Hence, since $x_n \to x$, $U_\delta(x) \subseteq U_\varepsilon(x_n)$ ultimately as $n \to \infty$. Applying F we get

$$F \circ U_\delta(x) \subseteq F \circ U_\varepsilon(x_n) \text{ ultimately}. \tag{16}$$

Since $(y, x) \in F$ and $(x, x) \in U_\delta$, $y \in F(x) \subseteq F \circ U_\delta(x)$. Since $U_\delta(x)$ is open, $F \circ U_\delta(x)$ is open by hypothesis. Therefore, since $y_n \to y$ and $y \in F \circ U_\delta(x)$, $y_n \in F \circ U_\delta(x)$ ultimately. Hence by (16), $y_n \in F \circ U_\varepsilon(x_n)$ ultimately, which contradicts (13).

Finally, Theorem 1 follows from Theorem 5 under Lemmas 8 and 9 since every connected metric space is well-chained.

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s-Fibrations

by

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Abstract. The concept of s-fibration is introduced which generalized the notions of Hurewicz fibrations and approximate fibrations. Many results about Hurewicz fibrations which are not true for approximate fibrations are proved for s-fibrations. For example, a homotopy classification theorem for s-fibrations over then n-sphere is proved.

1. Introduction. A mapping $f\colon E \to B$ between compact metric spaces is an *approximate fibration* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $h\colon X \to E$ and $H\colon X \times [0, 1] \to B$ are maps with $d\big(H(x, 0), fh(x)\big) < \delta$, then there exists $G\colon X \times [0, 1] \to E$ such that $G(x, 0) = h(x)$ and $d\big(H(x, t), fG(x, t)\big) < \varepsilon$ for all $x \in X$ and $t \in [0, 1]$. Coram and Duvall [2] introduced approximate fibrations as a generalization of cell-like mappings [10] and showed that the uniform limit of a sequence of Hurewicz fibrations is an approximate fibration. By using shape theoretic concepts, they also showed that approximate fibrations possessed many properties shared by Hurewicz fibrations.

One notable exception is that the pullback of an approximate fibration need not be an approximate fibration. In this work, we define the concept of s-fibrations which we show generalizes the concepts of approximate fibrations and Hurewicz fibrations. Pullbacks behave properly and many other results about Hurewicz fibrations carry over. For example, a homotopy classification theorem for s-fibrations over the n-sphere is proved (Theorem 11.1). As a consequence, information about cell-like decompositions of ANR's is obtained (Theorem 12.1).

T. B. Rushing has informed the authors that S. Mardešić and he [11] have also generalized the theory of approximate fibrations but that the overlap between these works is little. R. Goad [6] has also a generalization of approximate fibration but, again, there is no overlap with this work.

2. Definitions. We shall assume that the reader is familiar with [12]. Since our results are valid for a larger category of spaces than that considered in [12], our definitions will sometimes differ.

A directed set $(\Gamma, \leqslant)$ is *closure-finite* provided for every $\gamma \in \Gamma$, the set of predecessors of γ is finite. A *tower of topological spaces* $\underline{E} = \{E_\gamma, e_{\gamma\beta}, \Gamma\}$ is an inverse system of topological spaces where Γ is a closure-finite directed set of indices and the bonding maps, $e_{\gamma\beta}\colon E_\beta \to E_\gamma$, $\gamma \leqslant \beta$, are continuous. A *tower of maps* between two towers, $\underline{f}\colon \underline{E} \to \underline{E}' = \{E'_\gamma, e'_{\gamma\beta}, \Gamma'\}$ consists of an increasing function $f\colon \Gamma' \to \Gamma$ and a collection of continuous maps $f_\gamma\colon E_{f(\gamma)} \to E'_\gamma$ such that for $\gamma \leqslant \beta$, $f_\gamma e_{f(\gamma)f(\beta)} \simeq e'_{\gamma\beta} f_\beta$(–i.e., $f_\gamma e_{f(\gamma)f(\beta)}$ is homotopic to $e'_{\gamma\beta} f_\beta$).

Composition of two towers of maps, $\underline{f} \circ \underline{g}$ can be defined (see [12]), *id* will denote the identity tower of maps. Two towers of maps $\underline{f}, \underline{g}\colon \underline{E} \to \underline{E}'$ are *homotopic*, $\underline{f} \simeq \underline{g}$, provided for every $\beta \in \Gamma'$, there exists $\alpha \in \Gamma$, $\alpha \geqslant f(\beta), g(\beta)$, such that $f_\beta e_{f(\beta)\alpha} \simeq g_\beta e_{g(\beta)\alpha}$.

A *map* from a tower $\underline{E}$ to a topological space B is a collection $\underline{p} = \{p_\alpha;\ \alpha \in \Gamma\}$ of continuous maps $p_\alpha\colon E_\alpha \to B$ such that for all $\beta \geqslant \alpha$, $p_\alpha e_{\alpha\beta} = p_\beta$. Let $\underline{p}\colon \underline{E} \to B$ and $\underline{p}'\colon \underline{E}' \to B'$ be maps and let $g\colon B \to B'$ be a continuous map. A tower of maps, $\underline{f}\colon \underline{E} \to \underline{E}'$, is $(\underline{p}, \underline{p}', g)$-*preserving* if the homotopy in the definition of tower of maps between $f_\gamma e_{f(\gamma)f(\beta)}$ and $e'_{\gamma\beta} f_\beta$, say φ_t, $t \in [0, 1]$, can be chosen such that $p'_\gamma \varphi_t = g p_{f(\beta)}$ for all t.

Let $\underline{p}\colon \underline{E} \to B$ be a map. The triple $\xi = (\underline{p}, \underline{E}, B)$, or, more simply, $\underline{p}$, is called an *s-fibration* if, given $\alpha \in \Gamma$, there exists $\beta \geqslant \alpha$ such that whenever X is a topological space and $g\colon X \to E_\beta$ and $H\colon X \times [0, 1] \to B$ are maps with $p_\beta g(x) = H(x, 0)$, then there exists $G\colon X \times [0, 1] \to E_\alpha$ such that $p_\alpha G = H$ and $e_{\alpha\beta} g(x) = G(x, 0)$, $x \in X$.

An order preserving function $\varphi\colon \Gamma \to \Gamma$ such that $\varphi(\alpha) \geqslant \alpha$ for all $\alpha \in \Gamma$ is called a *t-function* for the s-fibration $\underline{p}\colon \underline{E} \to B$ if $\varphi(\alpha)$ can be substituted for β in the definition. By Lemma 5 of [12], we can find a t-function for an s-fibration.

If $x \in B$, the *fibre of p* at x is the tower of spaces $\underline{p}^{-1}(x) = \{p_\alpha^{-1}(x), e_{\beta\alpha} | p_\alpha^{-1}(x), \Gamma\}$.

Let $\sigma\colon \Gamma \to \Gamma$ be an order-preserving function such that $\sigma(\alpha) \geqslant \alpha$ for all α; the *shift map*, $\underline{\sigma}\colon \underline{E} \to \underline{E}$, induced by σ is the towers of maps $\underline{\sigma}$ defined by $\sigma_\alpha = e_{\alpha\sigma(\alpha)}$. Note that $\underline{\sigma} \simeq \underline{\mathrm{id}}$. Given two towers of maps, $\underline{f}, \underline{g}\colon \underline{E} \to \underline{E}'$, we write $\underline{f} \equiv \underline{g}$ if there exist shift maps $\underline{\sigma}, \underline{\sigma}'\colon \underline{E}' \to \underline{E}'$ and $\underline{\tau}, \underline{\tau}'\colon \underline{E} \to \underline{E}$ such that $\underline{\sigma f \tau} = \underline{\sigma' g \tau'}$. Trivially, if $\underline{f} \equiv \underline{g}$, then $\underline{f} \simeq \underline{g}$.

Let $\underline{E} = \{E_\alpha, e_{\alpha\beta}, \Gamma\}$ be a tower of spaces and let Y be a space; then $\underline{E} \times Y$ will denote the tower of spaces $\{E_\alpha \times Y, e_{\alpha\beta} \times \mathrm{id}, \Gamma\}$. If $\underline{p}\colon = \{p_\alpha\}\colon \underline{E} \to B$ is a map then $\underline{p} \times \mathrm{id} = \{p_\alpha \times \mathrm{id}\}\colon \underline{E} \times Y \to B \times Y$ is also a map.

Let $\underline{p}\colon \underline{E} \to B$ and $\underline{p}'\colon \underline{E}' \to B'$ be s-fibrations. Suppose that $f\colon B \to B'$ is a continuous map and $\underline{F}\colon \underline{E} \to \underline{E}'$ is a $(\underline{p}, \underline{p}', f)$-preserving tower of maps. If $x \in B$, then $\underline{F}$ induces a tower of maps, $\underline{F} | \underline{p}^{-1}(x)\colon \underline{p}^{-1}(x) \to \underline{p}'^{-1}(f(x))$, given by the collection, $\{F_\gamma | p_{F(\gamma)}^{-1}(x)\colon p_{F(\gamma)}^{-1}(x) \to p_\gamma'^{-1}(x), \gamma \in \Gamma\}$. The pair $(\underline{F}, f)$ is called a *bundle map* if, for each $x \in B$, there exists a tower of maps $\underline{G}_x\colon \underline{p}'^{-1}(f(x)) \to \underline{p}^{-1}(x)$ such that $(\underline{F} | \underline{p}^{-1}(x)) \circ \underline{G}_x \simeq \mathrm{id} | \underline{p}'^{-1}(f(x))$ and $\underline{G}_x \circ (\underline{F} | \underline{p}^{-1}(x)) \simeq \mathrm{id} | \underline{p}^{-1}(x)$. In the terminology of [12], $\underline{F} | \underline{p}^{-1}(x)$ is a homotopy equivalence for each $x \in B$. We say that the bundle map $\underline{F}$ *covers the map* f.

A *bundle homotopy* is a bundle map $(\underline{F}, f)$, $\underline{F}\colon \underline{E} \times [0, 1] \to \underline{E}'$ covering $f\colon B \times [0, 1] \to B'$. If $(\underline{F}, f)$ is a bundle homotopy, for $i \in [0, 1]$, let $\underline{F}_i\colon \underline{E} \to \underline{E}'$ be the tower of maps $\{F_{i\alpha}\}$ where $F_{i\alpha}(x) = F_\alpha(x, i)$. Two bundle maps $(\underline{f}_0, f_0)$ and $(\underline{f}_1, f_1)$ are *bundle homotopic*, denoted $\underline{f}_0 \simeq_b \underline{f}_1$, if there exists a bundle homotopy $(\underline{F}, f)$ such that $(\underline{F}_i, f_i) = (\underline{\sigma}^i f_i \underline{\tau}^i, f_i)$ for $i = 0, 1$ where $\underline{\sigma}^i \underline{\tau}^i$ are shift maps on $\underline{E}'$.

Let $p\colon \underline{E} \to B$ and $\underline{p}'\colon \underline{E}' \to B$ be s-fibrations. A bundle map $\underline{F}\colon \underline{E} \to \underline{E}'$ covering the identity map is a (*bundle*) *equivalence* if there exists a bundle map $\underline{G}\colon \underline{E}' \to \underline{E}$ covering the identity such that $\underline{F} \circ \underline{G}$ and $\underline{G} \circ \underline{F}$ are bundle homotopic to the identity with bundle homotopies covering the projections $B \times [0, 1] \to B'$ and $B \times [0, 1] \to B$, respectively. In Section 10, we will show that a bundle map covering the identity is an equivalence. $\underline{G}$ will be called a (*bundle*) *inverse* of $\underline{F}$. Note that a bundle map which is bundle homotopic to a bundle equivalence is also a bundle equivalence.

3. Approximate fibrations. Let $f\colon E \to B$ be a continuous map between the compact metric spaces E and B. Let d and d' denote the metrics on E and B respectively and define $\varrho((e, b), (e', b')) = \max\{d(e, e'), d'(b, b')\}$ for points $(e, b), (e', b') \in E \times B$. Let $\Gamma f = \{(e, f(e)\}$ be the graph of f and let E_i denote the $(1/i)$-neighborhood of Γf in $E \times B$, where i denotes a positive integer. Let $e_{ij}\colon E_j \to E_i$ denote inclusion, $j \geqslant i$. Define $p_i\colon E_i \to B$ by $p_i(e, x) = x$; note that $\underline{p} = \{p_i\}\colon \underline{E} = \{E_i\} \to B$ is a map.

THEOREM 3.1. *f is an approximate fibration if and only if p is an s-fibration.*

Proof. Suppose that f is an approximate fibration and let the positive integer i be given. For $\varepsilon = 1/i$, let δ be given from the definition of approximate fibration. Let δ' be chosen such that if $x, y \in E$, $d(x, y) < \delta'$, then $d(f(x), f(y)) < \delta/2$. Choose a positive integer $j \geqslant i$ such that $1/j \leqslant \delta', \delta/2$.

Let $g\colon X \to E_j$ and $H\colon X \times [0, 1] \to B$ be continuous maps such that $p_j g(x) = H(x, 0)$. Thus $g(x) = (g'(x), H(x, 0))$ for some function $g'\colon X \to E$. Since $g(x) \in E_j$, there exists $e \in E$ such that $\varrho[(e, f(e)), g(x)] < 1/j \leqslant \delta'$. Hence $d(e, g'(x)) < \delta'$ and $d'(f(e), fg'(x)) < \delta/2$; thus

$$d'(fg'(x), H(x, 0)) \leqslant d'(fg'(x), f(e)) + d'(f(e), H(x, 0)) < \delta\,.$$

By hypothesis, there exists a homotopy $G'\colon X \times [0, 1] \to E$ such that $G'(x, 0) = g'(x)$ and

$$d(fG'(x, t), H(x, t)) < \varepsilon\,.$$

Define $G\colon X \times [0, 1] \to E_i$ by $G(x, t) = (G'(x, t), H(x, t))$; G is the desired homotopy.

Now suppose that $\underline{p}$ is an s-fibration and let $\varepsilon > 0$ be given. Let δ' be chosen such that if $d(x, y) < \delta'$, then $d'(f(x), f(y)) < \varepsilon/2$. Choose a positive integer i such that $1/i \leqslant \varepsilon/2, \delta'$. Let j be the positive integer given by the definition of s-fibration. Let $\delta = 1/j$ and let $g\colon X \to E$ and $H\colon X \times [0, 1] \to B$ be continuous maps such that $d'(fg(x), H(x, 0)) < \delta$. Define $g'(x) = (g(x), H(x, 0)) \in E_j$; thus there exists $G'\colon X \times [0, 1] \to E_i$ such that $G'(x, 0) = g'(x)$ and $p_i G'(x, t) = H(x, t)$. Suppose that $G'(x, t) = (G(x, t), H(x, t))$ for some G. G is the desired function; for, since

$G'(x,t)\in E_i$, there exists $e\in E$ such that $\varrho[(e,f(e)),G'(x,t)]<1/i$. Hence $d(e,G(x,t))<1/i\leqslant\delta'$ and $d'(f(e),fG(x,t))<\varepsilon/2$; therefore

$$d'(fG(x,t),H(x,t))\leqslant d'(fG(x,t),f(e))+d'(f(e),H(x,t))<\varepsilon .$$

Note that E is the inverse limit of $\{E_i\}$ and f is the inverse limit of $\{p_i\}$. Let $f\colon E\to B$ be an approximate fibration which is not a Hurewicz fibration [2]; then the constant sequence $\{f\}\colon\{E\}\to B$ is not an s-fibration. Thus the inverse systems which we can associate to f and E in order to prove an analogue of Theorem 3.1 form a proper subset of those systems whose inverse limits are f and E. Note also that $f^{-1}(b)$ is also the inverse limit of $\underline{p}^{-1}(b)$.

