THEOREM 12.1. *Let $p\colon E \to S^n$ and $q\colon E' \to S^n$ be cell-like mappings of compact* ANR*'s onto S^n. Then for each $\varepsilon>0$ there exist mappings $h\colon E \to E'$ and $g\colon E' \to E$ such that $d(qh, p)<\varepsilon$, $d(pg, q)<\varepsilon$ and the composites hg and gh are homotopic to the identity.*

Theorem 12.1 follows also from [1] and [15] in the case when $E = E' = S^n$, $n \neq 4$.

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The Bing–Borsuk conjecture is stronger than the Poincaré conjecture

by

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Abstract. It is shown that the existence of a fake 3-cell implies the existence of a 3-dimensional homogeneous compact ANR-space which is not a manifold.

We say that the space X is *homogeneous*, if for every pair of points $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$. We are concerned with the following conjecture:

CONJECTURE 1 (Bing, Borsuk [4]). *Every n-dimensional homogeneous compact* ANR*-space is an n-dimensional manifold.*

In dimensions 1 and 2 this conjecture was proved by Bing and Borsuk in [4]. Here we prove that in dimension 3 Conjecture 1 is stronger than the Poincaré conjecture.

CONJECTURE 2 (Poincaré). *Every homotopy 3-sphere is homeomorphic to a 3-sphere.*

By a homotopy 3-sphere we mean a closed 3-dimensional manifold which has a homotopy type of 3-sphere. We shall use the term fake 3-cell for a compact contractible 3-manifold which is not homeomorphic to a 3-cell. It is known ([6], p. 26) that (2) is equivalent to the statement that there are no fake 3-cells. Our main goal may be formulated as follows:

THEOREM 3. *If there exists a fake 3-cell F, then there exists a 3-dimensional homogeneous compact* ANR*-space K which is not a manifold.*

The proof of Theorem 3 consists of several parts: first we shall construct the space K (assuming the existence of the fake 3-cell), then we shall prove that $K \in$ ANR, that K is homogeneous, that K is not a manifold, and finally that $\dim K = 3$. All the time we shall assume the existence of a fixed fake 3-cell F with a given triangulation (by [3] F can be triangulated) and with a fixed orientation. Moreover, we can assume that there exists a homotopy 3-sphere H such that F is obtained from H by removing from it a single open 3-simplex, in particular that the boundary ∂F is equal to the boundary of a 3-simplex (see [6], p. 26).

I. The construction of K. We shall construct a sequence of 3-manifolds K_i and maps $\alpha_{ij}\colon K_i \to K_j$ such that $K = \varprojlim\{K_i, \alpha_{ij}\}$. We define K_i inductively. For K_1 we take the boundary of a 4-simplex, and we fix on it an orientation. Let us suppose we have defined K_{i-1} as a manifold with a given triangulation and a fixed orientation. Then for each closed 3-simplex σ of K_{i-1} we perform the following construction: we take the second barycentric subdivision of σ, we choose one of it is closed 3-simplexes σ' such that $\sigma' \subset \operatorname{Int}\sigma$, and we remove the interior of σ'. Now we take a copy F_σ of a fake 3-cell F and we construct the space $(\sigma \setminus \operatorname{Int}\sigma') \cup_{f_\sigma} F_\sigma$ where $f_\sigma\colon \partial F_\sigma \to \partial\sigma'$ is a simplicial homeomorphism which reverses the orientation (both $\partial\sigma' \subset K_{i-1}$ and ∂F_σ have a triangulation of a boundary of a 3-simplex, and have an orientation induced by the orientations of K_{i-1} and F_σ, respectively). Performing this construction for all 3-simplexes of K_{i-1}, we get a new manifold K_i, with an orientation determined by the orientation of K_{i-1} (i.e. the orientations induced on $\sigma \setminus \operatorname{Int}\sigma'$ by the orientations of K_i and K_{i-1} are equal for each σ). The triangulation of K_i is determined by the second barycentric subdivision on each set of the form $\sigma \setminus \operatorname{Int}\sigma'$, and it is a fixed triangulation of F on each set F_σ. K_i is in fact homeomorphic to a connected sum $K_{i-1} \# H_1 \# H_2 \# \ldots \# H_{n_i}$ where each H_i is a copy of the homotopy 3-sphere H (see [6], p. 24]. We define $\alpha_{i,i-1}\colon K_i \to K_{i-1}$ as follows: for each 3-simplex σ of K_{i-1} we take $\alpha_{i,i-1}|(\sigma\setminus\operatorname{Int}\sigma') = \operatorname{id}\sigma\setminus\operatorname{Int}\sigma'$ and on each $F_\sigma \subset K_i$ we let $\alpha_{i,i-1}|F_\sigma\colon F_\sigma \to \sigma'$ be any simplicial map which, restricted to $\partial\sigma' = \partial F_\sigma$, is equal to the identity map. Finally, we take

$$\alpha_{ij} = \alpha_{j+1,j} \cdot \alpha_{j+2,j+1} \cdots \alpha_{i,i-1}$$

for $j<i$. By $\alpha_i\colon K \to K_i$ we shall denote the natural projection of $K = \varprojlim\{K_i, \alpha_{ij}\}$ into K_i.

