{F}}$, for $\sigma \not\subseteq i$, so $a_{i-1}^w(\partial F) \supseteq \partial F$ is a homeomorphism of 2-spheres which induces an orientation on $\partial F$.

For every orientable 3-manifold $M$ we let $M^*$ to be a family of metric spaces such that $X \in M^*$ iff there exists a good family $\mathcal{F}$ of 3-cells in $M$ such that $X = M(S(\mathcal{F}) \cup \cup F \cup 1)$ for every $Z \in \mathcal{F}$ there is a homeomorphism $g_Z: F \to F_Z$; $F_\emptyset = \emptyset$ for $Z \not\subseteq Z_0$, and $\text{dim}(\text{dim}(\mathcal{F})) = \emptyset = g_Z(\mathcal{F})$ where $Z \in \mathcal{F}$, and moreover $a_{i-1}^w(F): \partial F \to \partial Z$ is an orientation-reversing homeomorphism; 3) $X$ is equipped with a metric in which for every $s > 0$ the set $Z \in \mathcal{F}$: diam $F_Z > s$ is finite. Of course each $X \in M^*$ contains the oriented boundary $\partial M$. We shall denote it by $\partial M$ or $\partial M$ alternatively.

LEMMA 6. If $M$ and $N$ are orientable 3-manifolds, $h: M \to N$ is an orientation-preserving homeomorphism, and $X_1, X_2 \in M^*$ and $X_3 \in N^*$, then there is a homeomorphism $h_i: X_1 \to X_3$ such that $h_i(\partial M) = h(\partial M)$.

Proof. We have two good families of 3-cells: $\mathcal{F} \in M$ and $\mathcal{F} \in N$ such that $X_1 = M(S(\mathcal{F}) \cup \cup F \cup 1)$ and $X_2 = N(S(\mathcal{F}) \cup \cup F \cup 1$ and two families of homeomorphisms $g_Z: F \to F_Z$ and $g_F: F \to F_Z$ where $Z \in \mathcal{F}$ and we put $s_F(T) = (g_F(T), 0)$ for each $x \in M(S(\mathcal{F}) \cup \cup F \cup 1$ and we get the required homeomorphism $h_i: X_1 \to X_3$; 4) $X \in M(S(\mathcal{F}) \cup \cup F \cup 1$ and we get the required homeomorphism $h_i: X_1 \to X_3$.

LEMMA 7. Let $P$ be a polyhedron of $K_0$ (the fixed triangulation of $K_0$) which is a 3-manifold. Then $a_i^w(\mathcal{F}) \in M^*$ in particular $K_0 \in K_0$ for every natural $n$.

Proof. It follows from the construction in Section I that $K_0 = \cup_{i=1}^{n} (K_i \subset K_{i+1} = K_0 = \cup_{i=1}^{n} (K_i \subset K_{i+1})$ where $K_i$ is some finite family of 3-cells in $K_i$. If we put $F_i = \cup_{i=1}^{n} (K_i \subset K_{i+1}$ and $F = (Z \in \mathcal{F} \subset Z \in \mathcal{F} \subset Z \in \mathcal{F}$), then $F_i$ is a good family of 3-cells in $P$. Since $a_i^w(F_3) = (K_0, F_3)$ for $i=1$, the projection $a_i^w(F_3) = (P(S(\mathcal{F})))$ and $P(S(\mathcal{F})))$; in the sequel these sets will be identified, i.e. $a_i^w(F) = P(S(\mathcal{F})) \cup \cup F \cup 1$ and $F_3$ is a copy of the fake 3-cell $F$ attached in the 1st step of construction I instead of the removed interior of $Z \in \mathcal{F}$. Of course $F_3$ is a copy of $F$, $\partial F_3 = \partial Z$ and this identification reverses the orientation for each $Z \in \mathcal{F}$. It is easy to check that in every metric coin-
The Bing-Borsuk conjecture is stronger than the Poincaré conjecture.

Let us suppose that $K$ is a 3-manifold. By Lemma 7, $K$ is homogenous. We can consider a family $\mathcal{S}$ of 3-cells in $K$, and there exists a family of homeomorphisms $(\gamma_z: F \to F_z)$ for every $z$. For every $z$ the set $Z \subset K$, diam$(F_z)$ is finite. This means that there exists a $Z \subset K$ and a 3-cell $Q$ contained in $K = K_\mathcal{S}(Z) \cup F_z$ such that $F_z = \text{int}Q$. It is easy to see that the quotient space $K_\mathcal{S}(Z)$ is homeomorphic to $K$, and we denote $\pi_0: K \to K_\mathcal{S}(Z)$ by the quotient map. We consider a family $\mathcal{S}$ of 3-cells in $K_\mathcal{S}(Z)$ defined as follows: $Z \in \mathcal{S}$ if there exists a $Z \subset K$ such that $\pi_0(Z) = Y$. It is easy to check that $\mathcal{S}$ is a good family of 3-cells in $K_\mathcal{S}(Z)$. Now we consider the space $K/F_z = (K_\mathcal{S}(Z) \cup F_z)/F_z$, and the corresponding quotient map $\pi$. We have $K/F_z = \pi_0(K_\mathcal{S}(Z))/F_z$, and $\pi_0(F_z) = (F_z)/F_z$. We identify $\pi_0(K_\mathcal{S}(Z))/F_z$ and $\pi_0(K_\mathcal{S}(Z))/F_z$. Then the family $(\pi_0: F \to \pi_0(F_z))$ is a family of homeomorphisms satisfying the requirements of the definition of the family $(K_\mathcal{S}(Z), F_z)$. So $K/F_z \in (K_\mathcal{S}(Z), F_z)$. By Lemma 6 and from the fact that $K_\mathcal{S}(Z)$ and $K$ are homeomorphic we deduce that $K/F_z$ and $K$ are homeomorphic. Since $F_z$ is contained in a 3-cell $Q \subset K$, we can easily deduce from [5], Theorems 3 and 5, that $F_z$ is a cellular set in $K$.