The difficulty with the above construction is that the mappings p_i are not proper. We will now modify the above construction to alleviate this problem. The proof of the following is left to the reader.

PROPOSITION 3.2. *Let $f\colon E\to B$ be a continuous map between the compact metric spaces E and B. Form the inverse sequence $\{E_i,e_{ij}\}$ as above and let $\{A_i\}$ be a sequence of subsets of $E\times B$ such that for each i there exists j and k for which $A_j\subseteq E_i$ and $E_k\subseteq A_i$. Then $p\colon E\to B$ is an approximate fibration if and only if $\{q_i\}\colon\{A_i\}\to B$ is an s-fibration where $q_i(e,x)=x$.*

Let $Q=\prod_{k=1}^{\infty}[0,1]_k$ be the Hilbert cube and let $N_i=\prod_{k=1}^{\infty}[0,1/i]_k$. Define $n_i\colon E_i\times N_i\to B$ by $n_i(e,t)=p_i(e)$.

PROPOSITION 3.3. *$\underline{p}\colon\underline{E}\to B$ is an s-fibration if and only if $\{n_i\}\colon\{E_i\times N_i\}\to B$ is an s-fibration.*

THEOREM 3.4. *Let E and B be compact* ANR's *and let $f\colon E\to B$ be a continuous map. Then there exists a tower of compact* ANR's *$\{\tilde{E}_i\}$, and a map $q\colon\{\tilde{E}_i\}\to B$ such that f is an approximate fibration if and only if q is an s-fibration.*

Proof. Consider the map $\{n_i\}\colon\{E_i\times N_i\}\to B$ as in Proposition 3.3. Recall that an ANR Y is convenient [14] if given a compactum $X\subseteq Y$ and a neighborhood V of X in Y, there exists a compact ANR $M\subseteq V$ such that $X\subseteq\operatorname{int}M$. By [2] $E\times B\times Q$ is a convenient ANR. Thus it is possible to find a sequence of compact ANR's $\{A_i\}$ such that for each i there exists j and k for which $A_j\subseteq E_i\times N_i$ and $E_k\times N_k\subseteq A_i$. Theorem 3.4 now follows from 3.1 and 3.2.

Let $p\colon E\to B$ and $p'\colon E'\to B$ be approximate fibrations where E, E' and B are compact ANR's. Suppose that there exists an embedding $f\colon E\to E'$ such that $p'f=p$. Let $\{n_i\}\colon\{E_i\times N_i\}\to B$ and $\{n_i'\}\colon\{E_i'\times N_i\}\to B$ be defined as above. Given i choose $j>i$ such that if $d(x,y)<1/j$ then $d(f(x),f(y))<1/i$ for $x,y\in E$. Define $f_i\colon E_j\times N_j\to E_i'\times N_i$ by $f_i(e,b,t)=(f(e),b,t)$ where $(e,b,t)\in E\times B\times Q$. It is easily checked that $\{f_i\}\colon\{E_i\times N_i\}\to\{E_i'\times N_i\}$ is a tower of maps which is $(\{n_i\},\{n_i'\},\mathrm{id})$-preserving.

THEOREM 3.5. *Suppose that for each $x\in B$, $p^{-1}(x)$ and $p'^{-1}(x)$ are* ANR's *and $f|p^{-1}(x)\colon p^{-1}(x)\to p'^{-1}(x)$ is a homotopy equivalence. Then $\{f_i\}\colon\{E_i\times N_i\}\to\{E_i'\times N_i\}$ is a bundle map.*

Proof. It must be shown that $\{f_i\}|\{n_i\}^{-1}(x)\colon\{n_i\}^{-1}(x)\to\{n_i'\}^{-1}(x)$ is a homotopy equivalence. We would like to apply Theorem 12 of [12], but the inverse systems $\{n_i\}^{-1}(x)$, $\{n_i'\}^{-1}(x)$ do not satisfy the hypotheses that they consist of compact ANR's.

Let V_i be the $(1/i)$-neighborhood of $p^{-1}(x)$ in E; then $V_i\times\{x\}\times N_i\subseteq n_i^{-1}(x)$. There exists $\delta>0$ such that if $d(x,p(e))<\delta$, then $d(p^{-1}(x),p^{-1}(p(e)))<1/i$. Choose δ' such that if $d(y,z)<\delta'$, then $d(p(y),p(z))<\delta/2$. Choose $j\geqslant i$ such that $1/j\leqslant\delta',\ \delta/2$; then if $(y,x)\in p_j^{-1}(x)$, then $d(x,p(y))<\delta$ and hence $y\in V_i$. Thus $p_j^{-1}(x)\times N_j\subseteq V_i\times\{x\}\times N_i$ and the towers $\{n_i\}^{-1}(x)$ and $\{V_i\times\{x\}\times N_i\}$ are homotopy equivalent. Since $V_i\times\{x\}\times N_i$ is a convenient ANR, there exists a compact ANR $M_i\subseteq V_i\times\{x\}\times N_i$ which is a neighborhood of $p^{-1}(x)\times\{x\}\times\{0\}$. Note that $\{n_i\}^{-1}(x)$ and $\{M_i\}$ are homotopy equivalent when $\{M_i\}$ is a nested sequence.

Similarly construct $\{M_i'\}$ corresponding to $\{n_i'\}^{-1}(x)$. Now apply Theorem 12 of [12] to get that $\{M_i\}$ and $\{M_i'\}$ are homotopy equivalent and hence, $\{n_i\}^{-1}(x)$ $\{n_i'\}^{-1}(x)$ are also homotopy equivalent.

Let $f\colon E\to B$ and $f'\colon E'\to B$ be approximate fibrations where E, E', and B are compact ANR's. Let $\underline{q}\colon\underline{E}\to B$ and $\underline{q}'\colon\underline{E}'\to B$ be the s-fibrations associated to f and f', respectively, as in Theorem 3.1.

THEOREM 3.6. *If there exists an equivalence $\underline{F}\colon\underline{E}\to\underline{E}'$, then for each $\varepsilon>0$, there exist maps $h\colon E\to E'$ and $g\colon E'\to E$ such that*

(i) $d'(f'h(x),f(x))<\varepsilon$,

(ii) $d'(fg(x),f'(x))<\varepsilon$,

(iii) *gh and hg are homotopic to the identity.*

Proof. Consider the projections

$$p_1\colon E\times B\to E,\qquad p_1'\colon E'\times B\to E',$$
$$p_2\colon E\times B\to B,\qquad p_2'\colon E'\times B\to B.$$

Choose δ such that if $d(x,y)<\delta$, $d(x',y')<\delta$, then $d(p_2(x),p_2(y))<\varepsilon$, $d(p_2(x'),p_2(y'))<\varepsilon$. By Proposition 3.4 of [7] there exist δ-homotopies $h_t\colon E\times B\to E\times B$, $h_t'\colon E'\times B\to E'\times B$ such that

(a) $h_0=\mathrm{id}$, $h_0'=\mathrm{id}$,

(b) $h_t|\Gamma f=\mathrm{id}$, $h_t'|\Gamma f'=\mathrm{id}$,

(c) there exists i such that $h_1(E_i)\subseteq\Gamma f$ and $h_1'(E_i')\subseteq\Gamma f'$.

Let $\underline{G}\colon\underline{E}'\to\underline{E}$ be an inverse for $\underline{F}$ and define $g\colon E'\to E$ to be the composition.

$$E'\xrightarrow{\mathrm{id}\times f'}E'_{G(i)}\xrightarrow{G_i}E_i\xrightarrow{h_1}\Gamma f\xrightarrow{p_1}E .$$

Then

$$\begin{aligned}d'(fg(x),f'(x))&=d'(fp_1h_1G_i(\mathrm{id}\times f'(x),f'(x))\\&=d'(p_2h_1G_i(\mathrm{id}\times f')(x),p_2G_i(\mathrm{id}\times f')(x))\\&=d'(p_2h_1(z),p_2(z))\end{aligned}$$

where $z=G_i(\mathrm{id}\times f')(x)$. Since $d(h_1(z),z)<\delta$, $d'(p_2h_1(z),p_2(z))<\varepsilon$.

Define $h\colon E \to E'$ to be the composition

$$E \xrightarrow{\mathrm{id}\times f} E_{F(i)} \xrightarrow{F_i} E_i' \xrightarrow{h_1'} \Gamma f' \xrightarrow{p_1'} E' .$$

Similarly, $d'(f'h(x), f(x)) < \varepsilon$.

Choose $j > i$ such that $F_i G_{F(i)} e_{GF(i),j} \simeq e_{ij}$. (Here we use the fact that $\underline{F} \circ \underline{G} \simeq \mathrm{id}$.) Then

$$\begin{aligned} hg &= p_1' h_1' F_i(\mathrm{id}\times f) p_1 h_1 G_i(\mathrm{id}\times f') \\ &= p_1' h_1' F_i h_1 G_i(\mathrm{id}\times f') \\ &\simeq p_1' h_1' F_i h_1 G_{F(i)}(\mathrm{id}\times f') \\ &\simeq p_1' h_1' F_i h_0 G_{F(i)}(\mathrm{id}\times f') \\ &= p_1' h_1' F_i G_{F(i)}(\mathrm{id}\times f') \\ &\simeq p_1' h_1'(\mathrm{id}\times f') = \mathrm{id}. \end{aligned}$$

Here we used the fact that the bonding maps for $\underline{E}$ are inclusion maps. Similarly $gh \simeq \mathrm{id}$.

4. Homotopy theory of the space of shape equivalences. A well-known result in the theory of locally trivial fibre bundles is that there is a bijection between the equivalence classes of such bundles over the n-sphere with a suitable fibre F and the $n-1$ homotopy group of the space of homeomorphisms of F with the compact-open topology [17]. In Section 11, we shall prove the analogue of this theorem for s-fibrations. However, since there is no natural topology for the set of shape equivalences (or in the terminology of [12], homotopy equivalences) of the fibre, we shall use the formalism of semi-simplicial theory in order to develop a homotopy theory.

Let $\underline{F}$ be a tower of spaces and let $\mathcal{SE}(\underline{F})$ be semi-simplicial set of homotopy equivalences of $\underline{F}$; an n-simplex of $\mathcal{SE}(\underline{F})$ is a bundle equivalence $\underline{f}\colon \underline{F}\times\Delta^n \to \underline{F}\times\Delta^n$ where $\pi\colon \underline{F}\times\Delta^n \to \Delta^n$ is the product bundle with F as fibre and Δ^n is an n-cell. If Δ^n is triangulated as an ordered n-simplex and $\partial_1\colon \Delta^n \to \Delta^n$ is the boundary operator which omits the ith vertex, then define $\partial_i \underline{f} = \underline{f}\,|\,\underline{F}\times\partial_i\Delta^n$. The ith degeneracy operator is defined analogously. It is easily seen that $\mathcal{SE}(\underline{F})$ is a Kan complex [13]. Composition of bundle equivalences makes $\mathcal{SE}(F)$ into a Kan monoid complex ([13], p. 68).

If S^n is the n-sphere and $X_0 \in S^n$, then a based map of S^n into $\mathcal{SE}(\underline{F})$ is a bundle equivalence $\underline{f}\colon \underline{F}\times S^n \to \underline{F}\times S^n$ such that $\underline{f}\,|\,\underline{F}\times\{X_0\} \equiv \underline{\mathrm{id}}$. Two such maps, $\underline{f}_0$ and $\underline{f}_1$, are homotopic rel X_0 if there exists a bundle equivalence $\underline{H}\colon \underline{F}\times S^n\times[0,1] \to \underline{F}\times S^n\times[0,1]$ (as bundles over $S^n\times[0,1]$) such that $\underline{H}\,|\,\underline{F}\times S^n\times\{i\} \equiv \underline{f}_i\times\{i\}$, $i = 0, 1$ and $\underline{H}\,|\,\underline{F}\times\{X_0\}\times[0,1] \equiv \underline{\mathrm{id}}$; we write $\underline{f}_0 \simeq_0 \underline{f}_1 \cdot \simeq_0$ is an equivalence relation and the set of equivalence classes forms a group, $\pi_n(\mathcal{SE}(\underline{F}))$, called the *n-th homotopy group* of $\mathcal{SE}(\underline{F})$. Since $\mathcal{SE}(F)$ is a semi-simplicial monoid the group operation on $\pi_n(\mathcal{SE}(\underline{F}))$ can be defined by composition, $[\underline{f}][\underline{g}] = [\underline{f}\circ\underline{g}]$ ([13], p. 68) and $\pi_n(\mathcal{SE}(\underline{F}))$ is Abelian for $n > 0$ ([13], p. 68).

There are natural actions of $\pi_0(\mathcal{SE}(\underline{F}))$ and $\pi_1(\mathcal{SE}(\underline{F}))$ on $\pi_n(\mathcal{SE}(\underline{F}))$. The latter action is the one normally studied; the topological treatment in [8], pp. 131–134 can be followed in our circumstance. The conclusion which will be useful for us is that we can ignore basepoints if we only consider mappings of S^n, $n > 0$, into the path component of $\mathcal{SE}(\underline{F})$ which contains the identity. I.e., $\pi_n(\mathcal{SE}(\underline{F}))$ is isomorphic to the group of homotopy classes of maps of S_n into $\mathcal{SE}(\underline{F})$ with the property that if $\underline{f}\colon \underline{F}\times S^n \to \underline{F}\times S^n$ represents such a map, then $\underline{f}\,|\,\underline{F}\times\{X_0\}$ is homotopic to the identity ([8], p. 133). We shall abuse notation and use $\pi_n(\mathcal{SE}(\underline{F}))$ for the latter group.

Let $S_0 = \{X_0, X_1\}$ and let $\underline{g}\colon \underline{F}\times S^0 \to \underline{F}\times S^0$ represent an element of $\pi_0(\mathcal{SE}(\underline{F}))$. Define $\underline{g}'\colon \underline{F} \to \underline{F}$ by the composition

$$\underline{F} \xrightarrow{\times\{X_1\}} \underline{F}\times\{X_1\} \xrightarrow{g} \underline{F}\times\{X_1\} \to \underline{F} .$$

Let $\underline{f}$ represent an element of $\pi_n(\mathcal{SE}(\underline{F}))$, $n > 0$. Define

$$\underline{g}^*(\underline{f}) = ((\underline{g}')^{-1}\times\mathrm{id}) \circ \underline{f} \circ (\underline{g}'\times\mathrm{id}) .$$

($(\underline{g}')^{-1}$ denotes some homotopy inverse of $\underline{g}'$.) It is easily checked this defines an action of $\pi_0(\mathcal{SE}(\underline{F}))$ on $\pi_n(\mathcal{SE}(\underline{F}))$ (using the abuse of notation as noted above). Let $\pi_n^*(\mathcal{SE}(\underline{F}))$ denote the orbit space of this action; i.e. the quotient space obtained from $\pi_n(\mathcal{SE}(\underline{F}))$ by identifying $[\underline{f}]$ with $[\underline{g}^*(\underline{f})]$ for all $[\underline{g}] \in \pi_0(\mathcal{SE}(\underline{F}))$.