II. Proof that $K \in \mathrm{ANR}$. Let K_i be as in I. Since $\dim K_i = 3$ we can consider K_i as a polyhedron lying in the Euclidean space E^7. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_{n_i}\}$ be the family of 3-simplexes of K_i, and let $\{\bar\sigma_1, \bar\sigma_2, \ldots, \bar\sigma_{n_i}\}$ be the family of 7-simplexes in E^7 such that for $1 \leqslant j \leqslant k \leqslant n_i$ $\bar\sigma_j$ contains σ_j as a 3-dimensional face and for $j \neq k$ we have $\bar\sigma_j \cap \bar\sigma_k = \sigma_j \cap \sigma_k$. Moreover, we can require that $\operatorname{diam}\sigma_j = \operatorname{diam}\bar\sigma_j$ for $1 \leqslant j \leqslant n_i$. We put $B_i = \bar\sigma_1 \cup \bar\sigma_2 \ldots \cup \bar\sigma_{n_i}$. Of course $B_i \supset K_i$. Let us notice that K_{i+1} can be realized as a subpolyhedron of B_i. In fact, each fake 3-cell F_{σ_j} attached in the construction of K_{i+1} instead of the removed simplex σ_j', can be realized as a subpolyhedron of $\bar\sigma_j$ such that $\partial F_{\sigma_j} = \sigma_j \cap F_{\sigma_j}$, and that each simplex of F_{σ_j} has diameter smaller than $\frac{1}{2}\operatorname{diam}\sigma_j$. Moreover, we can claim that $K_{i+1} \subset B_{i+1} \subset B_i$. So we can assume that $B_i \supset B_2 \supset B_3 \supset \ldots$ It is easy to check that $\bigcap_{i=1}^{\infty} B_i$ is homeomorphic to K. For each i polyhedra K_i and K_{i+1} and consequently B_i and B_{i+1} have the same homotopy type, so there exists a retraction $r_i\colon B_i \to B_{i+1}$ (see [10], p. 39). Moreover, we can claim that for each $\bar\sigma_j \subset B_i$, $r_i(\bar\sigma_j) \subset \bar\sigma_j$. The last condition implies that $\{r_i \cdot r_{i-1} \cdot \ldots \cdot r_1\}$ is a sequence of maps convergent to the retraction $r\colon B_1 \to \bigcap_{i=1}^{\infty} B_i$, which proves that $K \in \mathrm{ANR}$.

III. Some auxiliary remarks. Let $\mathscr{Z}$ be the family of 3-cells contained in the interior of a given 3-manifold M. By $S(\mathscr{Z})$ we shall denote the sum of all interiors of 3-cells $Z \in \mathscr{Z}$. In all the cases where we say that M is orientable we assume that we have a fixed orientation on M and the induced orientations on ∂M, and on Z and ∂Z for every $Z \in \mathscr{Z}$.

We shall say that a family $\mathscr{Z}$ of 3-cells in the interior of a given 3-manifold M is *good* if: 1) for every $Z_1, Z_2 \in \mathscr{Z}$, $Z_1 \neq Z_2$ we have $Z_1 \cap Z_2 = \emptyset$, 2) each $Z \in \mathscr{Z}$ is a tame 3-cell in M, i.e. (M, Z) is homeomorphic to a polyhedral pair, 3) $\mathscr{Z}$ is a null-family in M, i.e. for every $\varepsilon > 0$ the set $\{Z \in \mathscr{Z}\colon \operatorname{diam} Z > \varepsilon\}$ is finite, 4) $S(\mathscr{Z})$ is dense in M. The following proof of the lemma is due to H. Toruńczyk:

LEMMA 4. *Let M and N be orientable 3-manifolds let $h\colon M \to N$ be an orientation-preserving homeomorphism, let $\mathscr{Y}$ and $\mathscr{Z}$ be two good families of 3-cells contained in the interior of M and N respectively and, for every $(Y, Z) \in \mathscr{Y} \times \mathscr{Z}$, let $\varphi_Y^Z\colon \partial Y \to \partial Z$ be an orientation-preserving homeomorphism. Then there exists a bijective function $p\colon \mathscr{Y} \to \mathscr{Z}$ and a homeomorphism $h'\colon M \setminus S(\mathscr{Y}) \to N \setminus S(\mathscr{Z})$ such that $h'|\partial M = h|\partial M$, and $h'|\partial Y = \varphi_Y^{p(Y)}$ for every $Y \in \mathscr{Y}$.*

Proof. Without loss of generality we can assume that $N = M$, $h = \mathrm{id}_M$, and $\operatorname{diam} M \leqslant 1$. Each φ_Y^Z can be extended to a homeomorphism $\psi_Y^Z\colon Y \to Z$ for $Y \in \mathscr{Y}$ and $Z \in \mathscr{Z}$. Let $\psi = \{\psi_Y^Z\colon Y \in \mathscr{Y}, Z \in \mathscr{Z}\}$, $H(M)$ be the set of all homeomorphisms of M which are an identity on ∂M,

$$\mathscr{Z}_n = \{Z \in \mathscr{Z}\colon \operatorname{diam} Z \geqslant 2^{-n}\},\quad \mathscr{Y}_n = \{Y \in \mathscr{Y}\colon \operatorname{diam} Y \geqslant 2^{-n}\}$$

and, for any $f \in H(M)$ and any family $\mathscr{T}$ of subsets of M, $f(\mathscr{T}) = \{f(T)\colon T \in \mathscr{T}\}$. We shall construct inductively homeomorphisms $f_n, g_n \in H(M)$, $n = 1, 2, \ldots$ such that the following conditions are satisfied:

