Hence, since \( x_n \rightarrow x \), \( U_q(x) \subseteq U(x_n) \) ultimately as \( n \rightarrow \infty \). Applying \( F \) we get
\[
F \ast U_q(x) \subseteq F \ast U(x_n) \text{ ultimately}.
\]
Since \( y, x \in F \) and \( (x, x) \in U_p, y \in F(x) \subseteq F \ast U_q(x) \). Since \( U_q(x) \) is open, \( F \ast U_q(x) \) is open by hypothesis. Therefore, since \( y_n \rightarrow y \) and \( y \in F \ast U_q(x) \), \( y_n \in F \ast U_q(x) \) ultimately. Hence by (16), \( y_n \in F \ast U_q(x) \) ultimately, which contradicts (13).

Finally, Theorem 1 follows from Theorem 5 under Lemmas 8 and 9 since every connected metric space is well-chained.

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\( s \)-Fibrations

by

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Abstract. The concept of \( s \)-fibration is introduced which generalized the notions of Hurewicz fibrations and approximate fibrations. Many results about Hurewicz fibrations which are not true for approximate fibrations are proved for \( s \)-fibrations. For example, a homotopy classification theorem for \( s \)-fibrations over then \( n \)-sphere is proved.

1. Introduction. A mapping \( f : E \rightarrow B \) between compact metric spaces is an approximate fibration if, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( h : X \rightarrow E \) and \( H : X \times [0, 1] \rightarrow B \) are maps with \( d(h(x, 0), fh(x)) < \delta \), then there exists \( G : X \times [0, 1] \rightarrow E \) such that \( G(x, 0) = h(x) \) and \( d(h(x, t), G(x, t)) < \varepsilon \) for all \( x \in X \) and \( t \in [0, 1] \). Coram and Duvall [2] introduced approximate fibrations as a generalization of cell-like mappings [10] and showed that the uniform limit of a sequence of Hurewicz fibrations is an approximate fibration. By using shape theoretic concepts, they also showed that approximate fibrations possessed many properties shared by Hurewicz fibrations.

One notable exception is that the pullback of an approximate fibration need not be an approximate fibration. In this work, we define the concept of \( s \)-fibrations which we show generalizes the concepts of approximate fibrations and Hurewicz fibrations. Pullbacks behave properly and many other results about Hurewicz fibrations carry over. For example, a homotopy classification theorem for \( s \)-fibrations over the \( n \)-sphere is proved (Theorem 11.1). As a consequence, information about cell-like decompositions of ANR's is obtained (Theorem 12.1).

T. B. Rushing has informed the authors that S. Mačvičić and he [11] have also generalized the theory of approximate fibrations but that the overlap between these works is little. R. Goad [6] has also a generalization of approximate fibration but, again, there is no overlap with this work.

2. Definitions. We shall assume that the reader is familiar with [12]. Since our results are valid for a larger category of spaces than that considered in [12], our definitions will sometimes differ.
A directed set \((\Gamma, \leq)\) is closure-finite provided for every \(\gamma \in \Gamma\), the set of pre-decessors of \(\gamma\) is finite. A tower of topological spaces \(E = (E_\alpha, \varphi_{\alpha\beta}, \Gamma)\) is an inverse system of topological spaces where \(\Gamma\) is a closure-finite directed set of indices and the bonding maps, \(\varphi_{\alpha\beta} : E_\alpha \to E_\beta\), are continuous. A tower of maps between two towers, \(f: E \to E' = (E'_\alpha, \varphi'_{\alpha\beta}, \Gamma)\), consists of an increasing function \(f: \Gamma \to f(\Gamma)\) and a collection of continuous maps \(f_\alpha: E_\alpha \to E'_\alpha\) such that \(\varphi_{\alpha\beta} \circ f_\alpha = f_{\beta} \circ \varphi_{\alpha\beta}\) for \(\gamma \in \beta\). Composition of two towers of maps, \(f \circ g\), can be defined (see [12]), \(i\delta\) will denote the identity tower of maps. Two towers of maps \(f, g: E \to E'\) are homotopic, \(f \simeq g\), provided for every \(\alpha \in \Gamma\), there exists \(\alpha \in \Gamma, \alpha \geq \gamma(\beta, \beta)(\varphi, \varphi(\beta))\), such that \(f_\alpha \varphi_{\alpha\beta} = g_\alpha \varphi_{\alpha\beta}\).