The following are easily shown.

PROPOSITION 4.1. *Let* $\underline{f}\colon \underline{F}_0 \to \underline{F}_1$ *be a homotopy equivalence; then* $\underline{f}$ *induces a bijection* $\underline{f}^*\colon \pi_n^*(\mathcal{SE}(\underline{F}_0)) \to \pi_n^*(\mathcal{SE}(\underline{F}_1))$ *for all* n.

PROPOSITION 4.2. *Let* $\underline{f}, \underline{g}\colon \underline{E} \to \underline{E}'$ *be towers of maps.* $\underline{f} \simeq \underline{g}$ *if and only if there exists a tower of maps* $\underline{K}\colon \underline{E}\times[0,1] \to \underline{E}'$ *such that* $\underline{K}_0 \equiv \underline{f}$ *and* $\underline{K}_1 \equiv \underline{g}$.

5. Covering homotopy theorem. The main result of this section is a covering homotopy theorem for towers of maps into s-fibrations.

First we have need of the following two results whose proofs are exactly the same as the proofs of the corresponding results for Hurewicz fibrations [16], pp. 100–101.

PROPOSITION 5.1. *Suppose that* $p\colon E \to B$ *is an* s*-fibration with* t*-function* φ. *Let* $F_0, F_1\colon X\times[0,1] \to E_{\varphi(\alpha)}$ *be maps such that there exist homotopies* $H\colon X\times[0,1]\times[0,1] \to B$ *and* $G\colon X\times\{0\}\times[0,1] \to E_{\varphi(\alpha)}$ *with* $H(x,t,0) = p_{\varphi(\alpha)}F_0(x,t)$, $H(x,t,1) = p_{\varphi(\alpha)}F_1(x,t)$, $G(x,0,0) = F_0(x,0)$, $G(x,0,1) = F_1(x,0)$ *and* $p_{\varphi(\alpha)}G(x,0,t) = H(x,0,t)$. *Then there exists a homotopy* $H'\colon X\times[0,1]\times[0,1] \to E_\alpha$ *such that* $p_\alpha H' = H$, $H'\,|\,X\times\{0\}\times[0,1] = e_{\alpha\varphi(\alpha)}G$, $H'(x,t,0) = e_{\alpha\varphi(\alpha)}F_0(x,t)$ *and* $H'(x,t,1) = e_{\alpha\varphi(\alpha)}F_1(x,t)$.

COROLLARY 5.2. *Suppose that* $\underline{p}\colon \underline{E} \to B$ *is an* s*-fibration with* t*-function* φ. *Let* $F_0, F_1\colon X\times[0,1] \to E_{\varphi(\alpha)}$ *be maps such that* $F_0\,|\,X\times\{0\} = F_1\,|\,X\times\{0\}$ *and* $p_{\varphi(\alpha)}F_0 = p_{\varphi(\alpha)}F_1$. *Then there exists a homotopy* $H\colon X\times[0,1]\times[0,1] \to E_\alpha$ *such that* $H(x,t,0) = e_{\alpha\varphi(\alpha)}F_0(x,t)$, $H(x,t,1) = e_{\alpha\varphi(\alpha)}F_1(x,t)$, $H(x,0,t) = e_{\alpha\varphi(\alpha)}F_0(x,0) = e_{\alpha\varphi(\alpha)}F(x,0)$ *and* $p_\alpha H(x,t,s) = p_{\varphi(\alpha)}F_0(x,t)$.

THEOREM 5.3. *Suppose that* $\underline{p}\colon \underline{E} \to B$ *is an* s-*fibration with* t-*function* φ. *Let* $\underline{h}\colon \underline{D} \to \underline{E}$ *be a tower of maps and let* $\underline{H}\colon \underline{D}\times[0,1] \to B$ *be a map such that* $\underline{h}$ *is* $(\underline{H}, \underline{p}, \mathrm{id})$-*preserving: then there exists a tower of maps* $\underline{G}\colon \underline{D}\times[0,1]\to \underline{E}$ *such that* $p_\alpha G_\alpha(x,t) = H_{G(\alpha)}(x,t)$, $G_\alpha(x,0) = e_{\alpha\varphi^2(\alpha)}h_{\varphi^2(\alpha)}(x)$ *and* G *is* $(\underline{H}, \underline{p}, \mathrm{id})$-*preserving.*

Proof. Let $\alpha\in\Gamma$ and consider the homotopy $H_{h\varphi^2(\alpha)}$ and the map $h_{\varphi^2(\alpha)}$. Since $p_{\varphi^2(\alpha)}h_{\varphi^2(\alpha)}(x) = H_{h\varphi^2(\alpha)}(x,0)$, there exists a homotopy $G'_\alpha\colon D_{h\varphi^2(\alpha)}\times[0,1]\to E_{\varphi(\alpha)}$ such that $G'_\alpha(x,0) = e_{\varphi(\alpha)\varphi^2(\alpha)}h_{\varphi^2(\alpha)}(x)$ and $p_{\varphi(\alpha)}G'_\alpha = H_{h\varphi^2(\alpha)}$. Let $G(\alpha) = h_{\varphi^2(\alpha)}$ and define $G_\alpha(x,t) = e_{\alpha\varphi(\alpha)}G'(x,t)$.

First we show that $\{G_\alpha\}\colon \underline{D}\times[0,1]\to\underline{E}$ is a tower of maps. Suppose that $\beta\geqslant\alpha$. Let $d_{\gamma\delta}$ denote the bonding maps in $\underline{D}$. Consider the homotopies $G'_\alpha\circ(d_{G(\alpha)G(\beta)}\times\mathrm{id})$, $e_{\varphi(\alpha)\varphi(\beta)}\circ G'_\beta\colon D_{G(\beta)}\times[0,1]\to E_{\varphi(\alpha)}$;

$$\begin{aligned} G'_\alpha\circ(d_{G(\alpha)G(\beta)}\times\mathrm{id})(x,0) &= e_{\varphi(\alpha)\varphi^2(\alpha)}h_{\varphi^2(\alpha)}d_{G(\alpha)G(\beta)}(x)\\ &\simeq e_{\varphi(\alpha)\varphi^2(\alpha)}e_{\varphi^2(\alpha)\varphi^2(\beta)}h_{\varphi^2(\beta)}(x)\\ &= e_{\varphi(\alpha)\varphi(\beta)}G'_\beta(x,0)\end{aligned}$$

$$\begin{aligned} p_{\varphi(\alpha)}G'_\alpha\circ(d_{G(\alpha)G(\beta)}\times\mathrm{id}) &= H_{h\varphi^2(\alpha)}\circ(d_{G(\alpha)G(\beta)}\times\mathrm{id}) = H_{G(\beta)}\\ &= p_{\varphi(\beta)}G'_\beta = p_{\varphi(\alpha)}e_{\beta(\alpha)\varphi(\beta)}G'_\beta\,. \end{aligned}$$

We can apply Proposition 5.1 to get a homotopy $G_\alpha\circ(d_{G(\alpha)G(\beta)}\times\mathrm{id})\simeq e_{\alpha\varphi(\beta)}G_\beta$ so that $\{G_\alpha\}$ is a tower of maps which is (H, p, id)-preserving.

$$p_\alpha G_\alpha(x,t) = p_\alpha e_{\alpha\varphi(\alpha)}G'_\alpha(x,t) = p_{\varphi(\alpha)}G'_\alpha(x,t) = H_{G(\alpha)}(x,t)$$

and

$$G_\alpha(x,0) = e_{\alpha\varphi(\alpha)}G'_\alpha(x,0) = e_{\alpha\varphi(\alpha)}e_{\varphi(\alpha)\varphi^2(\alpha)}h_{\varphi^2(\alpha)}(x)\,.$$

By using Theorem 5.3, one can prove the analogues of Proposition 5.1 and Corollary 5.2 as in [16], pp. 100–101.

PROPOSITION 5.4. *Let* $\underline{p}\colon\underline{E}\to B$ *be an* s-*fibration and let* $\underline{F}, \underline{F}'\colon \underline{D}\times[0,1]\to\underline{E}$ *be towers of maps such that there exist a map* $\underline{H}\colon \underline{D}\times[0,1]\times[0,1]\to B$ *and a tower of maps* $\underline{G}\colon \underline{D}\times\{0\}\times[0,1]\to\underline{E}$ *for which*

$$\underline{G}\,|\,\underline{D}\times\{0\}\times\{0\} \equiv \underline{F}\,|\,\underline{D}\times\{0\}\,,\qquad \underline{G}\,|\,\underline{D}\times\{0\}\times\{1\}\equiv\underline{F}'\,|\,\underline{D}\times\{0\}$$

and $\underline{G}$ *is* $(\underline{H}, \underline{p}, \mathrm{id})$-*preserving. Then there exists a tower of maps* $\underline{G}'\colon \underline{D}\times[0,1]\times\times[0,1]\to\underline{E}$ *such that*

$$\underline{G}'\,|\,\underline{D}\times\{0\}\times[0,1]\equiv\underline{G}\,,\qquad \underline{G}'\,|\,\underline{D}\times[0,1]\times\{1\}\equiv F'\,,\qquad \underline{G}'\,|\,\underline{D}\times[0,1]\times\{0\}\equiv\underline{F}$$

and $\underline{G}'$ *is* $(\underline{H}, \underline{p}, \mathrm{id})$-*preserving.*

PROPOSITION 5.5. *Let* $\underline{p}\colon\underline{E}\to B$ *be an* s-*fibration and let* $\underline{F}$ *and* $\underline{F}'\colon\underline{D}\times[0,1]\to\underline{E}$ *be towers of maps such that there exists a tower of maps* $\underline{G}\colon\underline{D}\times\{0\}\times[0,1]\to\underline{E}$ *with*

$$\underline{G}\,|\,\underline{D}\times\{0\}\times\{0\}\equiv\underline{F}\,|\,\underline{D}\times\{0\},\qquad \underline{G}\,|\,\underline{D}\times\{0\}\times\{1\}\equiv\underline{F}'\,|\,\underline{D}\times\{0\}$$

and $\underline{p}\circ\underline{F} = \underline{p}\circ\underline{F}'$ *and* $\underline{G}$ *is* $(\underline{p}\circ\underline{F}, \underline{p}, \mathrm{id})$-*preserving. Then there exists a tower of maps* $\underline{H}\colon\underline{D}\times[0,1]\times[0,1]\to\underline{E}$ *such that* $\underline{H}$ *is* $(\underline{p}\circ\underline{F}, \underline{p}, \mathrm{id})$-*preserving for all* $t\in[0,1]$,

$$\underline{H}\,|\,\underline{D}\times\{0\}\times[0,1]\equiv\underline{G},\quad \underline{H}\,|\,\underline{D}\times[0,1]\times\{0\}\equiv\underline{F}\quad \textit{and}\quad \underline{H}\,|\,\underline{D}\times[0,1]\times\{0\}\equiv\underline{F}'.$$

6. Shape invariance of fibres. As the first major application of the covering homotopy Theorem 5.3, we shall prove that if $\underline{p}\colon\underline{E}\to B$ is an s-fibration with B path-connected then for $b, b'\in B$, $\underline{p}^{-1}(b)$ and $\underline{p}^{-1}(b')$ are homotopy equivalent. This is a generalization of an analogous result for approximate fibrations proved by Coram and Duvall [2]. The proof is analogous to the proof of the corresponding result for Hurewicz fibrations ([16], p. 101). Next we apply these techniques to simplify the detection of bundle maps.

Let $\underline{p}\colon\underline{E}\to B$ be an s-fibration and let $w\colon[0,1]\to B$ be a path. Consider the inclusion $\underline{h}\colon \underline{p}^{-1}(w(0))\to\underline{E}$ and define $\underline{H}\colon\underline{p}^{-1}(w(0))\times[0,1]\to B$ by $\underline{H} = \tilde{w}\circ(\underline{p}\times\mathrm{id})$ where $\tilde{w}(x,t) = w(t)$. By Theorem 5.3, there exists a tower of maps $\underline{G}\colon\underline{p}^{-1}(w(0))\times\times[0,1]\to\underline{E}$ such that $p_\alpha G_\alpha = H_{G(\alpha)}$ for all α and $G_\alpha(x,0) = e_{\alpha\varphi^2(\alpha)}h_{\varphi^2(\alpha)}(x)$ where φ is the t-function of $\underline{p}$. Define $\underline{f}\colon\underline{p}^{-1}(w(0))\to\underline{p}^{-1}(w(1))$ by the composition

$$\underline{p}^{-1}(w(0))\xrightarrow{\underline{\mathrm{id}}\times\{1\}}\underline{p}^{-1}(w(0))\times[0,1]\overset{\underline{G}}{\to}\underline{E}\,.$$

Let $w'\colon[0,1]\to B$ be a path such that $w'(i) = w(i)$ for $i = 0, 1$ and such that w is homotopic rel$\{0,1\}$ to w'. If we construct $\underline{G}'$ and $\underline{f}'$ corresponding to w' as above, then it follows from Proposition 5.4 that the towers of maps $\underline{f}, \underline{f}'\colon\underline{p}^{-1}(w(0))\to\underline{p}^{-1}(w(1))$ are homotopic.

Hence we have a functor L from the fundamental groupoid of B ([16], p. 101) to the category whose objects are towers of spaces and whose morphisms are homotopy classes of towers of maps.

Let w, w' be paths in B such that $w(1) = w'(0)$. Recall that $w*w'$ is the path defined by

$$w*w'(t) = \begin{cases} w(2t) & \text{for}\quad t\in[0,\tfrac12]\\ w'(2t-1) & \text{for}\quad t\in[\tfrac12,1]\,.\end{cases}$$

Suppose that $w'(t) = w'(0)$ for $0\leqslant t\leqslant\frac12$. Construct $\underline{G}$, $\underline{f}$, $\underline{G}'$ and $\underline{f}'$ as above. By the assumption on w', we may also assume that $G'_\alpha(x,t) = e_{\alpha G'(\alpha)}(x)$ for $0\leqslant t\leqslant\frac12$. Let $G''\colon\Gamma\to\Gamma$ be an increasing function such that $G''(\alpha)\geqslant G(\alpha), fG'(\alpha)$ [12]. Let $k_t\colon p^{-1}_{G''(\alpha)}(w(0))\to p_\alpha^{-1}(w(1))$ be a homotopy such that

$$k_0 = f_\alpha e_{G(\alpha)G''(\alpha)}(x)\qquad\text{and}\qquad k_1 = e_{\alpha G'(\alpha)}f_{G'(\alpha)}e_{fG'(\alpha)G''(\alpha)'}\,.$$

Define $G''_\alpha\colon p^{-1}_{G''(\alpha)}(w(0))\to E_\alpha$ by

$$G''_\alpha(x,t) = \begin{cases} G_\alpha(e_{G(\alpha)G''(\alpha)}(x), 2t) & \text{for}\quad 0\leqslant t\leqslant\frac12\,,\\ k_{4t-2}(x) & \text{for}\quad \frac12\leqslant t\leqslant\frac34\,,\\ G'_\alpha(f_{G'(\alpha)}e_{fG'(\alpha)G''(\alpha)}(x), 2t-1) & \text{for}\quad \frac34\leqslant t\leqslant 1\,.\end{cases}$$

Note that $G''_\alpha(x, 0) = e_{\alpha G''(\alpha)}(x)$ and $p_\alpha D''_\alpha(x, t) = w * w'(t)$. Consider

$$f''_\alpha(x) = G''_\alpha(x, 1) = G'_\alpha\big(f_{G'(\alpha)} e_{fG'(\alpha)G''(\alpha)}(x), 1\big) = f'_\alpha \circ f_{G'(\alpha)} \circ e_{fG'(\alpha)G''(\alpha)}(x)\,.$$

We have shown the following.