(a_n) if $Y \in \mathscr{Y}_n$, then there is a $Z \in \mathscr{Z}$ such that $f_n(Y) = g_n(Z)$ and $g_n^{-1} f_n | Y \in \psi$,

(a_n)′ if $Z \in \mathscr{Z}_n$, then there is a $Y \in \mathscr{Y}$ such that $f_n(Y) = g_n(Z)$ and $g_n^{-1} f_n | Y \in \psi$,

(b_n) $\operatorname{diam} f_n(Y) < 2^{-n}$ for every $Y \in \mathscr{Y} \setminus (\mathscr{Y}_n \cup f_n^{-1} g_n(\mathscr{Z}_n))$,

(b_n)′ $\operatorname{diam} g_n(Z) < 2^{-n}$ for every $Z \in \mathscr{Z} \setminus (\mathscr{Z}_n \cup g_n^{-1} f_n(\mathscr{Y}_n))$,

(c_n) $f_n | Y = f_{n-1} | Y$ for every $Y \in \mathscr{Y}_{n-1} \cup f_{n-1}^{-1} g_{n-1}(\mathscr{Z}_{n-1})$,

(c_n)′ $g_n | Z = g_{n-1} | Z$ for every $Z \in \mathscr{Z}_{n-1} \cup g_{n-1}^{-1} f_{n-1}(\mathscr{Y}_{n-1})$,

(d_n) $\operatorname{dist}(f_n, f_{n-1}) \leqslant 2^{-n+2}$, $\operatorname{dist}(f_n^{-1}, f_{n-1}^{-1}) \leqslant 2^{-n+2}$,

(d_n)′ $\operatorname{dist}(g_n, g_{n-1}) \leqslant 2^{-n+3}$, $\operatorname{dist}(g_n^{-1}, g_{n-1}^{-1}) \leqslant 2^{-n+3}$.

Then $f = \lim f_n$ and $g = \lim g_n$ are in $H(M)$ (see [2], p. 121), $h'' = g^{-1} f$ is a homeomorphism such that $h''(M \setminus S(\mathscr{Y})) = M \setminus S(\mathscr{Z})$ and $h''|Y \in \psi$ for every $Y \in \mathscr{Y}$. So we can take $h' = h''|M \setminus S(\mathscr{Y})$ and $p(Y) = h''(Y)$ for every $Y \in \mathscr{Y}$.

In the construction of f_n and g_n we shall need the following

SUBLEMMA. *Let $\mathscr{T}$ be a good family of 3-cells in M, let $\mathscr{T}_1$ be a finite subfamily of $\mathscr{T}$, and let $u \in H(M)$. Then for any $\varepsilon > \delta > 0$ there exists a $v \in H(M)$ such that:*

$$v|T = u|T \quad \textit{for any } T \in \mathscr{T}_1,$$

$\operatorname{diam} v(T) < \delta$ *for any* $T \in \mathcal{T} \setminus \mathcal{T}_1$ *with* $\operatorname{diam} T < \varepsilon$ *and* $\operatorname{diam} u(T) < \varepsilon$; *and* $\operatorname{dist}(v, u) < \varepsilon$, and $\operatorname{dist}(v^{-1}, u^{-1}) < \varepsilon$.

Proof of the sublemma. For any element T of the finite family

$$\mathcal{T}_0 = \{T \in \mathcal{T} \setminus \mathcal{T}_1 \colon \operatorname{diam} T < \varepsilon \text{ and } \delta \leqslant \operatorname{diam} u(T) < \varepsilon\}$$

we find an open $\mathcal{T}$-saturated neighbourhood U_T homeomorphic to an open subset of R^3, such that $\operatorname{diam} U_T < \varepsilon$ and $\operatorname{diam} u(U_T) < \varepsilon$. Moreover, we can require that $U_{T_1} \cap U_{T_2} = \emptyset$ for any $T_1, T_2 \in \mathcal{T}_0$, $T_1 \neq T_2$, and that $U_T \cap T' = \emptyset$ for any $T \in \mathcal{T}_0$, $T' \in \mathcal{T}_1$. By Lemma 2 from [9] there is a $v' \in H(M)$ such that $v'(x) = x$ for $x \in M \setminus \bigcup_{T \in \mathcal{T}_0} U_T$ and $\operatorname{diam}(uv'(T)) < \delta$ for any $T \in \mathcal{T}$ with $T \subset \bigcup_{T \in \mathcal{T}_0} U_T$. It is easy to see that $v = uv'$ satisfies all the conditions of the sublemma.