We shall prove that $F_z$ is cellular for every $Z \subset K$. For this purpose we take any $Z \subset K$, and we consider the space $K_\mathcal{S}(Z)$, and the space $K_\mathcal{S}(Z)$ is a good family of 3-cells in $K_\mathcal{S}(Z)$, so we have $K/F_z \in (K_\mathcal{S}(Z), F_z)$, and the same argument $K/F_z \in (K_\mathcal{S}(Z), F_z)$. Of course $(\gamma_z: F \to F_z)$ is a homeomorphism preserving the orientation (note that $\partial F_z = F_z = K$. There exists a homeomorphism $h: K_\mathcal{S}(Z) \to K_\mathcal{S}(Z)$, such that $hF = (\gamma_z: F \to F_z)$, and so by Lemma 6 there exists a homeomorphism $h: K_\mathcal{S}(Z) \to K_\mathcal{S}(Z)$, such that $hF = (\gamma_z: F \to F_z)$. It can be extended to a homeomorphism $h: K \to K$ if we put $h(x) = (\gamma_z: F \to F_z)(x)$ for $x \in F_z$. It is easy to see that $h(F_z) = F_z$. This implies that $F_z$ is cellular in $K$ for any $Z \subset K$.
whose non-degenerate elements are the sets $F_Z$ for all $Z \in \mathcal{Z}_n$. The quotient space $K/G_n$ is homeomorphic to $K_n/G_n$, where $G_n$ is the decomposition of $K_n$ of its non-degenerate elements from the good family $\mathcal{Z}_n$. By Lemma 4, $K_n/G_n$, and so $K/G_n$ is homeomorphic to the 3-manifold $K_n$. Since we have supposed that $K$ is a 3-manifold and $G_n$ is shown to be a set of elements which are cellular in $K$, it follows from the theorem of Armentrout (see [1], p. 66, Theorem 2) that $K$ is homeomorphic to $K_n$ for every natural $n$. This implies that $K_n$ is homeomorphic to $K_n$. $K_n$ is a 3-sphere and $K_n$ contains the fake 3-cell $F_n$ which is impossible (see [5], Theorem 5). This proves that $K$ is not a 3-manifold.

VI. Proof that $\dim K = 3$. $\dim K \leq 3$, because $\dim K_n = 3$ for every $n$ and $K = \lim K_n$, $\leq_{K_n}$ and $\dim K \geq 3$ because $H^3(K, Z) \neq 0$ (see [7], p. 152).

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A theorem on the weak topology of $C(X)$ for compact scattered $X$

by

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Abstract. We prove that if a function space $C(X)$, where $X$ is a compact scattered space, is $K$-analytic under the weak topology, then $C(X)$ is a WCG space, i.e. $X$ is an Eberlein compact. This result is related to a recent author's example of a non-WCG space $C(X)$ with $X$ compact scattered, which is Lindelöf in the weak topology, a recent example of Talagrand of a non-WCG space $C(X)$, which is $K$-analytic in the weak topology, and the recent theorem of Talagrand that every WCG Banach space is $K$-analytic in the weak topology.

1. Introduction. It was an old problem of Corson [6] whether the WCG Banach spaces (the terminology will be explained in the next section) are exactly the Banach spaces which are Lindelöf in their weak topology. An example of a Banach space which is Lindelöf in the weak topology but not WCG was given by Rosenthal [12] and, on the other hand, Talagrand [15] proved that a WCG Banach space is $\mathcal{K}$-analytic (which is much more than the Lindelöf property) in the weak topology. It was still open after these works if the Corson's problem has an affirmative solution in the class of function spaces $C(X)$. Problem 6, 6', [4], Problem 7. Recently, the author [11] and independently, about the same time, Talagrand [16] constructed the appropriate counterexamples. The content of these examples is however quite different. The function space $C(X)$ in the author's example is not $\mathcal{K}$-analytic in the weak topology, while the compact $X$ is scattered, whereas the Talagrand's space $C(X)$ is $\mathcal{K}$-analytic in the weak topology, but the compact $X$ is not scattered.

The aim of this paper is to show that if a function space $C(X)$ is $\mathcal{K}$-analytic in the weak, or pointwise topology and the compact $X$ is scattered, then $C(X)$ is a WCG-space, or equivalently — $X$ is an Eberlein compact.

It is worth while to mention that one can exploit the Talagrand's example to show (1) that in fact there is no topological property which is invariant under continuous mappings, closed hereditarily and characterizes the Eberlein compacts as the compacts whose function space in the weak (or pointwise) topology has this

(1) By means, for example, of the reasonings given in [11], the proof of Lemma 1; cf. also [2].