A map from a tower \(E\) to a topological space \(B\) is a collection \(p = \{p_\alpha: \alpha \in \Gamma\}\) of continuous maps \(p_\alpha: E_\alpha \to B\) such that for all \(\alpha \geq \beta\), \(p_\beta \varphi_{\alpha\beta} = p_\alpha\). Let \(p: E \to B\) and \(p': E' \to B'\) be maps and let \(g: B \to B'\) be a continuous map. A tower of maps, \((E, f, g)\)-preserving if the homotopy in the definition of tower of maps between \(f_\alpha \varphi_{\alpha\beta} = g_\beta \varphi_{\alpha\beta}\) and \(p_\alpha = g_\beta, \alpha \geq \beta(\alpha(\beta, \beta)(\varphi, \varphi(\beta)))\), can be chosen such that \(f_\alpha \varphi_{\alpha\beta} = g_\beta \varphi_{\alpha\beta}\) for all \(\alpha \in \Gamma\).

Let \(g: E \to B\) be a map. The triple \((g, E, B)\), or, more simply, \(g\), is called an \(s\)-fibration if, given \(x \in X\), \(g(x) = x\) for all \(\alpha \geq x(\alpha, \alpha)(\varphi, \varphi(\beta))\). Let \(x \in X\), \(g(x) = x\) for all \(\alpha \geq x(\alpha, \alpha)(\varphi, \varphi(\beta))\). Let \(\alpha = (\gamma, \beta)\) be an order-preserving function such that \(\alpha(\beta, \beta)(\varphi, \varphi(\beta))\) for all \(\alpha \in \Gamma\). The shift map \(g: E \to E'\), induced by \(\alpha\), is the tower of maps \(\alpha\varphi_{\alpha\beta} = \alpha(\beta, \beta)(\varphi, \varphi(\beta))\). Note that \(x = \alpha(\beta, \beta)(\varphi, \varphi(\beta))\). Given two towers of maps, \(f, g: E \to E'\), we write \(f \simeq g\) if there exist maps \(h, g': E \to E'\), such that \(g \circ h = g' \circ f\). Trivially, if \(f \simeq g\), then \(g \simeq f\).

A bundle homotopy is a bundle map \((E, f, g): E \times [0, 1] \to E'\) covering \(f: B \times [0, 1] \to B'\). If \((E, f, g)\) is a bundle homotopy, for \(i \in [0, 1]\), let \(E_i = E \to E'\) be the tower of maps \(\{E_i\}\) where \(E_i(x) = F_i(x, t)\). Two bundle maps \((f, f_i)\) and \((f, f_i)\) are bundle homotopic, \(f \simeq f_i\), if there exists a bundle homotopy \((E_i, f, g, f_i)\) such that \(E_i(f)(x, t) = F_i(x, t, f)\) for \(i = 0, 1\), where \(f_i(\gamma)\) are shift maps on \(E'\).

Let \(g: E \to B\) and \(g': E' \to B'\) be \(s\)-fibrations. A bundle map \((E, f, g)\) covering the identity map \((B, B', \beta, \beta)\) is \((E, f, g)\)-equivariant if there exists a bundle map \((E, f, g)\) covering the identity map \(E \to E\) such that \(E \times G \to G = E\) are bundle homotopic to the identity with bundle homotopies covering the projections \(B \times [0, 1] \to B'\). In Section 10, we will show that a bundle map covering the identity is an equivalence. \(G\) will be called a \((s)\)-inverse of \(E\). Note that a bundle map which is bundle homotopy to a bundle equivalence is also a bundle equivalence.

3. Approximate fibrations. Let \(f: E \to B\) be a continuous map between the compact metric spaces \(E\) and \(B\). Let \(d, d'\) denote the metrics on \(E, B\) respectively and define \(g((e, b), (e', b')) = \max\{d(e, e'), d'(b, b')\}\) for points \((e, b), (e', b')\) \(\in E \times B\). Let \(f = \alpha_{E, B}\) be the graph of \(f\) and let \(E\) denote the \((1/0)\)-neighborhood of \(f\) in \(E \times B\), where \(f\) denotes a positive integer. Let \(e_1: E_1 \to E\) denote inclusion, \(\beta\). Define \(p_1: E_1 \to B\) or \(p_1(x) = x\); note that \(p_1 = \{p_1\}: E \to E_1\) is a map.

THEOREM 3.1. \(f\) is an approximate fibration if and only if \(f\) is an \(s\)-fibration.

Proof. Suppose that \(f\) is an approximate fibration and let the positive integer \(i\) be given. For \(s = i/0\), \(\delta\) be given from the definition of approximate fibration. Let \(B^i\) be chosen such that if \(x, y \in E, d(x, y) < \delta\), then \(d_1(f(x), f(y)) < \delta^2\). Choose a positive integer \(i > 1\) such that \(1/j < \delta, \delta^2\).