The inductive construction: we put $f_0 = g_0 = \mathrm{id}_M$. Suppose that for some $n \geqslant 1$ f_{n-1} and g_{n-1} are already constructed. Using (b_n) and the sublemma with $\varepsilon = 2^{-n+1}$, $\delta = \min\{\operatorname{diam} f_{n-1}(Y) \colon Y \in \mathcal{Y}_n \setminus \mathcal{Y}_{n-1}\}$, $\mathcal{T} = \mathcal{Z}$, $\mathcal{T}_1 = \mathcal{Z}_{n-1} \cup g_{n-1}^{-1} f_{n-1}(\mathcal{Y}_{n-1})$ and $u = g_{n-1}$, we get a new homeomorphism $v \in H(M)$ such that: $v|Z = g_{n-1}|Z$ for $Z \in \mathcal{T}_1$, $\operatorname{diam}(v(Z)) < \delta$ for $Z \in \mathcal{Z} \setminus \mathcal{T}_1$, and $\operatorname{dist}(v, g_{n-1}) < 2^{-n+1}$, $\operatorname{dist}(v^{-1}, g_{n-1}^{-1}) < 2^{-n+1}$. Given $Y \in \mathcal{Y}_n \setminus (f_{n-1}^{-1} g_{n-1}(\mathcal{Z}_{n-1}) \cup \mathcal{Y}_{n-1})$, we can use (b_{n-1}) to get an $\mathcal{Y}$-saturated neighbourhood U_Y of Y which is homeomorphic to an open subset of R^3 and satisfies $\operatorname{diam} U_Y < 2^{-n+1}$ and $\operatorname{diam} f_{n-1}(U_Y) < 2^{-n+1}$; we require in addition that the U_Y's are pairwise disjoint and miss the elements of $\mathcal{Y}_{n-1} \cup f_{n-1}^{-1} g_{n-1}(\mathcal{Z}_{n-1})$. Note that $f_{n-1}(Y)$ is not contained in any element of $v(\mathcal{Z})$ and hence there is a $Z_Y \in \mathcal{Z} \setminus \mathcal{Z}_{n-1} \setminus g_{n-1}^{-1} f_{n-1}(Y_{n-1})$ with $v(Z_Y) \subset f_{n-1}(Y)$. Since every $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$ is a tame cell in M, it follows from Annulus, Theorem [8], that we can find a $w \in H(W)$ such that: $w(x) = x$ if x is in none of the U_Y's, $wf_{n-1}(Y) = v(Z_Y)$ and $w^{-1} v f_{n-1} | Y \in \psi$ where Y runs over $\mathcal{Y}_n \setminus \mathcal{Y}_{n-1} \setminus f_{n-1}^{-1} g_{n-1}(\mathcal{Z}_{n-1})$. Applying the sublemma again with $\varepsilon = 2^{-n+2}$, $\delta = \min\{2^{-n}, \operatorname{diam} v(Z) \colon Z \in \mathcal{Z}_n \setminus \mathcal{Z}_{n-1}\}$, $\mathcal{T} = \mathcal{Y}$, $\mathcal{T}_1 = \mathcal{Y}_n \cup f_{n-1}^{-1} g_{n-1}(\mathcal{Z}_{n-1})$ and $u = wf_{n-1}$, we get a homeomorphism $f_n \in H(M)$ such that $f_n | Y = wf_{n-1} | Y$ for $Y \in \mathcal{T}_1$, $\operatorname{diam} f_n(Y) < \delta$ for every $Y \in \mathcal{Y} \setminus \mathcal{T}_1$, and $\operatorname{dist}(f_n, wf_{n-1}) < 2^{-n+1}$ and $\operatorname{dist}(f_n^{-1}, (wf_{n-1})^{-1}) < 2^{-n+1}$. It is easy to see that, with v in place of g_n, conditions (a_n), (b_n), (c_n) and (d_n) are satisfied. Now, given $Z \in \mathcal{Z}_n \setminus \mathcal{Z}_{n-1} \setminus f_n^{-1} v(\mathcal{Y}_{n-1})$, the set $v(Z)$ is not contained in any member of $f_n(\mathcal{Y})$ and hence there is a $Y_Z \in \mathcal{Y} \setminus \mathcal{Y}_n \setminus f_n^{-1} v(\mathcal{Z}_{n-1})$ with $f_n(Y_Z) \subset v(Z)$. Using [8], the Annulus Theorem, and the sublemma again as in the construction of w and f_n above, we get $g_n \in H(M)$ such that $g_n | Z = v | Z$ if $Z \in \mathcal{Z}_{n-1} \cup v^{-1} f_n(\mathcal{Y}_n)$, and conditions $(\mathrm{a}_n)'$, $(\mathrm{b}_n)'$, $(\mathrm{c}_n)'$ and $(\mathrm{d}_n)'$ are satisfied. This completes the inductive construction and the proof of Lemma 4.

Next we shall prove

LEMMA 5. *If $\mathcal{Z}$ is a good family of 3-cells in the interior of a 3-manifold M, and G is a decomposition of M whose non-degenerate elements are exactly the elements of $\mathcal{Z}$, then M/G is homeomorphic to M.*

Proof. By Lemma 4 we can assume that there is a triangulation τ of M, such

that the 2-skeleton of τ misses all the elements of $\mathcal{Z}$. For every 3-simplex σ of τ it follows from the proof of Theorem 2 in [9] that there is a homeomorphism $h_\sigma \colon \sigma \to \pi(\sigma)$ with $h_\sigma(x) = \pi(x)$ for each $x \in \partial\sigma$ where $\pi \colon M \to M/G$ is a projection. The h_σ's glued together yield the required homeomorphism $h \colon M \to M/G$.