PROPOSITION 6.1. $L[w * w'] = \Gamma[w] \circ L[w']$. ("$\circ$" denotes composition of homotopy classes; — i.e. $[f] \circ [g] = [f \circ g]$.)

THEOREM 6.2. *Let* $\underline{p}\colon \underline{E} \to B$ *be an s-fibration and let* b_0, b_1 *lie in the same path component of* B. *Then there exists a homotopy equivalence* $\underline{p}^{-1}(b_0) \to \underline{p}^{-1}(b_1)$.

THEOREM 6.3. *Let* $\underline{p}\colon \underline{E} \to B$ *and* $\underline{p}'\colon \underline{E}' \to B'$ *be s-fibrations, let* $k\colon B \to B'$ *be a continuous map and let* $\underline{K}\colon \underline{E} \to \underline{E}'$ *be a* $(\underline{p}, \underline{p}', k)$*-preserving tower of maps. If there exists* $b_0 \in B$ *such that* $\underline{K}|\underline{p}^{-1}(b_0)\colon \underline{p}^{-1}(b_0) \to \underline{p}'^{-1}(k(b_0))$ *is a homotopy equivalence, then for all* b *which lie in the path component of* B *which contains* b_0, $\underline{K}|\underline{p}^{-1}(b)\colon \underline{p}^{-1}(b) \to \underline{p}^{-1}(k(b))$ *is a homotopy equivalence. In particular, if* B *is path-connected, then* K *is a bundle map.*

Proof. Let $w\colon [0, 1] \to B$ be a path such that $w(0) = b_0$ and $w(1) = b$. Construct $\underline{H}$, $\underline{G}$ and $\underline{f}$ as above. Then kw is a path in B' such that $kw(0) = k(b_0)$ and $kw(1) = k(b)$; construct the corresponding $\underline{H}'$, $\underline{G}'$ and $\underline{f}'$ as above. Let $\underline{K}_0 = \underline{K}|\underline{p}^{-1}(b_0)$ and $\underline{K}_1 = \underline{K}|\underline{p}^{-1}(b)$ and let $\underline{g}\colon \underline{p}'^{-1}(k(b_0)) \to \underline{p}^{-1}(b_0)$ be the homotopy inverse of $\underline{K}_0$.

Let $\underline{G}'' = \underline{K} \circ \underline{G} \circ (\underline{g} \times \mathrm{id})\colon \underline{p}'^{-1}(k(b_0)) \times [0, 1] \to E'$ and let $\underline{f}''\colon \underline{p}'^{-1}(k(b_0)) \to \underline{p}'^{-1}(k(b))$ be the composition

$$\underline{p}'^{-1}(k(b_0)) \xrightarrow{\mathrm{id} \times \{1\}} \underline{p}'^{-1}(k(b_0)) \times [0, 1] \xrightarrow{\underline{G}''} \underline{E}'\,.$$

Note that

$$\underline{p}'\underline{G}'' = \underline{p}'\underline{K}\underline{G}(\underline{g} \times \mathrm{id}) = k\underline{p}\underline{G}(\underline{g} \times \mathrm{id}) = k\underline{H}(\underline{g} \times \mathrm{id}) = k\tilde{w}(\underline{p} \times \mathrm{id}) = \underline{p}'\underline{G}'$$

and

$$\underline{G}''|\underline{p}'^{-1}(k(b_0) \times \{0\} = \underline{K} \circ (\underline{g} \times \mathrm{id})|\underline{p}'^{-1}(k(b_0)) \times \{0\}$$

which is homotopic to the identity map of $\underline{p}'^{-1}(k(b_0)) \times \{0\}$. By Proposition 5.5, this homotopy can be extended to a bundle homotopy between $\underline{G}''$ and $\underline{G}'$ which covers $\underline{p}'\underline{G}'$. In particular, $\underline{f}'$ and $\underline{f}''$ are homotopic; hence $\underline{f}''$ is a homotopy equivalence. But since $\underline{f}'' = \underline{K}_1 \circ \underline{f} \circ \underline{g}$, and since both $\underline{f}$ and $\underline{g}$ are homotopy equivalences, $\underline{K}_1$ is a homotopy equivalence.

7. Path lifting property. W. Hurewicz defined the concept of path lifting property of a map and showed its equivalence with the covering homotopy property [9]. We develop an analogous theory in this section which will be useful in the next two sections.

If $\underline{E} = \{E_\alpha, e_{\alpha\beta}, \Gamma\}$ is a tower of spaces, let $\underline{E}^{[0,1]} = \{E_\alpha^{[0,1]}, \tilde{e}_{\alpha\beta}, \Gamma\}$ denote the tower of spaces where $E_\alpha^{[0,1]}$ is the collection of paths in E_α with the compact-open topology and $\tilde{e}_{\alpha\beta}(\varphi)(t) = e_{\alpha\beta}(\varphi(t))$.

Let $\underline{p}\colon \underline{E} \to B$ be a map. For $\alpha \in \Gamma$, let

$$\tilde{B}_\alpha = \{(e, w) \in E_\alpha \times B^{[0,1]} \mid w(0) = p_\alpha(e)\}$$

and define $b_{\alpha\beta}\colon \tilde{B}_\beta \to \tilde{B}_\alpha$ by $b_{\alpha\beta}(e, w) = (e_{\alpha\beta}(e), w)$. $\underline{\tilde{B}} = \{\tilde{B}_\alpha, b_{\alpha\beta}, \Gamma\}$ is a tower of spaces. Define $\underline{\tilde{p}}\colon \underline{E}^{[0,1]} \to \underline{\tilde{B}}$ by $\tilde{p}_\alpha(w) = (w(0), p_\alpha \circ w)$; it is straightforward to check that $\underline{\tilde{p}}\colon \underline{E}^{[0,1]} \to \underline{\tilde{B}}$ is a tower of maps.

Define $\underline{q}\colon \underline{\tilde{B}} \to B^{[0,1]}$ to be projection along the second coordinate. Let $\underline{r}\colon \underline{E}^{[0,1]} \to B^{[0,1]}$ be the map induced by $\underline{p}$. Note that $\underline{\tilde{p}}$ is $(\underline{r}, \underline{q}, \mathrm{id})$-preserving. A *lifting function for* $\underline{p}$ is a $(\underline{q}, \underline{r}, \mathrm{id})$-preserving tower of maps $\underline{\lambda}\colon \underline{\tilde{B}} \to \underline{E}^{[0,1]}$ such that $\underline{\tilde{p}} \circ \underline{\lambda} = \mathrm{id}$.

THEOREM 7.1. $\underline{p}\colon \underline{E} \to B$ *is an s-fibration if and only if there exists a lifting function for* p.

Proof. Suppose that $\underline{p}$ is an s-fibration and let φ be the t-function for $\underline{p}$. Define $f_\alpha\colon \tilde{B}_{\varphi^2(\alpha)} \to E_{\varphi^2(\alpha)}$ by $f_\alpha(e, w) = e$ and $F_\alpha\colon \tilde{B}_{\varphi^2(\alpha)} \times [0, 1] \to B$ by $F_\alpha((e, w), t) = w(t)$. Note that $F_\alpha((e, w), 0) = w(0) = p_{\varphi^2(\alpha)}(e) = p_{\varphi^2(\alpha)} f_\alpha(e, w)$. Hence there exists a homotopy $F'_\alpha\colon \tilde{B}_{\varphi^2(\alpha)} \times [0, 1] \to E_{\varphi(\alpha)}$ such that $p_{\varphi(\alpha)} F'_\alpha = F_\alpha$ and

$$F'_\alpha((e, w), 0) = e_{\varphi(\alpha)\varphi^2(\alpha)} f_\alpha(e, w) = e_{\varphi(\alpha)\varphi^2(\alpha)}(e)\,.$$

Define $\lambda_\alpha\colon \tilde{B}_{\varphi^2(\alpha)} \to E_\alpha^{[0,1]}$ by $\lambda_\alpha(e, w)(t) = e_{\alpha\varphi(\alpha)} F'_\alpha((e, w), t)$. It follows from Corollary 5.2 as in the proof of Theorem 5.3 that $\{\lambda_\alpha\}\colon \underline{\tilde{B}} \to \underline{E}^{[0,1]}$ is a tower of maps which is $(\underline{q}, \underline{r}, \mathrm{id})$-preserving. Consider

$$\begin{aligned}\tilde{p}_\alpha \circ \lambda_\alpha(e, w) &= \big(\lambda_\alpha(e, w)(0), p_\alpha \circ \lambda_\alpha(e, w)\big)\\ &= \big(e_{\alpha\varphi(\alpha)} F'_\alpha((e, w), 0), w\big)\\ &= \big(e_{\alpha\varphi^2(\alpha)}(e), w\big)\\ &= b_{\alpha\varphi^2(\alpha)}(e, w)\,.\end{aligned}$$

Hence $\{\lambda_\alpha\}$ is a lifting function for $\underline{p}$.

Now suppose that there exists a lifting function $\underline{\lambda} = \{\lambda_\alpha\}$ for $\underline{p}$. Let $\alpha \in \Gamma$ and let $f\colon X \to E_{\lambda(\alpha)}$ and $F\colon X \times [0, 1] \to B$ be maps such that $p_{\lambda(\alpha)} f(x) = F(x, 0)$. Define $g\colon X \to B^{[0,1]}$ by $g(x)(t) = F(x, t)$ and define $H\colon X \times [0, 1] \to E_\alpha$ by $H(x, t) = \lambda_\alpha(f(x), g(x))(t)$. H is the desired lifting.

A lifting function $\underline{\lambda} = \{\lambda_\alpha\}$ is *regular* if for every $x \in E_{\lambda(\alpha)}$, $\lambda_\alpha(x, p_{\lambda(\alpha)} x) = e_{\alpha\lambda(\alpha)}(x)$; — i.e. degenerate paths are lifted into degenerate paths. By using exactly the same proof as in [9], p. 957 one can show the following.

COROLLARY 7.2. *Let* B *be metric;* $\underline{p}\colon \underline{E} \to B$ *is an s-fibration if and only if there exists a regular lifting function for* $\underline{p}$.

We need the following proposition which is an analogue of a result of E. Fadell [4].

PROPOSITION 7.3. *Let* $\underline{p}\colon \underline{E} \to B$ *be an s-fibration and let* $\underline{\lambda}\colon \underline{\tilde{B}} \to \underline{E}^{[0,1]}$ *be a lifting function for* $\underline{p}$. *Define* $\underline{\tilde{\lambda}}\colon \underline{E}^{[0,1]} \to \underline{E}^{[0,1]}$ *by*

$$\tilde{\lambda}_\alpha(w) = \lambda_\alpha(w(0), p_{\lambda(\alpha)} \circ w)$$

where $w \in E_{\lambda(\alpha)}^{[0,1]}$. *Then there exists a homotopy* $\underline{\Phi}\colon \underline{E}^{[0,1]} \times [0, 1] \to \underline{E}^{[0,1]}$ *such that* $\underline{\Phi}|\underline{E}^{[0,1]} \times \{0\} \equiv \underline{\mathrm{id}}$, $\underline{\Phi}|\underline{E}^{[0,1]} \times \{1\} \equiv \underline{\tilde{\lambda}}$ *and* $\underline{\Phi}$ *covers the projection*

$$B^{[0,1]} \times [0, 1] \to B^{[0,1]}\,.$$

Proof. Let φ be the t-function for $\underline{p}$ and let $w \in E^{[0,1]}_{\lambda\varphi(\alpha)}$. Define $\sigma = p_{\lambda\varphi(\alpha)} \circ w$ and $\sigma^{1-s} \in B^{[0,1]}$ by

$$\sigma^{1-s}(t) = \begin{cases} \sigma(s+t) & \text{for} \quad t \in [0, 1-s], \\ \sigma(1) & \text{for} \quad t \in [1-s, 1]. \end{cases}$$

Define $H_\alpha\colon E^{[0,1]}_{\lambda\varphi(\alpha)} \times [0, 1] \to E^{[0,1]}_\alpha$ by

$$H_\alpha(w, s)(t) = \begin{cases} e_{\alpha\lambda\varphi(\alpha)} w(t) & \text{for} \quad t \in [0, s], \\ e_{\alpha\varphi(\alpha)} \lambda_{\varphi(\alpha)}[w(s), \sigma^{1-s}](t-s) & \text{for} \quad t \in [s, 1]. \end{cases}$$

By Proposition 5.1, $\underline{\Phi} = \{H_\alpha\}\colon \underline{E}^{[0,1]} \times [0, 1] \to \underline{E}^{[0,1]}$ is a tower of maps; it is easily checked that $\underline{\Phi}$ has the desired properties.

8. Pullbacks. Let $\xi = (\underline{p}, \underline{E}, B)$ be an s-fibration and let $f\colon B' \to B$ be a continuous map. For $\alpha \in \Gamma$, let $E'_\alpha = \{(b, x) \in B' \times E_\alpha \mid f(b) = p_\alpha(x)\}$ with the subspace topology.

If $\beta \geqslant \alpha$, define

(i) $e'_{\alpha\beta}\colon E'_\beta \to E'_\alpha$ by $e_{\alpha\beta}(b, x) = (b, e_{\alpha\beta}(x))$,

(ii) $F_\alpha\colon E'_\alpha \to E_\alpha$ by $F_\alpha(b, x) = x$,

(iii) $F\colon \Gamma \to \Gamma$ by $F(\alpha) = \alpha$,

(iv) $p'_\alpha\colon E'_\alpha \to B'$ by $p'_\alpha(b, x) = b$.

It is easily checked that $\underline{p}' = \{p'_\alpha\}\colon \underline{E}' = \{E'_\alpha\} \to B'$ is a map.

Theorem 8.1. *$\underline{p}'\colon \underline{E}' \to B'$ is an s-fibration with the same t-function as $\underline{p}$ and $(\underline{F}, f)$ is a bundle map.*

Proof. Given $\alpha \in \Gamma$, choose $\beta = \varphi(\alpha)$ where φ is the t-function for p. Let $g\colon X \to E'_\beta$ and $H\colon X \times [0, 1] \to B'$ be maps such that $p'_\beta g(x) = H(x, 0)$. Find $\tilde{G}\colon X \times [0, 1] \to E_\alpha$ such that $p_\alpha \tilde{G} = fH$ and $e_{\alpha\beta} F_\beta g(x) = \tilde{G}(x, 0)$. $G(x, t) = (H(x, t), \tilde{G}(x, t))$ defines the desired homotopy. Hence $\underline{p}'$ is an s-fibration. Since $F_\alpha | p'^{-1}_\alpha(x)\colon p'^{-1}_\alpha(x) \to p^{-1}_\alpha(f(x))$ is a homeomorphism, $(\underline{F}, f)$ is a bundle map.