Let \(g: X \to E\) and \(H: X \times [0, 1] \to B\) be continuous maps such that \(g(x) = H(x, 0)\). Then \(g(x) = (g(x), H(x, 0))\) for some function \(g': X \to E\).

Since \(g(x) \in E\), there exists \(e \in E\) such that \(g((e, f(x)) < 1/j < \delta\). Hence \(d((e, f(x)), e) < \delta^2\).

By hypothesis, there exists a homotopy \(G: X \times [0, 1] \to E\) such that \(G(x, 0) = g(x)\) and \(d_1(F(x), f(x)) < \delta\).

Define \(G: X \times [0, 1] \to E\) by \(G(x, t) = (G(x, t), H(x, t))\); \(G\) is the desired homotopy.

Now suppose that \(p\) is an \(s\)-fibration and let \(e > 0\) be given. Let \(B^i\) be chosen such that if \(d(x, y) < \delta\), then \(d_1(f(x), f(y)) < \delta^2\). Choose a positive integer \(i\) such that \(1/j < \delta^2, \delta\).

Let \(\delta = 1/j\) and let \(g: X \to E\) and \(H: X \times [0, 1] \to B\) be continuous maps such that \(d_1(f(x), H(x, 0)) < \delta\).

Define \(g(x) = (g(x), H(x, 0)) \in E\); thus there exists \(G': X \times [0, 1] \to E\), such that \(G'(x, 0) = g(x)\) and \(p_1(G'(x, t)) = H(x, t)\).

Since \(G'(x, t) = (G(x, t), H(x, t))\) for some \(G\), \(G\) is the desired function; for, since
$G(x, t) \in C_{E_i}$ there exists $e \in E$ such that $q((e, f(e)), G(x, t)) < 1/l$. Hence $d(e, G(x, t)) < 1/l + l$ and $d(f(e), G(x, t)) < 1/2$; therefore

$$d'(f(e), G(x, t)) = d'(f(e), G(x, t)) < 1/l.$$  

Note that $E$ is the inverse limit of $(\mathcal{E}_k)$ and $f$ is the inverse fibration of $(p_k)$. Let $f: E \to B$ be an approximate fibration which is not a Hurewicz fibration [2]. Then the constant sequence $(\mathcal{F})$: $(E) \to (B)$ is not an s-fibration. Thus the inverse systems which we can associate to $f$ and $E$ in order to prove an analogue of Theorem 3.1 form a proper subset of those systems whose inverse limits are $f$ and $E$. Note also that $f^{-1}(b)$ is also the inverse limit of $p_k^{-1}(b)$.

The difficulty with the above construction is that the mappings $p_k$ are not proper. We will now modify the above construction to alleviate this problem. The proof of the following is left to the reader.

**Proposition 3.2.** Let $f: E \to B$ be a continuous map between the compact metric spaces $E$ and $B$. Form the inverse sequence $(\mathcal{E}_k)_{k=0}^\infty$ as above and let $(\mathcal{A}_i)$ be a sequence of subsets of $E \times B$ such that for each $i$, $j$, and $k$ for which $A_j \subseteq E_i$ and $E_k \subseteq A_i$. Then $p: E \to B$ is an approximate fibration if and only if $(q_k): (A_i) \to B$ is an s-fibration where $q_k(x, s) = x$.

Let $Q = \prod_{k=0}^\infty \prod_{i=1}^\infty [0, 1]_{i/k}$. Define $n_i: E_i \times N_i \to B$ by $n_i(e, t) = p_i(e)$. Then $n_i: E_i \times N_i \to B$ is an s-fibration if and only if $n_i: (A_i) \to B$ is an s-fibration.

**Proposition 3.3.** $f: E \to B$ is an s-fibration if and only if $k: E \to B$ such that $f$ is an approximate fibration if and only if $q$ is an s-fibration.

**Theorem 3.4.** Let $E$ and $B$ be compact ANRs and let $f: E \to B$ be a continuous map. Then there exists a tower of compact ANR's $(\mathcal{E}_k)$ and a map $q: (\mathcal{E}_k) \to B$ such that $f$ is an approximate fibration if and only if $q$ is an s-fibration.