Let K_i and $F_\sigma \subset K_i$ be the sets described in Section I for a fixed natural i and for a fixed 3-simplex σ of K_{i-1}. We denote $(\tilde{F}, \partial\tilde{F}) = (\alpha_i^{-1}(F_\sigma), \alpha_i^{-1}(\partial F_\sigma))$, where $\alpha_i \colon \varprojlim(K_i, \alpha_{ij}) \to K_i$ is the a natural projection. It is easy to see that, if we take any other natural number i' and simplex σ', we shall get a pair $(\tilde{F}', \partial\tilde{F}')$ homeomorphic to $(\tilde{F}, \partial\tilde{F})$. $\alpha_{ij} | \partial F_\sigma = \mathrm{id}_{\partial F_\sigma}$ for $j \geqslant i$; so $\alpha_i^{-1} | \partial F_\sigma \colon \partial F_\sigma \to \partial\tilde{F}$ is a homeomorphism of 2-spheres which induces an orientation on $\partial\tilde{F}$.

For every orientable 3-manifold M we let M^* to be a family of metric spaces such that: $X \in M^*$ iff there exists a good family $\mathcal{Z}$ of 3-cells in M such that $X = M \setminus S(\mathcal{Z}) \cup \bigcup_{Z \in \mathcal{Z}} \tilde{F}_Z$ and 1) for every $Z \in \mathcal{Z}$ there is a homeomorphism $g_Z \colon \tilde{F} \to \tilde{F}_Z$; 2) $\tilde{F}_{Z_1} \cap \tilde{F}_{Z_2} = \emptyset$ for $Z_1 \neq Z_2$ and $(M \setminus S(\mathcal{Z})) \cap \tilde{F}_Z = \partial Z = g_Z(\partial\tilde{F})$ where $Z \in \mathcal{Z}$, and moreover $g_Z | \partial\tilde{F} \colon \partial\tilde{F} \to \partial Z$ is an orientation-reversing homeomorphism; 3) X is equipped with a metric in which for every $\varepsilon > 0$ the set $\{Z \in \mathcal{Z} \colon \operatorname{diam} \tilde{F}_Z > \varepsilon\}$ is finite. Of course each $X \in M^*$ contains the oriented boundary ∂M of M. We shall denote it by ∂M or ∂X alternatively.

LEMMA 6. *If M and N are orientable 3-manifolds, $h \colon M \to N$ is an orientation-preserving homeomorphism, and $X_1 \in M^*$ and $X_2 \in N^*$, then there is a homeomorphism $\bar{h} \colon X_1 \to X_2$ such that $\bar{h} | \partial M = h | \partial M$.*

Proof. We have two good families of 3-cells: $\mathcal{Y}$ in M and $\mathcal{Z}$ in N, such that $X_1 = M \setminus S(\mathcal{Y}) \cup \bigcup_{Y \in \mathcal{Y}} \tilde{F}_Z$ and $X_2 = N \setminus S(\mathcal{Z}) \cup \bigcup_{Z \in \mathcal{Z}} \tilde{F}_Z$ and two families of homeomorphisms $\{g_Y \colon \tilde{F} \to \tilde{F}_Y\}_{Y \in \mathcal{Y}}$ and $\{g_Z \colon \tilde{F} \to \tilde{F}_Z\}_{Z \in \mathcal{Z}}$. If we put $\varphi_Y^Z = (g_Z g_Y^{-1}) | \partial Y$ for $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$, we shall get by Lemma 4 a bijective function $p \colon \mathcal{Y} \to \mathcal{Z}$ and a homeomorphism $h' \colon M \setminus S(\mathcal{Y}) \to N \setminus S(\mathcal{Z})$ satisfying all the conditions of Lemma 4. Putting $\bar{h}(x) = h'(x)$ for all $x \in M \setminus S(\mathcal{Y})$ and $\bar{h}(x) = (g_{p(Y)} g_Y^{-1})(x)$ for $x \in \tilde{F}_Y$, $Y \in \mathcal{Y}$, we get the required homeomorphism.

LEMMA 7. *Let P be a subpolyhedron of K_n (in the fixed triangulation of K_n) which is a 3-manifold. Then $\alpha_n^{-1}(P) \in P^*$. In particular $K \in K_n^*$ for every natural n.*

Proof. It follows from the construction in Section I that $K_n \cap K_{n+i} = K_n \setminus S(\mathcal{Z}_{ni})$ where $\mathcal{Z}_{ni}$ is some finite family of 3-cells in K_n. If we put $\mathcal{Z} = \bigcup_{i=1}^{\infty} \mathcal{Z}_{ni}$ and $\mathcal{Z}_P = \{Z \in \mathcal{Z} \colon Z \subset P\}$, then $\mathcal{Z}_P$ is a good family of 3-cells in P. Since $\alpha_{mn} | P \setminus S(\mathcal{Z}_P) = \mathrm{id}_{P \setminus S(\mathcal{Z}_P)}$ for $m \geqslant n$, the projection α_n establishes a homeomorphism between $\alpha_n^{-1}(P \setminus S(\mathcal{Z}_P))$ and $P \setminus S(\mathcal{Z}_P)$; in the sequel these sets will be identified, i.e. $\alpha_n^{-1}(P) = P \setminus S(\mathcal{Z}_P) \cup \bigcup_{Z \in \mathcal{Z}_P} \tilde{F}_Z$ where $\tilde{F}_Z = \alpha_i^{-1}(F_Z)$ and F_Z is a copy of the fake 3-cell F attached in the ith step of construction I instead of the removed interior of a 3-cell $Z \in \mathcal{Z}_P$. Of course $\tilde{F}_Z$ is a copy of $\tilde{F}$, $\partial\tilde{F}_Z = \partial Z$ and this identification reverses the orientation for each $Z \in \mathcal{Z}_P$. It is easy to check that in any metric coincident with the topology of K the set $\{Z \in \mathscr{Z}_P\colon \operatorname{diam} \tilde{F}_Z > \varepsilon\}$ is finite for every $\varepsilon > 0$. So $\alpha_n^{-1}(P) \in P^*$.