$f^*\xi = (\underline{p}', \underline{E}', B')$ is called the pullback of ξ by f.

Theorem 8.2. *Let $\xi = (\underline{p}, \underline{E}, B)$ be an s-fibration and let $f_0, f_1\colon X \to B$ be homotopic maps. Then $f_0^*\xi$ and $f_1^*\xi$ are bundle equivalent.*

Proof (see [16], p. 102). Let $f_0^*\xi = (\underline{p}^0, \underline{E}^0, X)$ and $f_1^*\xi = (\underline{p}^1, \underline{E}^1, X)$ and let $(\underline{F}^0, f_0)\colon f_0^*\xi \to \xi$ and $(\underline{F}^1, f_1)\colon f_1^*\xi \to \xi$ be the bundle maps constructed above. Let $F\colon X \times [0, 1] \to B$ be a homotopy such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. By Theorem 5.3, there exist towers of maps $\underline{G}^0\colon \underline{E}^0 \times [0, 1] \to \underline{E}$ and $\underline{G}^1\colon \underline{E}^1 \times [0, 1] \to \underline{E}$ such that

$$p_\alpha G^i_\alpha(x, t) = F\big(p^i_{G^i(\alpha)}(x), t\big), \quad i = 0, 1,$$
$$G^0_\alpha(x, 0) = e_{\alpha\varphi^2(\alpha)} F^0_{\varphi^2(\alpha)}(x), \quad G^1_\alpha(x, 1) = e_{\alpha\varphi^2(\alpha)} F^1_{\varphi^2(\alpha)}(x).$$

Define $\underline{g}^0\colon \underline{E}^0 \to \underline{E}^1$ and $\underline{g}^1\colon \underline{E}^1 \to \underline{E}^0$ by

$$g^0_\alpha(b, x) = \big(b, G^0_\alpha((b, x), 1)\big) \quad \text{for} \quad (b, x) \in E^0_{G^0(\alpha)},$$
$$g^1_\alpha(b, x) = \big(b, G^1_\alpha((b, x), 0)\big) \quad \text{for} \quad (b, x) \in E^1_{G^1(\alpha)}.$$

Note that

$$\underline{G}^0(\underline{g}^1 \times \mathrm{id}) | \underline{E}^1 \times \{0\} \equiv \underline{G}^1 | \underline{E}^1 \times \{0\}$$

and

$$\underline{p}\underline{G}^0(\underline{g}^1 \times \mathrm{id}) = F(\underline{p}^0 \times \mathrm{id})(\underline{g}^1 \times \mathrm{id}) = F(\underline{p}^1 \times \mathrm{id}) = \underline{p}\underline{G}^1.$$

By Proposition 5.5, $(\underline{G}^0(\underline{g}^1 \times \mathrm{id}), F)$ and $(\underline{G}^1, F)$ are bundle homotopic. Similarly, $(\underline{G}^0, F)$ and $(\underline{G}^1(\underline{g}^0 \times \mathrm{id}), F)$ are bundle homotopic. Since $\underline{G}^1(\underline{g}^0 \underline{g}^1 \times \mathrm{id}) \simeq_b \underline{G}^1$, the restriction of this bundle homotopy to $\underline{E}^1 \times \{1\}$ induces a bundle homotopy $\underline{F}^1 \underline{g}^0 \underline{g}^1 \simeq_b \underline{F}^1$ which covers f_1. This, in turn, induces a bundle homotopy $\underline{g}^0 \underline{g}^1 \simeq_b \mathrm{id}$ covering $\mathrm{id}\colon X \to X$. Similarly $\underline{g}^1 \underline{g}^0 \simeq_b \mathrm{id}$.

Corollary 8.3. *Let $\xi = (\underline{p}, \underline{E}, B)$ be an s-fibration and let B be contractible; then ξ is bundle equivalent to the trivial s-fibration $\underline{p}^{-1}(b_0) \times B \to B$ for any $b_0 \in B$.*

9. Homotopy extension theorem. The following is the main result of this section. The analogous theorem for Hurewicz fibrations was proved by Fadell [5]; this proof is very similar.

Theorem 9.1. *Let $\underline{p}\colon \underline{E} \to B$ and $\underline{p}'\colon \underline{E}' \to B$ be s-fibrations where B is a polyhedron and let A be a subpolyhedron of B. Let $X = (A \times [0, 1]) \cup (B \times \{0\})$ and $\underline{T} = (\underline{p} \times \mathrm{id})^{-1}(X)$ where $\underline{p} \times \mathrm{id}\colon \underline{E} \times [0, 1] \to B \times [0, 1]$. Let $\underline{\varphi}\colon \underline{T} \to \underline{E}'$ be a bundle map covering the map $X \to B$ defined by $(y, t) \to y$. Then there exists a bundle map $\underline{\Phi}\colon \underline{E} \times [0, 1] \to \underline{E}'$ covering the projection $B \times [0, 1] \to B$ such that $\underline{\Phi} | \underline{T} \equiv \underline{\varphi}$.*

Proof. Let U be an open set in $B \times [0, 1]$ such that $X \subseteq U$ and X is a strong deformation retract of U; let $H\colon U \times [0, 1] \to U$ be a homotopy such that $H(x, 0) = x$, $H(x, 1) \in X$ and $H(x, t) = x$ for all $x \in X$, $t \in [0, 1]$. Let $\tilde{H}\colon U \to U^{[0,1]}$ be defined by $\tilde{H}(x)(t) = H(x, t)$. Let $V_\alpha = (p_\alpha \times \mathrm{id})^{-1} U$. Consider the projections

$$\pi\colon B \times [0, 1] \to B, \quad \underline{\pi}\colon \underline{E} \times [0, 1] \to \underline{E}, \quad \varrho\colon B \times [0, 1] \to [0, 1].$$

Let $\tilde{\pi}\colon (B \times [0, 1])^{[0,1]} \to B^{[0,1]}$ be induced by π and define $\psi_\alpha\colon V_\alpha \to B^{[0,1]}$ by $\psi_\alpha = \tilde{\pi} \circ \tilde{H} \circ (p_\alpha \times \mathrm{id}) | V_\alpha$ and $\chi_\alpha\colon V_\alpha \to [0, 1]$ by $\chi_\alpha = \varrho \circ H_1 \circ (p_\alpha \times \mathrm{id}) | V_\alpha$. Define $\bar{\psi}_\alpha\colon V_\alpha \to B^{[0,1]}$ by $\bar{\psi}_\alpha(v)(t) = \psi((v)(1-t))$.

Let $\underline{\lambda}$ and $\underline{\lambda}'$ be regular lifting functions for $\underline{p}$ and $\underline{p}'$, respectively (cf. Corollary 7.2). Define $\Phi\colon \Gamma' \to \Gamma$ by $\Phi(\alpha) = \lambda\varphi\lambda'(\alpha)$ and define $\Phi'_\alpha\colon V_{\Phi(\alpha)} \to E'_\alpha$ by

$$\Phi'_\alpha(v) = \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}\big(\pi_{\Phi(\alpha)}(v), \psi_{\Phi(\alpha)}(v)\big)(1), \chi_{\Phi(\alpha)}(v)], \bar{\psi}_{\Phi(\alpha)}(v)\big)(1).$$

Let W be an open set in B such that $A \subseteq W$ and $A \times [0, 1] \subseteq W \times [0, 1] \subseteq U$. Let $f\colon B \to [0, 1]$ be a continuous function such that $f(A) = 1$ and $f(B - W) = 0$. Define $\Phi_\alpha\colon E_{\Phi(\alpha)} \times [0, 1] \to E'_\alpha$ by $\Phi_\alpha(y, t) = \Phi'_\alpha\big(y, f(p_{\Phi(\alpha)}(y))t\big)$.

We first show that $\{\Phi_\alpha\}\colon \underline{E} \times [0, 1] \to \underline{E}'$ is a tower of maps. Let $\alpha \leqslant \beta$, then

$$\begin{aligned} &\Phi_\alpha\big(e_{\Phi(\alpha)\Phi(\beta)}(y), t\big) \\ &\quad = \Phi'_\alpha\big(e_{\Phi(\alpha)\Phi(\beta)}(y), f\big(p_{\Phi(\alpha)} e_{\Phi(\alpha)\Phi(\beta)}(y)\big) \cdot t\big) \\ &\quad = \Phi'_\alpha(\bar{v}) \quad \text{where } \bar{v} = \big(e_{\Phi(\alpha)\Phi(\beta)}(y), f\big(p_{\Phi(\beta)}(y)\big) \cdot t\big) \end{aligned}$$

$$\begin{aligned}
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(\pi_{\Phi(\alpha)}(\bar{v}), \psi_{\Phi(\alpha)}(\bar{v}))(1), \chi_{\Phi(\sigma)}(v)], \bar{\psi}_{\Phi(\alpha)}(\bar{v})\big)(1)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(e_{\Phi(\alpha)\Phi(\beta)}(y), \psi_{\Phi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\\
&\quad \text{where } v = (y, fp_{\Phi(\beta)}(y)\cdot t)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)} \circ (e_{\Phi(\alpha)\Phi(\beta)} \times \mathrm{id})(y, \psi_{\varphi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\\
&\simeq \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[e_{\varphi\lambda'(\alpha)\varphi\lambda'(\beta)} \circ \lambda_{\varphi\lambda'(\beta)}(y, \psi_{\Phi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)} \circ (e_{\varphi\lambda'(\alpha)\varphi\lambda'(\beta)} \times \mathrm{id})[\lambda_{\varphi\lambda'(\beta)}(y, \psi_{\Phi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\\
&\simeq \lambda'_\alpha \circ (e_{\lambda'(\alpha)\lambda'(\beta)} \times \mathrm{id})\big(\varphi_{\lambda'(\beta)}[\lambda_{\varphi\lambda'(\beta)}(y, \psi_{\Phi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\\
&\simeq e_{\alpha\beta}\big[\lambda'_\beta\big(\varphi_{\lambda'(\beta)}[\lambda_{\varphi\lambda'(\beta)}(y, \psi_{\Phi(\beta)}(v))(1), \chi_{\Phi(\beta)}(v)], \bar{\psi}_{\Phi(\beta)}(v)\big)(1)\big]\\
&= e_{\alpha\beta}\Phi'_\beta(y, fp_{\Phi(\beta)}(y)t) = e_{\alpha\beta}\Phi_\beta(y, t)\,.
\end{aligned}$$

We now show that $\{\Phi_\alpha\}$ covers the projection $B\times[0,1]\to B$.

$$\begin{aligned}
p'_\alpha\Phi_\alpha(y,t) &= p'_\alpha\Phi'_\alpha\big(y, f(p_{\Phi(\alpha)}(y))t\big)\\
&= p'_\alpha\big[\lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(\pi_{\Phi(\alpha)}(v), \psi_{\Phi(\alpha)}(v))(1), \chi_{\Phi(\alpha)}(v), \bar{\psi}_{\Phi(\alpha)}(v)\big)(1)\big]\\
&= \bar{\psi}_{\Phi(\alpha)}(v)(1) \qquad \text{where } v = (y, fp_{\Phi(\alpha)}(y)t)\\
&= \psi_{\Phi(\alpha)}(v)(0)\\
&= (\tilde{\pi} \circ \tilde{H} \circ (p_{\Phi(\alpha)} \times \mathrm{id})(v))(0)\\
&= \tilde{\pi} \circ \tilde{H}(p_{\Phi(\alpha)}y, fp_{\Phi(\alpha)}(y)t)(0)\\
&= \pi H\big((p_{\Phi(\alpha)}y, f_{p\Phi(\alpha)}(y)t), 0\big) = \pi(p_{\Phi(\alpha)}y, fp_{\Phi(\alpha)}(y)t) = p_{\Phi(\alpha)}y\,.
\end{aligned}$$

Suppose that $(y,t)\in T_{\Phi(\alpha)}$; then

$$\begin{aligned}
\Phi_\alpha(y,t) &= \Phi'_\alpha\big(y, f(p_{\Phi(\alpha)}(y))t\big) = \Phi'_\alpha(y,t)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(y, \psi_{\Phi(\alpha)}(y,t))(1), \chi_{\Phi(\alpha)}(y,t)], \bar{\psi}_{\Phi(\alpha)}(y,t)\big)(1)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(y, \psi_{\Phi(\alpha)}(y,t))(1), t], \bar{\psi}_{\Phi(\alpha)}(y,t)\big)(1)\,.
\end{aligned}$$

Since $(y,t)\in T_{\Phi(\alpha)}$, $(p_{\Phi(\alpha)}(y), t)\in X$ and $H\big((p_{\Phi(\alpha)}(y), t), s\big) = (p_{\Phi(\alpha)}(y), t)$ for all $s\in[0,1]$. Thus $\psi_{\Phi(\alpha)}(y,t) = \tilde{\pi}\circ\tilde{H}\circ(p_{\Phi(\alpha)}\times\mathrm{id})(y,t)$ is the constant path which will denote by its image, $p_{\Phi(\alpha)}(y)$. Hence,

$$\begin{aligned}
\Phi_\alpha(y,t) &= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}[\lambda_{\varphi\lambda'(\alpha)}(y, p_{\Phi(\alpha)}(y))(1), t], p_{\Phi(\alpha)}(y)\big)(1)\\
&= \lambda'_\alpha\big(\varphi_{\lambda'(\alpha)}e_{\varphi\lambda'(\alpha),\Phi(\alpha)}[y, t], p_{\Phi(\alpha)}(y)\big)(1)\\
&= e'_{\alpha\lambda'(\alpha)}\varphi_{\lambda'(\alpha)}e_{\varphi\lambda'(\alpha),\Phi(\alpha)}(y,t)\,.
\end{aligned}$$

Thus $\underline{\Phi}|I \equiv \underline{\varphi}$. The fact that $\underline{\Phi}$ is a bundle map follows from Theorem 6.3.

COROLLARY 9.2. *Let $\xi = (\underline{p}, \underline{E}, B)$ be an s-fibration and let B be contractible. Let $x_0\in B$; then there exists a bundle equivalence $\underline{F}\colon \underline{E}\to \underline{p}^{-1}(b_0)\times B$ such that $\underline{F}|p^{-1}(b_0) \equiv \underline{\mathrm{id}}\times\{\mathrm{id}\}$.*

See Corollary 8.3.

10. Bundle maps and bundle equivalences. The main result in this section is that a bundle map of an s-fibration over a compact polyhedron covering the identity is a bundle equivalence. The proof is modelled on an argument of E. Fadell [5] to prove an analogous result for Hurewicz fibrations.