**Proof.** Consider the map $(\mathcal{A}_i): (E_i) \to B$ as in Proposition 3.3. Recall that an ANR $Y$ is convenient [14] if given a compactum $E$ and a neighborhood $V$ of $E$ in $Y$, there exists a compact ANR $M \subseteq V$ such that $M \subseteq V$. By [2] $E \times B \times Q$ is a convenient ANR. Thus it is possible to find a sequence of compact ANR's $(\mathcal{A}_i)$ such that for each $i$, $j$ and $k$ for which $A_j \subseteq E_i \times N_i$ and $E_k \subseteq A_i$. Theorem 3.4 now follows from 3.1 and 3.2.

Let $p: E \to B$ and $p': E' \to B$ be approximate fibrations where $E$, $E'$ and $B$ are compact ANR's. Suppose that there exists an embedding $f: E \to E'$ such that $p'f = p$. Let $(\mathcal{A}_i): (E_i \times N_i) \to B$ and $(\mathcal{A}_j): (E_j \times N_j) \to B$ be defined as above. Given $x$ in $E_i \times N_i$, let $d(x, y) < 1/l$ then $d(f(x), f(y)) < 1/l$. Define $f_i: E_i \times N_i \to E_i \times N_i$ by $f_i(e, b, t) = (f_i(e), b, t)$ where $e, b, t \in E \times B \times Q$. It is easily checked that $(f_i): (E_i \times N_i) \to (E_i \times N_i)$ is a tower of maps which is $(\mathcal{A}_i, \mathcal{A}_j)$, id-preserved.

**Theorem 3.5.** Suppose that for each $x \in B$, $p^{-1}(x)$ and $p'^{-1}(x)$ are ANR's and $f: p^{-1}(x) \to p'^{-1}(x)$ is a homotopy equivalence. Then $(f_i): (E_i \times N_i) \to (E_i \times N_i)$ is a bundle map.

**Proof.** It must be shown that $(f_i)^{-1}(\mathcal{A}_i)(p)^{-1}(x) \to (\mathcal{A}_j)^{-1}(p')^{-1}(x)$ is a homotopy equivalence. We would like to apply Theorem 12 of [12], but the inverse systems $(\mathcal{A}_i)^{-1}(p)^{-1}(x)$, $(\mathcal{A}_j)^{-1}(p')^{-1}(x)$ do not satisfy the hypotheses that they consist of compact ANR's.

Let $V$ be the $(1/l)$-neighborhood of $p^{-1}(x)$ in $E$; then $V \cap \{x\} \times N \subseteq \mathcal{A}_i(x)$.

There exists $\delta > 0$ such that if $d(x, p(e)) < \delta$, then $d(p^{-1}(x), p^{-1}(p(e))) < 1/l$. Choose $\delta > 0$ such that if $d(x', x) < \delta'$, then $d(p(x'), p(x)) < \delta/2$. Choose $\delta < 0$ such that $f(\delta') < \delta$, then $d(x', p(e)) < \delta$. Then $\delta' = \delta$ and the towers $(\mathcal{A}_i)^{-1}(x)$ and $(\mathcal{A}_j)^{-1}(x)$ are homotopy equivalent. Since $V \cap \{x\} \times N$ is a convenient ANR, there exists a compact ANR $M \subseteq V \cap \{x\} \times N$, which is a neighborhood of $p^{-1}(x) \cap \{x\}$.

Note that $(\mathcal{A}_i)^{-1}(x)$ and $(\mathcal{A}_j)$ are homotopy equivalent when $(\mathcal{A}_j)$ is a nested sequence.

Similarly construct $(\mathcal{M}_i)$ corresponding to $(\mathcal{A}_i)^{-1}(x)$. Now apply Theorem 12 of [12] to get that $(\mathcal{M}_i)$ and $(\mathcal{M}_j)$ are homotopy equivalent and hence, $(\mathcal{A}_i)^{-1}(x)$ are also homotopy equivalent.

Let $f; E \to B$ and $f': E' \to B$ be approximate fibrations where $E$, $E'$ and $B$ are compact ANR's. Let $g: E \to B$ and $g': E' \to B$ be the s-fibrations associated to $f$ and $f'$, respectively, as in Theorem 3.1.

**Theorem 3.6.** If there exists an equivalence $E \to E'$, then for each $e > 0$, there exist maps $h: E \to E'$ and $h': E' \to E$ such that

(i) $d'(f'(h(e), f(e))) < e$,

(ii) $d'(f(h'(e), f'(e))) < e$,

(iii) $gh$ and $h'g$ are homotopic to the identity.

**Proof.** Consider the projections $p_1: E \times B \to E$, $p_2: E \times B \to B$, $p_1: E' \times B \to E'$, $p_2: E' \times B \to B$.

Choose $\delta$ such that if $(x, y) < \delta$, then $d(x', y') < \delta/2$, $d(x, p(e)) < \delta$, and $d(p_1(x), p_1(y)) < \delta/2$.