LEMMA 8. *For every orientable* 3*-manifold* M *and every* $X \in M^*$, $X \in (M \# H_0)^*$ *where* H_0 *is some copy of the homotopy sphere* H.

Proof. We have $X = M \setminus S(\mathscr{Z}) \cup \bigcup_{Z \in \mathscr{Z}} \tilde{F}_Z$ for some good family $\mathscr{Z}$ of 3-cells in M. It follows from Lemma 7 that $\tilde{F}_{Z_1} \in F_{Z_1}^*$ for any $Z_1 \in \mathscr{Z}$. So $\tilde{F}_{Z_1} = F_0 \setminus S(\mathscr{Y}) \cup \bigcup_{Y \in \mathscr{Y}} \tilde{F}_Y$ where F_0 is a copy of F, and $\mathscr{Y}$ is a good family of 3-cells in it. Letting $\mathscr{T} = \mathscr{Y} \cup \mathscr{Z} \setminus \{Z_1\}$, we hence have $X = ((M \setminus \operatorname{Int} Z_1) \cup F_0) \setminus S(\mathscr{T}) \cup \bigcup_{T \in \mathscr{T}} \tilde{F}_T$, but $(M \setminus \operatorname{Int} Z_1) \cup F_0 = M \# H_0$ for some copy H_0 of H, and $\mathscr{T}$ is a good family of 3-cells in $M \# H_0$. This completes the proof.

IV. Proof that K is homogeneous. Let $x, y \in K$. We find two sequences $\{U_i\}_{i \in N}$ and $\{V_i\}_{i \in N}$ of open sets in K such that $x = \bigcap_{i \in N} U_i$, $y = \bigcap_{i \in N} V_i$, $U_i \supset \overline{U}_{i+1}$, $V_i \supset \overline{V}_{i+1}$ for every i, and each of U_i and V_i has the form $\alpha_j^{-1}(A)$ where A is some open polyhedral 3-cell in K_j for some natural $j = j(i)$. For example we can take A to be an open star of some vertex of K_j, for j suficiently big. Moreover, we require that for $i = 1$, $j = j(1) = 1$ and so U_1 and V_1 have the form $\alpha_1^{-1}(A)$ for some open 3-cell $A \subset K_1$, and that $j(k) < j(l)$ for any natural $k < l$. It is easy to see that the construction of the sets U_i and V_i is always possible.

By Lemma 7 $K \setminus U_1 \in (K_1 \setminus A_1)^*$ and $K \setminus V_1 \in (K_1 \setminus A_1')^*$ for some open 3-cells A_1 and A_1' in K_1. So by Lemma 6 there is a homeomorphism $h_1\colon K \setminus U_1 \to K \setminus V_1$ such that $h_1(\partial \overline{U}_i) = \partial \overline{V}_1$ and that $h_1|\partial \overline{U}_1\colon \partial \overline{U}_1 \to \partial \overline{V}_1$ is a homeomorphism preserving the orientation. Let us suppose we have constructed a homeomorphism $h_n\colon K \setminus U_n \to K \setminus V_n$ such that $h_n(\partial \overline{U}_n) = \partial \overline{V}_n$ and that a homeomorphism $h_n|\partial \overline{U}_n\colon \partial \overline{U}_n \to \partial \overline{V}_n$ preserves the orientation. For $p = j(n)$ and $q = j(n+1)$ we have polyhedral open 3-cells A_p, $A_p' \subset K_p$ and A_q, $A_q' \subset K_q$ such that

$$\overline{U}_n \setminus U_{n+1} = \alpha_p^{-1}(\overline{A}_p) \setminus \alpha_q^{-1}(A_q) = \alpha_q^{-1}(\alpha_{qp}^{-1}(\overline{A}_p) \setminus A_q)$$

and

$$\overline{V}_n \setminus V_{n+1} = \alpha_q^{-1}(\alpha_{qp}^{-1}(\overline{A}_p') \setminus A_q').$$

By Lemma 7

$$\overline{U}_n \setminus U_{n+1} \in (\alpha_{qp}^{-1}(\overline{A}_p) \setminus A_q)^* \quad \text{and} \quad \overline{V}_n \setminus V_{n+1} \in (\alpha_{qp}^{-1}(\overline{A}_p') \setminus A_q')^*.$$