LEMMA 10.1 *Let $\underline{p}\colon \underline{E}\to\Delta^n$ be an s-fibration where Δ^n is the n-cell and let $S = (\partial\Delta^n\times[0,1])\cup(\Delta^n\times\{0\})\cup(\Delta^n\times\{1\})$. Let $\underline{D} = (\underline{p}\times\mathrm{id})^{-1}(S)$. If $\underline{f}\colon \underline{D}\to\underline{D}$ is a bundle equivalence such that $\underline{f}|\underline{E}\times\{0\}\equiv\underline{\mathrm{id}}$, then there is a bundle equivalence $\underline{g}\colon \underline{D}\to\underline{D}$ such that $\underline{g}|(\underline{p}^{-1}(\partial\Delta^n)\times[0,1])\cup(\underline{E}\times\{0\})\equiv\underline{\mathrm{id}}$ and the bundle map $\underline{g}\circ\underline{f}\colon \underline{D}\to\underline{D}$ admits an extension $\underline{\varphi}\colon \underline{E}\times[0,1]\to\underline{E}\times[0,1]$ which is also a bundle equivalence.*

Proof. Let $\underline{F} = \underline{p}^{-1}(x_0)$ be the fibre of $\underline{p}$ where x_0 is a vertex of Δ^n. By Corollary 9.2, there exists a bundle equivalence $\underline{\beta}\colon \underline{E}\to\underline{F}\times\Delta^n$ such that $\underline{\beta}|\underline{p}^{-1}(x_0)\equiv\underline{\mathrm{id}}\times\{x_0\}$. We may choose $\underline{\beta}^{-1}$ so that $\underline{\beta}^{-1}|F\times\{x_0\}\to\underline{p}^{-1}(x_0)$ is (the shift of the) projection along the first factor. $\underline{\beta}$ induces bundle equivalences $\underline{\beta}\times\mathrm{id}\colon \underline{E}\times[0,1]\to\underline{F}\times\Delta^n\times[0,1]$ and $\underline{\beta}'\colon \underline{D}\to\underline{F}\times S$ where $\underline{\beta}' = \underline{\beta}\times\mathrm{id}|\underline{D}$.

Consider $\underline{f}' = \underline{\beta}'\circ\underline{f}\circ\underline{\beta}'^{-1}\colon \underline{F}\times S\to\underline{F}\times S$. If we choose $(x_0, 0)$ to be a base point for S, then $\underline{f}'$ represents an element of $\pi_n(\mathscr{S}\mathscr{E}(\underline{F}), \underline{\mathrm{id}})$. Let $\underline{g}'\colon \underline{F}\times S\to\underline{F}\times S$ represent the inverse of the class of $\underline{f}'$; we may assume that

$$\underline{g}'|\underline{F}\times((\partial\Delta^n\times[0,1])\cup(\Delta^n\times\{0\}))\equiv\underline{\mathrm{id}}\,.$$

Let $\underline{\tilde{g}} = \underline{\beta}'^{-1}\circ\underline{g}'\circ\underline{\beta}'$.

Since $\mathscr{S}\mathscr{E}(\underline{F})$ is a simplicial group and by the choice of $\underline{g}'$, there exists an extension $\underline{\varphi}'\colon \underline{F}\times\Delta^n\times[0,1]\to\underline{F}\times\Delta^n\times[0,1]$ of $\underline{g}'\circ\underline{f}'$. Let $\underline{\tilde{\varphi}} = (\underline{\beta}^{-1}\times\mathrm{id})\circ\underline{\varphi}'\circ(\underline{\beta}\times\mathrm{id})$. $\underline{\tilde{\varphi}}|\underline{D}\equiv\underline{\beta}'^{-1}\circ\underline{g}'\circ\underline{f}'\circ\underline{\beta}' = \underline{\beta}'^{-1}\circ\underline{g}'\circ\underline{\beta}'\circ\underline{f}\circ\underline{\beta}'^{-1}\circ\underline{\beta}'$ which is bundle homotopic to $\underline{\beta}'^{-1}\circ\underline{g}'\circ\underline{\beta}'\circ\underline{f} = \underline{\tilde{g}}\circ\underline{f}$. By Theorem 9.1, $\underline{\tilde{g}}\circ\underline{f}$ is extendable to a bundle map $\underline{\varphi}''\colon \underline{E}\times[0,1]\to\underline{E}\times[0,1]$.

$$\begin{aligned}
&\underline{\tilde{g}}|(\underline{p}^{-1}(\partial\Delta^n)\times[0,1])\cup(\underline{E}\times\{0\})\\
&\qquad = \underline{\beta}'^{-1}\circ\underline{q}'\circ\underline{\beta}'|(\underline{p}^{-1}(\partial\Delta^n)\times[0,1])\cup(\underline{E}\times\{0\})\\
&\qquad \equiv \underline{\beta}'^{-1}\circ\underline{\beta}'|(\underline{p}^{-1}(\partial\Delta^n)\times[0,1])\cup(E\times\{0\})\,.
\end{aligned}$$

Since $\underline{\beta}'^{-1}\circ\underline{\beta}'$ is bundle homotopic to $\underline{\mathrm{id}}$, we apply Theorem 9.1 to obtain a bundle map $\underline{g}\colon \underline{D}\to\underline{D}$ such that $\underline{g}|(\underline{p}^{-1}(\partial\Delta^n)\times[0,1])\cup(\underline{E}\times\{0\})\equiv\underline{\mathrm{id}}$ and $\underline{g}$ is bundle homotopic to $\underline{\tilde{g}}$. Since $\underline{g}\circ\underline{f}$ is bundle homotopic to $\underline{\tilde{g}}\circ\underline{f}$, the existence of the extension $\underline{\varphi}$ follows from a third application of Theorem 9.1.

LEMMA 10.2. *Let $\underline{p}\colon \underline{E}\to\Delta^n$ and $\underline{p}'\colon \underline{E}'\to\Delta^n$ be s-fibrations and let $\underline{H}\colon \underline{E}\to\underline{E}'$ be a bundle map covering the identity. Let $\underline{g}\colon \underline{p}'^{-1}(\partial\Delta^n)\to\underline{p}^{-1}(\partial\Delta^n)$ be a bundle map covering the identity such that $\underline{g}\circ\underline{H}|\underline{p}^{-1}(\partial\Delta^n)$ is bundle homotopic to $\underline{\mathrm{id}}|\underline{p}^{-1}(\partial\Delta^n)$ by the homotopy $\underline{\xi}\colon \underline{p}^{-1}(\partial\Delta^n)\times[0,1]\to\underline{p}^{-1}(\partial\Delta^n)$ which covers the projection $\partial\Delta^n\times[0,1]\to\partial\Delta^n$. Then there exists a bundle map $\underline{G}\colon \underline{E}'\to\underline{E}$ covering the identity such that $\underline{G}\circ\underline{H}$ is bundle homotopic to $\underline{\mathrm{id}}$ by a homotopy $\underline{\Phi}$ which extends $\underline{\xi}$ and covers the projection $\Delta^n\times[0,1]\to\Delta^n$.*

SUBLEMMA. *Let x_0 be a vertex of Δ^n. Define $C\colon \Delta^n\to(\Delta^n)^{[0,1]}$ by $C(x)(t) \doteq (1-t)x+tx_0$. Let $\underline{\lambda}$ and $\underline{\lambda}'$ be regular lifting functions for $\underline{p}$ and $\underline{p}'$, respectively. Let $\underline{H}^0 = \underline{H}|\underline{p}^{-1}(x_0)$ and let $\underline{G}'$ be a homotopy inverse for $\underline{H}^0$. Define $\underline{\tilde{H}}\colon \underline{E}\to\underline{E}'$ and $\underline{\tilde{G}}\colon \underline{E}'\to\underline{E}$ by*

$$\begin{aligned}\tilde{H}(\alpha) &= \lambda H\lambda'(\alpha) \quad \text{for} \quad \alpha\in\Gamma',\\ \tilde{G}(\alpha) &= \lambda' G'\lambda(\alpha) \quad \text{for} \quad \alpha\in\Gamma,\\ \tilde{H}_\alpha(y) &= \lambda'_\alpha\{H^0_{\lambda'(\alpha)}[\lambda_{H\lambda'(\alpha)}(y,\, C(p_{\lambda H\lambda'(\alpha)}(y)))(1)],\, \bar{C}(p_{\lambda H\lambda'(\alpha)}(y))\}(1),\\ \tilde{G}_\alpha(y) &= \lambda_\alpha\{G'_{\lambda(\alpha)}[\lambda'_{G'\lambda(\alpha)}(y,\, C(p'_{\lambda G'\lambda'(\alpha)}(y)))(1)],\, \bar{C}(p'_{\lambda G'\lambda'(\alpha)}(y))\}(1)\end{aligned}$$

where $\bar{C}(x)(t) = C(x)(1-t)$.

Then $\underline{\tilde{H}}$ *is a bundle equivalence which is bundle homotopic to* $\underline{H}$, $\underline{\tilde{G}}$ *is an inverse for* $\underline{\tilde{H}}$ *and* $\underline{\tilde{G}}|\underline{p}'^{-1}(\partial\Delta'')$ *is bundle homotopic to* $\underline{g}$.

Proof. Define $\underline{\tilde{\lambda}}'\colon (\underline{E}')^{[0,1]}\to(\underline{E}')^{[0,1]}$ as in Proposition 7.3 and let $\underline{\Phi}\colon (\underline{E}')^{[0,1]}\times[0,1]\to(\underline{E}')^{[0,1]}$ be the homotopy whose existence is guaranteed by this proposition. Define $\underline{\beta}\colon \underline{E}\to(\underline{E}')^{[0,1]}$ by

$$\beta(\alpha) = \lambda H(\alpha) \quad \text{for} \quad \alpha\in\Gamma',$$

$$\beta_\alpha(y)(t) = H_\alpha[\lambda_{H(\alpha)}(y,\, C(p_{\lambda H(\alpha)}(y)))(1-t)] \quad \text{for} \quad y\in E_{\lambda H(\alpha)}$$

and define $\underline{\Psi}\colon \underline{E}\times[0,1]\to\underline{E}'$ by

$$\Psi(\alpha) = \beta\Phi(\alpha) \quad \text{for} \quad \alpha\in\Gamma',$$

$$\Psi_\alpha(y,t) = \Phi_\alpha[\beta_{\Phi(\alpha)}(y),\, t](1) \quad \text{for} \quad y\in E_{\beta\Phi(\alpha)}.$$

Then

$$\begin{aligned}\Psi_\alpha(y,0) &= \Phi_\alpha[\beta_{\Phi(\alpha)}(y),\, 0](1) = e'_{\alpha\Phi(\alpha)}[\beta_{\Phi(\alpha)}(y)(1)]\\ &= e'_{\alpha\Phi(\alpha)} H_{\Phi(\alpha)}[\lambda_{H\Phi(\alpha)}(y,\, C(p_{\lambda H\Phi(\alpha)}(y))(0)]\\ &= e'_{\alpha\Phi(\alpha)} H_{\Phi(\alpha)} e_{H\Phi(\alpha),\beta\Phi(\alpha)}(y),\end{aligned}$$

$$\begin{aligned}\Psi_\alpha(y,1) &= \Phi_\alpha[\beta_{\Phi(\alpha)}(y),\, 1](1)\\ &= e'_{\alpha\varphi(\alpha)}(\tilde{\lambda}'_{\varphi(\alpha)}\beta_{\Phi(\alpha)}(y))(1)\\ &= e'_{\alpha\varphi(\alpha)}\big(\lambda'_{\varphi(\alpha)}(\beta_{\Phi(\alpha)}(y)(0),\, p'_{\Phi(\alpha)}\circ\beta_{\Phi(\alpha)}(y))(1)\big)\end{aligned}$$

$$= e'_{\alpha\varphi(\alpha)}\Big(\lambda'_{\varphi(\alpha)}\{H_{\Phi(\alpha)}[\lambda_{H\Phi(\alpha)}(y,\, C(p_{\lambda H\Phi(\alpha)}(y)))(1)],\, p'_{\Phi(\alpha)} H_{\Phi(\alpha)}[\lambda_{H\Phi(\alpha)}(y,\, C(p_{\lambda H\Phi(\alpha)}(y)))]\}\Big)$$

$$= e'_{\alpha\varphi(\alpha)}\Big(\lambda'_{\varphi(\alpha)}\{H_{\Phi(\alpha)}[\lambda_{H\Phi(\alpha)}(y,\, C(p_{\lambda H\Phi(\alpha)}(y)))(1)],\, \bar{C}(p_{\lambda H\Phi(\alpha)}(y))(1)\}\Big)$$

$$= e'_{\alpha\varphi(\alpha)}\tilde{H}_{\varphi(\alpha)}(y).$$

Thus $\underline{H}$ and $\underline{\tilde{H}}$ are bundle homotopic.

Consider $\lambda'\colon\Gamma'\to\Gamma'$; let $\delta\colon\underline{E}'\to\underline{E}'$ be the shift map induced by λ'. There exists a homotopy $\underline{K}\colon \underline{p}^{-1}(x_0)\times[0,1]\to\underline{p}^{-1}(x_0)$ such that $K_0 = \sigma G'\delta\delta H^0\sigma'$ and $K_1 = \tau$ where σ, σ' and τ are shift maps. Define $\underline{\tilde{K}}\colon \underline{E}\times[0,1]\to\underline{E}$ by

$$\tilde{K}(\alpha) = \lambda\sigma' H\lambda'\lambda' G'\sigma\lambda(\alpha) = \lambda K\lambda(\alpha) \quad \text{for} \quad \alpha\in\Gamma.$$

and

$$\tilde{K}_\alpha(y,t) = \lambda_\alpha\{K_{\lambda(\alpha)}[\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1),\, t],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1).$$

There is no loss of generality in assuming that $\sigma(\alpha)\geqslant\lambda(\alpha)$ and $\sigma'(\beta)\geqslant\lambda'(\beta)$ for all α and β. Note that

$$\begin{aligned}\tilde{K}_\alpha(y,0) &= \lambda_\alpha\{K_{\lambda(\alpha)}[\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1),\, 0],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1)\\ &= \lambda_\alpha\{e_{\lambda(\alpha)\sigma\lambda(\alpha)} G'_{\sigma\lambda(\alpha)} e'_{G\sigma\lambda(\alpha),\lambda'\lambda'G'\sigma\lambda(\alpha)} H^0_{\lambda'\lambda'G'\sigma\lambda(\alpha)} e_{H\lambda'\lambda'G'\sigma\lambda(\alpha),K\lambda(\alpha)}\\ &\qquad [\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1)],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1)\\ &= \lambda_\alpha\{e_{\lambda(\alpha)\sigma\lambda(\alpha)} G'_{\sigma\lambda(\alpha)} e'_{G'\sigma\lambda(\alpha),\lambda'G'\sigma\lambda(\alpha)} \tilde{H}_{\lambda'G'\sigma\lambda(\alpha)} e_{\lambda H\lambda'\lambda'G'\sigma\lambda(\alpha),K\lambda(\alpha)}\\ &\qquad [\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1)],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1)\\ &\simeq_b \lambda_\alpha\{e_{\lambda(\alpha)\sigma\lambda(\alpha)} G'_{\sigma\lambda(\alpha)} \lambda'_{G'\sigma\lambda(\alpha)}[\tilde{H}_{\lambda'G'\sigma\lambda(\alpha)} e_{\lambda H\lambda'\lambda'G'\sigma\lambda(\alpha),K\lambda(\alpha)}(y),\\ &\qquad C(p_{\lambda K\lambda(\alpha)}(y)](1),\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1).\end{aligned}$$

(for, we let, ignoring indices, $w(t) = \tilde{H}e[\lambda(y,\, C(p(y)))(t)]$; by Proposition 7.3, $e'w(1)\simeq_b\tilde{\lambda}'(w)(1)$)

$$\begin{aligned}&\simeq_b e_{\alpha\beta}\lambda_\beta\{G'_{\lambda(\beta)}\lambda'_{G'\lambda(\beta)}[\tilde{H}_{\lambda'G'\lambda(\beta)} e_{\lambda H\lambda'\lambda'G'\lambda(\beta),K\lambda(\alpha)}(y),\\ &\qquad C(p_{\lambda K\lambda(\alpha)}(y)](1),\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))(1) \quad \text{for some } \beta\\ &= e_{\alpha\beta}\tilde{G}_\beta\tilde{H}_{\lambda'G'\lambda(\beta)} e_{\lambda H\lambda'\lambda'G'\lambda(\beta),K\lambda(\alpha)}(y)\end{aligned}$$

and

$$\begin{aligned}\tilde{K}_\alpha(y,1) &= \lambda_\alpha\{K_{\lambda(\alpha)}[\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1),\, 1],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1)\\ &= \lambda_\alpha\{e_{\lambda(\alpha),K\lambda(\alpha)}[\lambda_{K\lambda(\alpha)}(y,\, C(p_{\lambda K\lambda(\alpha)}(y)))(1)],\, \bar{C}(p_{\lambda K\lambda(\alpha)}(y))\}(1)\\ &\simeq_b e_{\alpha,\lambda K\lambda(\alpha)}(y).\end{aligned}$$

[Again, this bundle homotopy follows from Proposition 7.3. This time, we consider the path, ignoring indices,

$$w(t) = \lambda\{e[\lambda(y,\, C_t(p(y)))(t)],\, \bar{C}_t(p(y))\}(1)$$

where $C_t(z)(s) = C(z)(ts)$.]