By Proposition 3.4 of [7] there exist $\delta$-homotopies $h: E \times B \to E \times B$ and $h': E' \times B \to E' \times B$ such that

(a) $h_0 = id$, $h'_0 = id$,

(b) $h|_{\Gamma'} = id$, $h'|_{\Gamma'} = id$,

(c) $(\mathcal{A}_i)$, $(\mathcal{A}_j)$, id-preserved.

Let $G: E \to B$ be an inverse for $E$ and define $g: E \to E$ to be the composition $G$. Then

$$E \frac{E \times B}{E \times B} \xrightarrow{p_1} E \xrightarrow{h} E \xrightarrow{p_1} E.$$  

Then

$$d'(f'h, f'(e)) = d'(f_1 h_1 G_1(\text{id} \times f', f'(e))) = d'(p_2 h_1 G_1(\text{id} \times f', f'(e))) = d'(p_2 h_1 G_1(\text{id} \times f', f'(e))) < e,$$  

where $x = G_1(\text{id} \times f')(e)$. Since $d(h_1(e), e) < \delta$, $d'(p_2 h_1(e), p_2(e)) < e$.

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Define $h : E \to E'$ to be the composition
\[ E \xrightarrow{f} E/F_0 \xrightarrow{p_1} E/\pi \xrightarrow{\phi'} E'. \]
Similarly, $d'(f' h(x), f(x)) \leq \epsilon$.

Choose $j > i$ such that $F_i G_{E/F_0} = \pi_g$. (Here we use the fact that $E \times G \simeq \text{id}$.) Then
\[
\begin{align*}
   h \circ l \circ f = & \ p_i h_i (f(id \times f) p_i h_i G_i (id \times f)) \\
   = & \ p_i h_i f_i G_i (id \times f) \\
   = & \ p_i h_i G_i (id \times f) \\
   = & \ p_i G_i (id \times f) \\
   = & \ id.
\end{align*}
\]

We used the fact that the bonding maps for $E$ are inclusion maps. Similarly $g \circ l \circ f = \text{id}$.

4. Homotopy theory of the space of shape equivalences. A well-known result in the theory of locally trivial fibre bundles is that there is a bijection between the equivalence classes of such bundles over the $n$-sphere with a suitable fibre $F$ and the $n-1$ homotopy group of the space of homeomorphisms of $F$ with the compact-open topology [17]. In Section 11, we shall prove the analogue of this theorem for $s$-fibrations. However, since there is no natural topology for the set of shape equivalences (or in the terminology of [12], homotopy equivalences) of the fibre, we shall use the formalism of semi-simplicial theory in order to develop a homotopy theory.

Let $E$ be a tower of spaces and let $\mathcal{S}(E)$ be the semi-simplicial set of homotopy equivalences of $E$; an $n$-simplex of $\mathcal{S}(E)$ is a bundle equivalence $f : E \times A^r \to E \times A^r$ where $\pi : E \times A^r \to A^r$ is the product bundle with $F$ as fibre and $A^r$ is an $n$-cell. If $E^r$ is triangulated as an ordered $n$-simplex and $\partial_i : A^r \to A^{r-1}$ is the boundary operator which omits the $i$th vertex, then define $\partial_i f = f(\partial_i E \times A^{r-1})$. The $i$th degeneracy operator is defined analogously. It is easily seen that $\mathcal{S}(E)$ is a Kan complex [13]. Composition of bundle equivalences makes $\mathcal{S}(E)$ into a Kan monoid complex ([13], p. 68).

If $E^r$ is the $n$-sphere and $E_0 \in E^r$, then a based map of $E^r \to \mathcal{S}(E)$ is a bundle equivalence $f : E \times S^r \to E \times S^r$ such that $f(E \times \{0\}) = \text{id}$. Two such maps $f_0$ and $f_1$ are homotopic rel $X_0$ if there exists a bundle equivalence $H : E \times S^r \times [0, 1] \to E \times S^r \times [0, 1]$ (as bundles over $S^r \times [0, 1]$) such that $H(E \times S^r \times \{t\}) = f_1(E \times S^r \times \{t\})$, $t = 0, 1$, and $H(E \times X_0) \times [0, 1]$ = $f_0(E \times X_0)$ = $f_1(E \times X_0)$ = $f_0(E \times X_0)$ = $f_1(E \times X_0)$. We write $f_0 \simeq f_1$ if $\pi_g \simeq f_1$, where $\pi_g$ is an equivalence relation and the set of equivalence classes forms a group, $\pi_g(\mathcal{S}(E))$, called the $r$th homotopy group of $\mathcal{S}(E)$. Since $\mathcal{S}(E)$ is a semi-simplicial monoid, the group operation on $\pi_g(\mathcal{S}(E))$ can be defined by composition, $[f](g) = f \circ g$ ([13], p. 68) and $\pi_0(\mathcal{S}(E))$ is Abelian for $n \geq 2$ ([13], p. 68).