From construction I it follows that there are orientation-preserving homeomorphisms from $\alpha_{qp}^{-1}(\overline{A}_p) \setminus A_q$ onto $D_1 \# D_2 \# H_1 \# \ldots \# H_{r_1}$ and from $\alpha_{qp}^{-1}(\overline{A}_p') \setminus A_q'$ onto $D_1 \# D_2 \# H_1 \# \ldots \# H_{r_2}$, where D_1 and D_2 are 3-cells, each H_i is a copy of H, and r_1 and r_2 are two natural numbers. So using Lemma 8 and Lemma 6, we get a homeomorphism $\bar{h}_{n+1}\colon \overline{U}_n \setminus U_{n+1} \to \overline{V}_n \setminus V_{n+1}$ such that $\bar{h}_{n+1}|\partial \overline{U}_n = h_n|\partial \overline{U}_n$ and that $\bar{h}_{n+1}|\partial U_{n+1}\colon \partial \overline{U}_{n+1} \to \partial \overline{V}_{n+1}$ is an orientation-preserving homeomorphism. Putting $h_{n+1}(x) = h_n(x)$ for $x \in K \setminus U_n$ and $h_{n+1}(x) = \bar{h}_{n+1}(x)$ for $x \in \overline{U}_n \setminus U_{n+1}$

we get a homeomorphism $h_{n+1}\colon K \setminus U_{n+1} \to K \setminus V_{n+1}$ such that $h_{n+1}(\partial \overline{U}_{n+1}) = \partial \overline{V}_{n+1}$ and that the homeomorphism $h_{n+1}|\partial \overline{U}_{n+1}\colon \partial \overline{U}_{n+1} \to \partial \overline{V}_{n+1}$ preserves the orientation. So we have inductively defined h_n for each $n \in N$, and we can define a homeomorphism $h\colon (K, x) \to (K, y)$ putting $h|(K \setminus U_n) = h_n|(K \setminus U_n)$ for each $n \in N$ and $h(x) = y$.

V. Proof that K is not a 3-manifold. Let us suppose that K is a 3-manifold. By Lemma 7 $K \in K_n^*$ for every natural n, i.e. $K = K_n \setminus S(\mathscr{Z}_n) \cup \bigcup_{Z \in \mathscr{Z}_n} \tilde{F}_Z$ where $\mathscr{Z}_n$ is a good family of 3-cells in K_n and there exists a family of homeomorphisms $\{g_Z\colon \tilde{F} \to \tilde{F}_Z\}_{Z \in \mathscr{Z}_n}$. For every $\varepsilon > 0$ the set $\{Z \in Z_n\colon \operatorname{diam}(\tilde{F}_Z) > \varepsilon\}$ is finite. This implies that there exists a $Z_0 \in \mathscr{Z}$ and a 3-cell Q contained in $K = K_n \setminus S(\mathscr{Z}_n) \cup \bigcup_{Z \in \mathscr{Z}_n} \tilde{F}_Z$, such that $\tilde{F}_{Z_0} \subset \operatorname{Int} Q$. It is easy to see that the quotient space K_n/Z_0 is homeomorphic to K_n. Let $\pi_0\colon K_n \to K_n/Z_0$ be the quotient map. We consider a family $\mathscr{Y}$ of 3-cells in K_n/Z_0 defined as follows: $Y \in \mathscr{Y}$ iff there exists a $Z \in \mathscr{Z} \setminus \{Z_0\}$ such that $\pi_0(Z) = Y$. It is easy to check that $\mathscr{Y}$ is a good family of 3-cells in K_n/Z_0. Now we consider the space

$$K/\tilde{F}_{Z_0} = (K_n \setminus S(\mathscr{Z}_n) \cup \bigcup_{Z \in \mathscr{Z}_n} \tilde{F}_Z)/\tilde{F}_{Z_0}$$

and the corresponding quotient map π_1. We have

$$K/\tilde{F}_{Z_0} = \pi_1(K_n \setminus S(\mathscr{Z}_n)) \cup \bigcup_{Z \in \mathscr{Z}_n \setminus \{Z_0\}} \pi_1(\tilde{F}_Z).$$

We identify $\pi_1(K_n \setminus S(\mathscr{Z}_n))$ and $\pi_0(K_n \setminus S(\mathscr{Z}_n)) = (K_n \setminus Z_0) \setminus S(\mathscr{Y})$. Then the family $\{\pi_1 \cdot g_z\colon \tilde{F} \to \pi_1(\tilde{F}_Z)\}_{Z \in \mathscr{Z} \setminus \{Z_0\}}$ is a family of homeomorphisms satisfying the requirements of the definition of the family $(K_n/Z_0)^*$. So $K/\tilde{F}_{Z_0} \in (K_n/Z_0)^*$, and from Lemma 6 and from the fact that K_n/Z_0 and K_n are homeomorphic we deduce that $K/\tilde{F}_{Z_0}$ and K are homeomorphic. Since $\tilde{F}_{Z_0}$ is contained in a 3-cell $Q \subset K$, we can easily deduce from [5], Theorems 3 and 5, that $\tilde{F}_{Z_0}$ is a cellular set in K.