Thus $\underline{\tilde{G}\tilde{H}}\simeq_b\underline{\mathrm{id}}$. Similarly, $\underline{\tilde{H}\tilde{G}}\simeq_b\underline{\mathrm{id}}$.

All that remains to be shown is that $\underline{\tilde{G}}|\underline{p}'^{-1}(\partial\Delta'')$ is bundle homotopic to $\underline{g}$. Note that $\underline{\mathrm{id}}\simeq\underline{g}(\underline{H}|\underline{p}'^{-1}(\partial\Delta''))\simeq\underline{g}(\underline{\tilde{H}}|\underline{p}'^{-1}(\partial\Delta''))$ and hence,

$$\underline{\tilde{G}}|\underline{p}'^{-1}(\partial\Delta'')\simeq\underline{g}(\underline{\tilde{H}}|\underline{p}'^{-1}(\partial\Delta''))(\underline{\tilde{G}}|\underline{p}'^{-1}(\partial\Delta''))\simeq\underline{g}.$$

Proof of Lemma 10.2. By the above sublemma, $\underline{\tilde{G}}|\underline{p}'^{-1}(\partial\Delta'')$ is bundle homotopic to $\underline{g}$; by Theorem 9.1, $\underline{g}$ extends to a bundle equivalence $\underline{G}'\colon\underline{E}\to\underline{E}$. Without loss of generality, we may assume that $\underline{G}' = \underline{\sigma g\tau}$ where $\underline{\sigma}$ and $\underline{\tau}$ are shift maps. Using the homotopy $\underline{\xi}$ given in the hypotheses, we can find a homotopy

$$\underline{\xi}'\colon \underline{p}^{-1}(\partial\Delta'')\times[0,1]\to\underline{p}^{-1}(\partial\Delta'')$$

such that $\underline{\xi}'_0 = \underline{\mu}$ and $\underline{\xi}'_1 = \underline{\omega G'H\zeta}$ where $\underline{\mu}$, $\underline{\omega}$ and $\underline{\zeta}$ are shift maps.

Let S be as in Lemma 10.1 and let $\underline{D} = (\underline{p}\times\mathrm{id})^{-1}(S)$. Suppose that $\underline{E} = \{E_\alpha,\, e_{\alpha\beta},\, \Gamma\}$; then $\underline{D} = \{D_\alpha,\, d_{\alpha\beta},\, \Gamma\}$ where $D_\alpha = (p_\alpha\times\mathrm{id})^{-1}(S)$ and

$d_{\alpha\beta} = e_{\alpha\beta} \times \mathrm{id} \mid D_\alpha$. Define $h\colon \Gamma \to \Gamma$ to be an increasing function such that $h(\alpha) \geqslant \zeta H G' \omega\varphi(\alpha)$, $\xi'\varphi(\alpha)$, $\varphi(\alpha)$ for all $\alpha \in \Gamma$; φ is the t-function of $\underline{p}$. Define $h_\alpha\colon D_{h(\alpha)} \to D_\alpha$ by

$$h_\alpha(y, t) = \begin{cases} \big(e_{\alpha,\omega\varphi(\alpha)} G'_{\omega\varphi(\alpha)} H_{G'\omega\varphi(\alpha)}(e_{HG'\omega\varphi(\alpha),h(\alpha)}(y)),\, t\big) \\ \qquad\qquad \text{for} \quad t = 1,\, p_{h(\alpha)}(y) \in \Delta^n, \\ \big(e_{\alpha\varphi(\alpha)} \xi'_{\varphi(\alpha)}(e_{\xi\varphi(\alpha),h(\alpha)}(y),\, t),\, t\big) \\ \qquad\qquad \text{for} \quad p_{h(\alpha)}(y) \in \partial\Delta^n,\, t \in [0, 1], \\ \big(e_{\alpha h(\alpha)}(y),\, t\big) \quad \text{for} \quad p_{h(\alpha)}(y) \in \Delta^n,\, t = 0. \end{cases}$$

It is easily checked that $\underline{h} = \{h_\alpha\}$ is a bundle equivalence of $\underline{D}$.

By Lemma 10.1, there is a bundle map $\underline{\gamma}\colon \underline{D} \to \underline{D}$ such that

$$\underline{\gamma} \mid (\underline{p}^{-1}(\partial\Delta^n) \times [0, 1]) \cup (\underline{E} \times \{0\}) \equiv \underline{\mathrm{id}}$$

and $\underline{\gamma}\underline{h}$ can be extended to a bundle map $\underline{\Phi}'\colon \underline{E} \times [0, 1] \to \underline{E} \times [0, 1]$. Let $\underline{\gamma}'$ be the composition

$$\underline{E} \xrightarrow{\times\{1\}} \underline{E} \times [0, 1] \xrightarrow{\underline{\gamma}} \underline{E} \times [0, 1] \xrightarrow{\underline{\pi}} \underline{E}$$

where $\underline{\pi}$ is projection. $\underline{G} = \underline{\gamma}'\underline{G}'$ and $\underline{\Phi} = \underline{\pi}\underline{\Phi}'$ are the desired bundle maps.

THEOREM 10.3 *Let $\underline{p}\colon \underline{E} \to B$ and $\underline{p}'\colon \underline{E}' \to B$ be s-fibrations such that B is a finite connected polyhedron B. If $\underline{h}\colon \underline{E} \to \underline{E}'$ is a bundle map covering the identity, then $\underline{h}$ is a bundle equivalence.*

The proof is by induction on the number of simplexes of a triangulation of B and by the use of Lemma 10.2 (cf. [5]).

11. Classification theorem. Let $s\mathscr{F}(X, \underline{F})$ denote the set of equivalence classes of s-fibrations over X whose fibre is homotopy equivalent to $\underline{F}$ where the equivalence relation is bundle equivalence.

Consider the n-sphere $S^n = B^+ \cup B^-$ where B^+ and B^- are n-cells with $B^+ \cap B^- = S^{n-1}$, an $(n-1)$-sphere. Suppose that $x_0 \in S^{n-1}$. Let $\underline{p}\colon \underline{E} \to S^n$ be an s-fibration such that $\underline{p}^{-1}(x_0)$ is homotopy equivalent to the tower of spaces, $\underline{F}$.

By Corollary 9.2, there exist equivalences $\underline{H}^+\colon \underline{E}^+ \to \underline{p}^{-1}(x_0) \times B^+$ and $\underline{H}^-\colon \underline{E}^- \to \underline{p}^{-1}(x_0) \times B^-$ where $\underline{H}^+ \mid \underline{p}^{-1}(x_0) \equiv \underline{\mathrm{id}} \times \{x_0\}$; $\underline{p}^+\colon \underline{E}^+ \to B^+$ and $\underline{p}^-\colon \underline{E}^- \to B^-$ are restrictions of $\underline{p}$ to B^+ and B^-, respectively. Note that $\underline{L} = \underline{H}^+ \circ (\underline{H}^-)^{-1} \mid \underline{p}^{-1}(x_0) \times S^{n-1}$ is an equivalence of $\underline{p}^{-1}(x_0) \times S^{n-1}$ to itself such that $\underline{L} \mid \underline{p}^{-1}(x_0) \times \{x_0\} \equiv \underline{\mathrm{id}} \times \mathrm{id}$. By assumption, there exists a homotopy equivalence $\underline{G}\colon \underline{p}^{-1}(x_0) \to \underline{F}$. Define $\mu\colon s\mathscr{F}(S^n, \underline{F}) \to \pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$ by sending the equivalence class of $\underline{p}\colon \underline{E} \to S^n$ to the homotopy class represented by

$$(\underline{G} \times \mathrm{id}) \circ \underline{L} \circ (\underline{G}^{-1} \times \mathrm{id})\colon \underline{F} \times S^{n-1} \to \underline{F} \times S^{n-1}.$$

Note that this map represents an element of $\pi_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$; however, in order to show that μ is well-defined, we have to pass to $\pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$. The main result of this section is the following.

THEOREM 11.1 *$\mu\colon s\mathscr{F}(S^n, \underline{F}) \to \pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$ is a bijection.*

PROPOSITION 11.2. *μ is independent of the choices of the equivalences $\underline{H}^+$, $\underline{H}^-$ and $\underline{G}$.*

Proof. Let $\tilde{\underline{G}}\colon \underline{p}^{-1}(x_0) \to \underline{F}$ be an equivalence; then

$$\begin{aligned} &(\underline{G} \times \mathrm{id}) \circ L \circ (\underline{G}^{-1} \times \mathrm{id}) \\ &\quad = (\underline{\mathrm{id}} \times \mathrm{id}) \circ \big((\underline{G} \times \mathrm{id}) \circ L \circ (\underline{G}^{-1} \times \mathrm{id})\big) \\ &\quad \simeq \big((\tilde{\underline{G}} \circ \underline{G}^{-1}) \times \mathrm{id}\big)(\underline{G} \circ \tilde{\underline{G}}^{-1} \times \mathrm{id})\big((\underline{G} \times \mathrm{id}) \circ L \circ (\underline{G}^{-1} \times \mathrm{id})\big) \end{aligned}$$

which, by using the action of $\pi_0(\mathscr{S}\mathscr{E}(\underline{F}))$ on $\pi_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$, is equivalent to

$$\begin{aligned} &\simeq \big((\tilde{\underline{G}} \circ \underline{G}^{-1}) \times \mathrm{id}\big) \circ \big((\underline{G} \times \mathrm{id}) \circ L \circ (\underline{G}^{-1} \times \mathrm{id})\big) \circ \big((\underline{G} \circ \tilde{\underline{G}}^{-1}) \times \mathrm{id}\big) \\ &\simeq (\tilde{\underline{G}} \times \mathrm{id}) \circ L \circ (\tilde{\underline{G}}^{-1} \times \mathrm{id}). \end{aligned}$$

The independence of the choices of H^+ and H^- follows by a similar, but much simpler, argument since the restriction of two equivalences $\underline{E}^+ \to \underline{p}^{-1}(x_0) \times B^+$ to the s-fibration over S^{n-1} are homotopic.

PROPOSITION 11.3. *μ is well-defined.*

Proof. Let $\tilde{\underline{p}}\colon \tilde{\underline{E}} \to S^n$ be an s-fibration such that there exists an equivalence $\underline{K}\colon \tilde{\underline{E}} \to \underline{E}$. Let $\tilde{\underline{G}} = \underline{G} \circ \underline{K} \mid \tilde{\underline{p}}^{-1}(x_0)$, $\tilde{\underline{H}}^+ = (\tilde{\underline{G}}^{-1} \circ \underline{G} \times \mathrm{id}) \circ \underline{H}^+ \circ \underline{K} \mid \underline{p}^{-1}(B^+)$, $\tilde{\underline{H}}^- = (\tilde{\underline{G}}^{-1} \circ \underline{G} \times \mathrm{id}) \circ \underline{H}^- \circ \underline{K} \mid \tilde{\underline{p}}^{-1}(B^-)$ and $\tilde{\underline{L}} = \tilde{\underline{H}}^+ \circ (\tilde{\underline{H}}^-)^{-1} \mid \tilde{\underline{p}}^{-1}(S^{n-1})$. Then

$$\begin{aligned} &(\tilde{\underline{G}} \times \mathrm{id}) \circ \tilde{\underline{L}} \circ (\tilde{\underline{G}}^{-1} \times \mathrm{id}) \\ &\simeq (\tilde{\underline{G}} \times \mathrm{id}) \circ (\tilde{\underline{G}}^{-1} \circ \underline{G} \times \mathrm{id}) \circ \underline{H}^+ \circ \underline{K} \circ \underline{K}^{-1} \circ (\underline{H}^-)^{-1} \circ (\underline{G}^{-1} \circ \tilde{\underline{G}} \times \mathrm{id}) \circ (\tilde{\underline{G}}^{-1} \times \mathrm{id}) \\ &\simeq (\underline{G} \times \mathrm{id}) \circ \underline{L} \circ (\underline{G}^{-1} \times \mathrm{id}). \end{aligned}$$

Now suppose that $\underline{F}$ is a singleton set $\{F\}$ where F is an ANR. We shall show that μ is a bijection by constructing its inverse $\lambda\colon \pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F})) \to s\mathscr{F}(S^n, \underline{F})$.

Let $\underline{h}\colon \underline{F} \times S^{n-1} \to \underline{F} \times S^{n-1}$ be a bundle equivalence which represents an element of $\pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F}))$ such that $\underline{h} \mid \underline{F} \times \{x_0\}$ is homotopic to $\underline{\mathrm{id}}$. If $\underline{h} = \{h\}$, then let E be the quotient space obtained from $(F \times B^+) \cup (F \times B^-)$ by identifying $x \in F \times S^{n-1}$ with $h(x)$. It is at this point that the hypotheses that $\underline{F}$ is a singleton is used; the author is unable to perform a similar construction to obtain $\underline{E}$ otherwise. Define $p\colon E \to S^n$ by $p(x, y) = y$, where $x \in F$ and $y \in S^n$.

PROPOSITION 11.5. *$p\colon E \to S^n$ is an approximate fibration.*

Proof. It is easily checked that p is completely movable and, hence, by [3] (see also [6]), p is an approximate fibration.

Let $\underline{p} = \{p_i\}\colon \underline{E} = \{E_i\} \to S^n$ be the s-fibration associated to p as in Proposition 3.3. Recall that the fibre of $\underline{p}$ is homotopy equivalent to $\underline{F}$.