There are natural actions of $\pi_0(\mathcal{S}(E))$ and $\pi_1(\mathcal{S}(E))$ on $\pi_n(\mathcal{S}(E))$. The latter action is the one normally studied; the topological treatment in [8], pp. 131-134 can be followed in our circumstance. The conclusion which will be useful for us is that we can ignore basepoints if we only consider mappings of $S^n$, $n > 0$, into the path component of $\mathcal{S}(E)$ which contains the identity. I.e., $\pi_0(\mathcal{S}(E))$ is isomorphic to the group of homotopy classes of maps of $S^n$ into $\mathcal{S}(E)$ with the property that if $f : E \times S^n \to E \times S^n$ represents such a map, then $f|E \times \{x_0\}$ is homotopic to the identity ([8], p. 133). We shall abuse notation and use $\pi_n(\mathcal{S}(E))$ for the latter group. Let $S_0 = \{x_0, x_1\}$ and let $g : E \times S^0 \to E \times S^0$ represent an element of $\pi_0(\mathcal{S}(E))$. Define $g' : E \to E$ by the composition
\[ E \xrightarrow{g} E \times S^0 \xrightarrow{f} E \times S^0 \xrightarrow{\pi_0} E. \]
Let $f$ represent an element of $\pi_0(\mathcal{S}(E))$, $n > 0$. Define
\[ g'^{-1}(f) = \{
\begin{cases} (g')^{-1} & \text{if } f = (g' \times id), \\ g' & \text{if } f = (g' \times id). \end{cases}
\]

($(g')^{-1}$ denotes some homotopy inverse of $g'$.) It is easily checked this defines an action of $\pi_n(\mathcal{S}(E))$ on $\pi_n(\mathcal{S}(E))$ (using the abuse of notation as noted above). Let $\pi_n(\mathcal{S}(E))$ denote the orbit space of this action; i.e, the quotient space obtained from $\pi_n(\mathcal{S}(E))$ by identifying $[f]$ with $[g(f)]$ for all $[g] \in \pi_n(\mathcal{S}(E))$.

The following are easily shown.

**Proposition 4.1.** Let $f : E_0 \to E_1$ be a homotopy equivalence; then $f$ induces a bijection $\pi_n(\mathcal{S}(E_0)) \to \pi_n(\mathcal{S}(E_1))$ for all $n$.

**Proposition 4.2.** Let $f, g : E \to E'$ be towers of maps. $f \simeq g$ if and only if there exists a tower of maps $\pi : E \times [0, 1] \to E'$ such that $\pi_0 = f$ and $\pi_1 = g$.

5. Covering homotopy theorem. The main result of this section is a covering homotopy theorem for towers of maps into $s$-fibrations.

First we have need of the following two results whose proofs are exactly the same as the corresponding results for Hurewicz fibrations [16], pp. 100-101.

**Proposition 5.1.** Suppose that $p : E \to B$ is an $s$-fibration with $t$-function $\varphi$. Let $F_0, F_1 : X \times [0, 1] \to E_{\varphi_0}$ be maps such that there exist homotopies $H : X \times [0, 1] \times [0, 1] \to B$ and $G : X \times [0] \times [0, 1] \to E_{\varphi_0}$ with $H(x, t, 0) = p_{E_0}(F_t(x, 0), t) = G(x, 0, 0) = F_0(x, 0), G(x, 0, 1) = F_1(x, 0) = G_0(x, 0) = H_0(x, t), 0 = p_{E_0}(F_t(x, 0), t).$ Then there exists a homotopy $H' : X \times [0, 1] \times [0, 1] \to E_{\varphi_0}$ such that $H'(x, t, 0) = p_{E_0}(F_t(x, 0), t), H(x, t, 1) = p_{E_0}(F_t(x, 0), t), G(x, 0, 0) = F_0(x, 0), G(x, 0, 1) = F_1(x, 0)$, and $p_{E_0}(F_t(x, 0), t) = H_0(x, t, 0) = p_{E_0}(F_t(x, 0), t).$