We shall prove that $\tilde{F}_Z$ is cellular for every $Z \in \mathscr{Z}_n$. For this purpose we take any $Z_1 \in \mathscr{Z}_n$ and we consider the space

$$K \setminus \operatorname{Int} \tilde{F}_{Z_1} = (K_n \setminus \operatorname{Int} Z_1) \setminus S(\mathscr{Z}_n \setminus \{Z_1\}) \cup \bigcup_{Z \in \mathscr{Z}_n \setminus \{Z_1\}} \tilde{F}_Z.$$

$\mathscr{Z}_n \setminus \{Z_1\}$ is a good family of 3-cells in $K_n \setminus \operatorname{Int} Z_1$, so we have $K \setminus \operatorname{Int} \tilde{F}_{Z_1} \in (K_n \setminus \operatorname{Int} Z_1)^*$, and by the same argument $K \setminus \operatorname{Int} \tilde{F}_{Z_0} \in (K_n \setminus \operatorname{Int} Z_0)^*$. Of course

$$(g_{Z_1}|\partial \tilde{F})(g_{Z_0}|\partial \tilde{F})^{-1}\colon \partial Z_0 \to \partial Z_1$$

is a homeomorphism preserving the orientation (note that $\partial Z_1 \subset \tilde{F}_{Z_1} \subset K$, and $\partial Z_0 \subset \tilde{F}_{Z_0} \subset K$). There exists a homeomorphism $h\colon K_n \setminus \operatorname{Int} Z_0 \to K_n \setminus \operatorname{Int} Z_1$ such that $h|\partial Z_0 = (g_{Z_1}|\partial \tilde{F})(g_{Z_0}|\partial \tilde{F})^{-1}$, and so by Lemma 6 there exists a homeomorphism $\bar{h}\colon K \setminus \operatorname{Int} \tilde{F}_{Z_0} \to K \setminus \operatorname{Int} \tilde{F}_{Z_1}$ such that $\bar{h}|\partial Z_0 = (g_{Z_1}|\partial \tilde{F})(g_{Z_0}|\partial \tilde{F})^{-1}$. $\bar{h}$ can be extended to a homeomorphism $h'\colon K \to K$ if we put $h'(x) = g_{Z_1} g_{Z_0}^{-1} x$ for $x \in \tilde{F}_{Z_0}$. It is easy to see that $h'(\tilde{F}_{Z_0}) = \tilde{F}_{Z_1}$. This implies that $\tilde{F}_{Z_1}$ is cellular in K for any $Z_1 \in \mathscr{Z}_n$.

Let us consider the decomposition G_n of the space $K = K_n \setminus S(\mathscr{Z}_n) \cup \bigcup_{Z \in \mathscr{Z}_n} \tilde{F}_Z$

whose non-degenerate elements are the sets $\tilde{F}_Z$ for all $Z \in \mathscr{Z}_n$. The quotient space K/G_n is homeomorphic to K_n/G'_n where G'_n is the decomposition of K_n whose non-degenerate elements form the good family $\mathscr{Z}_n$. By Lemma 4 K_n/G'_n, and so K/G_n is homeomorphic to the 3-manifold K_n. Since we have supposed that K is a 3-manifold and G_n was shown to consist of sets which are cellular in K, it follows from the theorem of Armentrout (see [1], p. 66, Theorem 2) that K is homeomorphic to K_n for every natural n. This implies that K_1 is homeomorphic to K_2. K_1 is a 3-sphere and K_2 contains the fake 3-cell F, which is impossible (see [5], Theorem 5). This proves that K is not a 3-manifold.

VI. Proof that $\dim K = 3$. $\dim K \leqslant 3$, because $\dim K_n = 3$ for every n and $K = \underleftarrow{\lim}\{K_n, \alpha_{mn}\}$ and $\dim K \geqslant 3$ because $H^3(K, Z) \neq 0$ (see [7], p. 152).

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A theorem on the weak topology of $C(X)$ for compact scattered X

by

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Abstract. We prove that if a function space $C(X)$, where X is a compact scattered space, is K-analytic under the weak topology, then $C(X)$ is a WCG space, i.e. X is an Eberlein compact. This result is related to a recent author's example of a non-WCG space $C(X)$ with X compact scattered, which is Lindelöf in the weak topology, a recent example of Talagrand of a non-WCG space $C(K)$, which is K-analytic in the weak topology, and the recent theorem of Talagrand that every WCG Banach space is K-analytic in the weak topology.

1. Introduction. It was an old problem of Corson [6] whether the WCG Banach spaces (the terminology will be explain in the next section) are exactly the Banach spaces which are Lindelöf in their weak topology. An example of a Banach space which is Lindelöf in the weak topology but not WCG was given by Rosenthal [12] and, on the other hand, Talagrand [15] proved that a WCG Banach space is $\mathscr{K}$-analytic (which is much more than the Lindelöf property) in the weak topology. It was still open after these works if the Corson's problem has an affirmative solution in the class of function spaces [9], Problem 6, 6′, [4], Problem 7. Recently, the author [11] and independently, about the same time, Talagrand [16] constructed the appropriate counterexamples. The content of these examples is however quite different. The function space $C(X)$ in the author's example is not $\mathscr{K}$-analytic in the weak topology, while the compact X is scattered, whereas the Talagrand's space $C(K)$ is $\mathscr{K}$-analytic in the weak topology, but the compact K is not scattered.

The aim of this paper is to show that if a function space $C(X)$ is $\mathscr{K}$-analytic in the weak, or pointwise topology and the compact X is scattered, then $C(X)$ is a WCG-space, or equivalently — X is an Eberlein compact.

It is worth while to mention that one can exploit the Talagrand's example to show ([1]) that in fact there is no topological property which is invariant under continuous mappings, closed hereditarily and characterizes the Eberlein compacts as the compacts whose function space in the weak (or pointwise) topology has this

([1]) By means, for example, of the reasonings given in [11], the proof of Lemma 1; cf. also [2].