Define $\lambda\colon \pi^*_{n-1}(\mathscr{S}\mathscr{E}(\underline{F})) \to s\mathscr{F}(S^n, \underline{F})$ by sending the class of $\underline{h}$ to the class of $\underline{p}\colon \underline{E} \to S^n$.

PROPOSITION 11.6. *λ is well-defined.*

Proof. Let us first consider the case when $\underline{h}$ is homotopic to $\underline{h}'$ by a homotopy $\underline{H}\colon \underline{F}\times S^{n-1}\times[0,1]\to\underline{F}\times S^{n-1}\times[0,1]$ where $\underline{H}|\underline{F}\times S^{n-1}\times\{0\}\equiv\underline{h}\times\{0\}$ and $\underline{H}|\underline{F}\times S^{n-1}\times\{1\}\equiv\underline{h}'\times\{1\}$. Let D be the quotient space obtained from $(F\times B^{+}\times[0,1])\cup(F\times B^{-}\times[0,1])$ by identifying $x\in F\times S^{n-1}\times[0,1]$ with $H(x)$. Define $\delta\colon D\to S^{n}\times[0,1]$ by $\delta(x,y)=y$ where $x\in F$ and $y\in S^{n}\times[0,1]$. As in Proposition 11.5, δ is an approximate fibration. Let $\underline{\delta}\colon\underline{D}\to S^{n}\times[0,1]$ be the associated s-fibration as given in Proposition 3.3.

By Theorems 3.5 and 10.3, the s-fibrations $\underline{p}\colon\underline{E}\to S^{n}$ and

$$\underline{\delta}|\underline{\delta}^{-1}(S^{n}\times\{0\})\colon\ \underline{\delta}^{-1}(S^{n}\times\{0\})\to S^{n}\times\{0\}$$

are equivalent (we identify S^{n} with $S^{n}\times\{0\}$). Let $\underline{\varphi}\colon\underline{E}\to\underline{D}$ be the composition $\underline{E}\to\underline{\delta}^{-1}(S^{n}\times\{0\})\subseteq\underline{D}$ where the first map is the equivalence given above; note that φ is a bundle map such that $\underline{\delta}\varphi=p$. By Theorem 5.3, there exists a tower of maps $\underline{\Phi}\colon\underline{E}\times[0,1]\to\underline{D}$ such that $\underline{\Phi}|\underline{E}\times\{0\}\equiv\underline{\varphi}$, $\underline{\delta}\underline{\Phi}=\underline{p}\times\mathrm{id}$ and $\underline{\Phi}$ is $(\underline{p}\times\mathrm{id},\underline{\delta},\mathrm{id})$-preserving. Since $\underline{\varphi}$ is a bundle map, by Theorem 6.3, $\underline{\Phi}$ is a bundle map covering id and, hence, by Theorem 10.3, $\underline{\Phi}$ is a bundle equivalence. In particular, $\underline{p}\times\mathrm{id}|(\underline{p}\times\mathrm{id})^{-1}(S^{n}\times\{1\})$ and $\underline{\delta}|\underline{\delta}^{-1}(S^{n}\times\{1\})$ are equivalent; again by using Theorems 3.5 and 10.3, these two s-fibrations are equivalent to $\underline{p}\colon\underline{E}\to S^{n}$ and $\underline{p}'\colon\underline{E}'\to S^{n}$, respectively, where $\underline{p}'$ and $\underline{E}'$ are defined analogous to $\underline{p}$ and $\underline{E}$, for $\underline{h}'$. Thus $\underline{p}$ and $\underline{p}'$ are equivalent.

Let $\underline{g}\colon\underline{F}\to\underline{F}$ be a homotopy equivalence; to complete the proof of the proposition, we must show that the s-fibrations associated to $\underline{h}$ and

$$\underline{h}'=(\underline{g}^{-1}\times\mathrm{id})\circ\underline{h}\circ(\underline{g}\times\mathrm{id})$$

are equivalent.

The function $F\xrightarrow{g^{-1}}F\xrightarrow{\times Q}F\times Q$ is homotopic to an embedding $\lambda_1\colon F\to F\times Q$ and there exists $\lambda_2\colon F\times Q\to F$ such that $\lambda_2\circ\lambda_1=\mathrm{id}$. Define $\lambda_1'(x,y,z)=\big(\lambda_1(x),z\big)$ and $\lambda_2'(x,y,z)=\big(\lambda_2(x,y),y,z\big)$ for $(x,y,z)\in F\times Q\times S^{n-1}$. Note that

$$\underline{g}\times\mathrm{id}\colon\ \underline{F}\times Q\times S^{n-1}\to\underline{F}\times Q\times S^{n-1}$$

and $\underline{g}^{-1}\times\mathrm{id}\colon\underline{F}\times Q\times S^{n-1}\to\underline{F}\times Q\times S^{n-1}$ are bundle homotopic (over S^{n-1}) to $\{\lambda_2'\}$ and $\{\lambda_1'\}$ respectively.

Define

$$h''\colon F\times Q\times S^{n-1}\to F\times Q\times S^{n-1}\quad\text{and}\quad F'''\colon F\times Q\times S^{n-1}\to F\times Q\times S^{n-1}$$

by

$$h''(x,y,z)=(g^{-1}\times\mathrm{id})\big(t\big(h(g(x),z),y\big)\big),\qquad h'''(x,y,z)=\lambda_1'\big(t\big(h(\lambda_2(x,y),z),y\big)\big)$$

where $\pi\colon F\times Q\to F$ is projection and $t\colon F\times S^{n-1}\times Q\to F\times Q\times S^{n-1}$ is given by $t(x,y,z)=(x,z,y)$. Form $\underline{p}''\colon\underline{E}''\to S^{n}$ and $\underline{p}'''\colon\underline{E}'''\to S^{n}$ corresponding to h'' and h''', respectively, as above. Since $\underline{h}''$ is bundle homotopic to $\underline{h}'''$ (over S^{n-1}), the s-fibrations $(\underline{p}'',\underline{E}'',S^{n})$ and $(\underline{p}''',\underline{E}''',S^{n})$ are bundle equivalent by the first part of this proof.

Define $\varphi\colon E'\to E''$ by $\varphi(x,y)=(x,0,y)$ where $(x,y)\in F\times(B^{+}\cup B^{-})$. By Theorems 3.5 and 10.3, φ induces a bundle equivalence $\underline{\varphi}\colon\underline{E}'\to\underline{E}''$. Define $\xi\colon E\to E'''$ by $\xi(x,y)=\big(\lambda_1(x),y\big)$ where $(x,y)\in F\times(B^{+}\cup B^{-})$. Again by Theorems 3.5 and 10.3, ξ induces a bundle equivalence $\underline{\xi}\colon\underline{E}\to\underline{E}'''$. Hence, $\underline{E}$ and $\underline{E}'$ are bundle equivalent and the proof of 11.6 is completed.

PROPOSITION 11.7. $\mu\lambda=\mathrm{id}$.

Proof. Let $\underline{h}\colon\underline{F}\times S^{n-1}\to\underline{F}\times S^{n-1}$ represent an element of $\pi_{n-1}^{*}(\mathscr{S}\mathscr{E}(\underline{F}))$. Let $\underline{p}\colon\underline{E}\to S^{n}$ be the s-fibration associated to $\underline{h}$ by λ as constructed above. Let $\varphi'\colon F\times B^{+}\to p^{-1}(B^{+})$ be the inclusion map and define $\varphi_i^{+}\colon F\times B^{+}\to E_i\times N_i$ by $\varphi_i^{+}(z)=\big(\varphi'(z),p\varphi'(z),0\big)$. Note that $\{\varphi_i^{+}\}=\underline{\varphi}^{+}\colon\underline{F}\times B^{+}\to\underline{p}^{-1}(B^{+})$ is a tower of maps which by Theorems 3.5 and 10.3 is a bundle equivalence. Define $\underline{\varphi}^{-}\colon\underline{F}\times B^{-}\to\underline{p}^{-1}(B^{-})$ similarly: note that $\underline{\varphi}^{+}\underline{h}=\underline{\varphi}^{-}$.

$\tilde{\underline{\varphi}}=(\underline{\varphi}^{+})^{-1}\underline{\varphi}^{-}|\underline{F}\times S^{n-1}$ is a bundle equivalence which represents the class associated to $\underline{p}$ by μ. Note that $\underline{\varphi}^{+}\tilde{\underline{\varphi}}$ is bundle homotopic to $\underline{\varphi}^{-}|\underline{F}\times S^{n-1}$ and since $\underline{\varphi}^{+}\underline{h}=\underline{\varphi}^{-}$, $\tilde{\underline{\varphi}}$ is bundle homotopic to $\underline{h}$.

PROPOSITION 11.8. $\lambda\mu=\mathrm{id}$.

Proof. Let $\underline{r}\colon\underline{R}\to S^{n}$ be an s-fibration; choose equivalences $\underline{H}^{+}\colon\underline{r}^{-1}(B^{+})\to\underline{r}^{-1}(x_0)\times B^{+}$, $\underline{H}^{-}\colon\underline{r}^{-1}(B^{-})\to\underline{r}^{-1}(x_0)\times B^{-}$ and $\underline{G}\colon\underline{r}^{-1}(x_0)\to\underline{F}$ as above. Let $\underline{h}=(\underline{G}\times\mathrm{id})\circ\underline{H}^{+}\circ(\underline{H}^{-})^{-1}\circ(\underline{G}^{-1}\times\mathrm{id})$ be the equivalence of $\underline{F}\times S^{n-1}$ associated to $\underline{r}$ by μ. Let $\underline{p}\colon\underline{E}\to S^{n}$ be the s-fibration associated to $\underline{h}$ by λ as constructed above. Define $\underline{\varphi}^{+}\colon\underline{F}\times B^{+}\to\underline{p}^{-1}(B^{+})$ and $\underline{\varphi}^{-}\colon\underline{F}\times B^{-}\to\underline{p}^{-1}(B^{-})$ as in the proof of Proposition 11.7.

Let $\underline{\xi}^{+}=\underline{\varphi}^{+}\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{+}\colon\underline{r}^{-1}(B^{+})\to\underline{p}^{-1}(B^{+})$ and $\underline{\xi}^{-}=\underline{\varphi}^{-}\circ(\underline{G}\times\mathrm{id})\circ\circ\underline{H}^{-}\colon\underline{r}^{-1}(B^{-})\to\underline{p}^{-1}(B^{-})$. We would like to define $\underline{\xi}\colon\underline{R}\to\underline{E}$ by using $\underline{\xi}^{+}$ and $\underline{\xi}^{-}$; unfortunately they do not agree on $\underline{r}^{-1}(S^{n-1})$. Note that

$$\begin{aligned}\underline{\xi}^{+}|\underline{r}^{-1}(S^{n-1})&=\underline{\varphi}^{+}\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{+}\\&\simeq_b\underline{\varphi}^{+}\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{+}\circ(\underline{H}^{-})^{-1}\circ(\underline{G}^{-1}\times\mathrm{id})\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{-}\\&=\underline{\varphi}^{+}\circ\underline{h}\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{-}=\underline{\varphi}^{-}\circ(\underline{G}\times\mathrm{id})\circ\underline{H}^{-}.\end{aligned}$$

By Theorem 9.1, this bundle homotopy can be extended to a bundle homotopy of $\underline{r}^{-1}(\underline{B}^{+})$. By using the end of this bundle homotopy $\underline{\xi}^{-}$ can be extended to $\underline{\xi}\colon\underline{R}\to\underline{E}$. Since $\underline{\xi}^{+}$ and $\underline{\xi}^{-}$ are equivalences, $\underline{\xi}$ is a bundle map by Theorem 6.3 and, hence, by Theorem 10.3 $\underline{\xi}$ is an equivalence.

12. An application. Let E and E' be compact ANR's such that there exist cell-like maps $p\colon E\to S^{n}$ and $q\colon E'\to S^{n}$; — i.e. for each $x\in S^{n}$, $p^{-1}(x)$ and $q^{-1}(x)$ have the shape of a point [13]. By [10], p and q are approximate fibrations and hence, by Theorem 3.4, the associated maps $\underline{p}\colon\underline{E}\to S^{n}$ and $\underline{q}\colon\underline{E}'\to S^{n}$ are s-fibrations. As in the proof of Theorem 3.5, the fibres $\underline{p}^{-1}(x)$ and $\underline{q}^{-1}(x)$ are homotopy equivalent to the inverse system consisting of a single point $\{x_0\}$. Note that $\pi_{n-1}(\mathscr{S}\mathscr{E}\{x_0\})$ is the trivial group. By Theorem 11.5, $\underline{p}\colon\underline{E}\to S^{n}$ and $\underline{q}\colon\underline{E}'\to S^{n}$ are bundle equivalent. The following result is a consequence of Theorem 3.6.

THEOREM 12.1. *Let* $p\colon E \to S^n$ *and* $q\colon E' \to S^n$ *be cell-like mappings of compact* ANR*'s onto* S^n. *Then for each* $\varepsilon > 0$ *there exist mappings* $h\colon E \to E'$ *and* $g\colon E' \to E$ *such that* $d(qh, p) < \varepsilon$, $d(pg, q) < \varepsilon$ *and the composites* hg *and* gh *are homotopic to the identity.*

Theorem 12.1 follows also from [1] and [15] in the case when $E = E' = S^n$, $n \neq 4$.

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The Bing–Borsuk conjecture is stronger than the Poincaré conjecture

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Abstract. It is shown that the existence of a fake 3-cell implies the existence of a 3-dimensional homogeneous compact ANR-space which is not a manifold.

We say that the space X is *homogeneous*, if for every pair of points $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$. We are concerned with the following conjecture:

CONJECTURE 1 (Bing, Borsuk [4]). *Every* n*-dimensional homogeneous compact* ANR*-space is an* n*-dimensional manifold.*

In dimensions 1 and 2 this conjecture was proved by Bing and Borsuk in [4]. Here we prove that in dimension 3 Conjecture 1 is stronger than the Poincaré conjecture.

CONJECTURE 2 (Poincaré). *Every homotopy* 3*-sphere is homeomorphic to a* 3*-sphere.*

By a homotopy 3-sphere we mean a closed 3-dimensional manifold which has a homotopy type of 3-sphere. We shall use the term fake 3-cell for a compact contractible 3-manifold which is not homeomorphic to a 3-cell. It is known ([6], p. 26) that (2) is equivalent to the statement that there are no fake 3-cells. Our main goal may be formulated as follows:

THEOREM 3. *If there exists a fake* 3*-cell* F, *then there exists a* 3*-dimensional homogeneous compact* ANR*-space* K *which is not a manifold.*

The proof of Theorem 3 consists of several parts: first we shall construct the space K (assuming the existence of the fake 3-cell), then we shall prove that $K \in \mathrm{ANR}$, that K is homogeneous, that K is not a manifold, and finally that $\dim K = 3$. All the time we shall assume the existence of a fixed fake 3-cell F with a given triangulation (by [3] F can be triangulated) and with a fixed orientation. Moreover, we can assume that there exists a homotopy 3-sphere H such that F is obtained from H by removing from it a single open 3-simplex, in particular that the boundary ∂F is equal to the boundary of a 3-simplex (see [6], p. 26).