**Corollary 5.2.** Suppose that $p : E \to B$ is an $s$-fibration with $t$-function $\varphi$. Let $F_0, F_1 : X \times [0, 1] \to E_{\varphi_0}$ be maps such that $F_0(X \times [0]) = F_1[X \times [0])$ and $p_{E_0}(F_0 = p_{E_0}(F_1$. Then there exists a homotopy $H : X \times [0, 1] \times [0, 1] \to E_{\varphi_0}$ such that $H(x, t, 0) = p_{E_0}(F_t(x, 0), t), H(x, t, 1) = p_{E_0}(F_t(x, 0), t), H(x, 0, t) = p_{E_0}(F_t(x, 0), t) = p_{E_0}(F_0(x, 0) = p_{E_0}(F_1(x, 0))$. The
THEOREM 5.3. Suppose that $p: E \to B$ is an $s$-fibration with $t$-function $\varphi$. Let $h: D \to E$ be a tower of maps and let $H: D \times [0, 1] \to B$ be a map such that $h$ is $(H, p, \text{id})$-preserving; then there exists a tower of maps $G: D \times [0, 1] \to E$ such that $p_G(x, t) = H_{t, \varphi}(x, t)$, $G(x, 0) = \varphi_{t, \varphi}(h_{t, \varphi}(x, 0))$, and $G$ is $(H, p, \text{id})$-preserving.

Proof. Let $x \in F$ and consider the homotopy $H_{t, \varphi}(x, t)$ and the map $h_{t, \varphi}(x, 0)$. Since $p_{t, \varphi}(h_{t, \varphi}(x, 0)) = H_{t, \varphi}(x, 0)$, there exists a homotopy $G_t: D \times [0, 1] \to E$ such that $G_t(x, 0) = \varphi_{t, \varphi}(h_{t, \varphi}(x, 0))$, and $p_{t, \varphi}(G_t(x, t)) = H_{t, \varphi}(x, t)$. Let $G(x) = h_{t, \varphi}(x, 0)$ and define $G(x, t) = \varphi_{t, \varphi}(h_{t, \varphi}(x, 0))$.

First we show that $G_t: D \times [0, 1] \to E$ is a tower of maps. Suppose that $\beta \geq x$. Let $d_{\beta}$ denote the bonding maps in $D$. Consider the homotopies $G_{t, \beta} = (d_{t, \beta}G_t \times \text{id})(x, 0) = e_{t, \beta}(x, 0)$, $G_{t, \beta} = (d_{\beta}G_t \times \text{id})(x, 0) = e_{\beta}G_t(x, 0)$, $p_{t, \beta}G_t = H_{t, \varphi}(x, 0)$, and $p_{t, \beta}G_t = H_{t, \varphi}(x, 0)$.

We can apply Proposition 5.1 to get a homotopy $G_t = (d_{t, \beta}G_t \times \text{id})e_{t, \beta}G_t$ so that $G_t$ is a tower of maps which is $(H, p, \text{id})$-preserving. We define $p_t G_t(x, t) = p_{t, \beta}G_t(x, t) = p_{t, \beta}G_t(x, t) = H_{t, \varphi}(x, t)$ and $G(x, 0) = e_{t, \beta}G_t(x, 0) = e_{t, \beta}G_t(x, 0)$.

By using Theorem 5.3, one can prove the analogues of Proposition 5.1 and Corollary 5.2 as in [16], pp. 100-101.

PROPOSITION 5.4. Let $p: E \to B$ be an $s$-fibration and let $E, E': D \times [0, 1] \to E$ be towers of maps such that there exist maps $H: D \times [0, 1] \to E$ and $G: D \times [0, 1] \to E$ for which

$G = H \circ G'$ and $G$ is $(H, p, \text{id})$-preserving. Then there exists a tower of maps $G': D \times [0, 1] \times D \times [0, 1] \to E$ such that

$G'(x, 0, 0) = G(x, 0)$, $G'(x, 0, 1) = G(x, 1)$, $G'(x, 1, 0) = G(x, 1)$, $G'(x, 1, 1) = G'(x, 1)$, and $G'$ is $(H, p, \text{id})$-preserving.

PROPOSITION 5.5. Let $p: E \to B$ be an $s$-fibration and let $E$ and $E': D \times [0, 1] \to E$ be towers of maps such that there exists a tower of maps $G: D \times [0, 1] \to E$ with

$G = H \circ G'$ and $G$ is $(H, p, \text{id})$-preserving.

Define $G_t: p_t G_t(x, 0) = e_{t, \beta}G_t(x, 0)$ and $k_t = e_{t, \beta}G_t(x, 0)$.

Define $G_t: p_t G_t(x, 0) = e_{t, \beta}G_t(x, 0)$ and $k_t = e_{t, \beta}G_t(x, 0)$.

and $p \circ E = p \circ E'$ and $G$ is $(p \circ E, p, \text{id})$-preserving. Then there exists a tower of maps $H: D \times [0, 1] \to E$ such that $H$ is $(p \circ E, p, \text{id})$-preserving for all $t \in [0, 1]$.

$H: D \times [0, 1] \times [0, 1] \to E$ and $H: D \times [0, 1] \times [0, 1] \to E'$.

6. Shape invariance of fibres. As the first major application of the covering homotopy Theorem 5.3, we shall prove that if $p: E \to B$ is an $s$-fibration with $B$ path-connected then for $b, b' \in B$, $p^{-1}(b)$ and $p^{-1}(b')$ are homotopy equivalent. This is a generalization of an analogous result for approximate fibrations proved by Coram and Duvall [2]. The proof is analogous to the proof of the corresponding result for Hurewicz fibrations (16), p. 101). Next we apply these techniques to simplify the definition of bundle maps.

Let $p: E \to B$ be an $s$-fibration and let $w: [0, 1] \to B$ be a path. Consider the inclusion $h: p^{-1}(w(0)) \to E$ and define $H: p^{-1}(w(0)) \times [0, 1] \to B$ by $H = w \circ (p \times \text{id})$ where $w(t)$, $t = w(t)$. By Theorem 5.3, there exists a tower of maps $G: p^{-1}(w(0)) \times [0, 1] \to E$ such that $p_G(x, t) = H_{t, \varphi}(x, t)$ and $G(x, 0) = \varphi_{t, \varphi}(p(x, 0))$ where $\varphi$ is the $t$-function of $p$. Define $E^{-1}(w(0)) = p^{-1}(w(0)) \cup F^{-1}(w(0))$ by the composition

$E^{-1}(w(0)) = \{(x, t) \in [0, 1] \times [0, 1] \to E \}$.

Let $w: [0, 1] \to B$ be a path such that $w(t) = w(t)$ for $t = 0$,

and such that $w$ is homotopic rel $[0, 1]$ to $w'$. If we construct $G'$ and $f'$ corresponding to $w'$ as above, then it follows from Proposition 5.4 that the towers of maps $f, f': p^{-1}(w(0)) \to p^{-1}(w(1))$ are homotopic. Hence we have a functor $L$ from the fundamental groupoid of $B$ (16), p. 101) to the category whose objects are towers of spaces and whose morphisms are homotopy classes of towers of maps.

Let $w, w'$ be paths in $B$ such that $w(t) = w(t)$ for $t \in [0, 1]$. Recall that $w \circ w'$ is the path defined by

$w \circ w'(t) = \begin{cases} w(t) & \text{for } t \in [0, 1] \\ w(t) & \text{for } t \in [1, 1]. \end{cases}$

Suppose that $w'(t) = w(t)$ for $0 \leq t \leq 1$. Construct $G, G'$ and $f'$ as above. By the assumption on $w'$, we may also assume that $G(x, t) = \varphi_{t, \varphi}(x)$ for $0 \leq t \leq 1$. Let $G': I \to F'$ be an increasing function such that $G'(x) = \varphi_{x, \varphi}(x)$, and $G'(x) = \varphi_{x, \varphi}(x)$. Let $k_t: p_t G_t(x, 0) = p_t G_t(x, 0)$ be a homotopy such that

$k_0 = f_{t, \varphi}(p_{t, \varphi}(x, 0))$ and $k_1 = e_{t, \beta}G_t(x, 0)$.

Define $G_t: p_t G_t(x, 0) = E_t$ by

$G_t(x, t) = \begin{cases} G(t, x) & \text{for } 0 \leq t \leq 1 \\ G(t, x) & \text{for } 1 \leq t \leq 2, \end{cases}$

$G_t(x, t) = \begin{cases} G(t, x) & \text{for } 0 \leq t \leq 1 \\ G(t, x) & \text{for } 1 \leq t \leq 2, \end{cases}$

$G_t(x, t) = \begin{cases} G(t, x) & \text{for } 0 \leq t \leq 1 \\ G(t, x) & \text{for } 1 \leq t \leq 2, \end{cases}$

$G_t(x, t) = \begin{cases} G(t, x) & \text{for } 0 \leq t \leq 1 \\ G(t, x) & \text{for } 1 \leq t \leq 2, \end{cases}$
Note that \( G''(x, 0) = e_{0_0} = 0(x) \) and \( p_x D''(x, t) = \frac{w * w(t)}{w} \). Consider

\[ f''(x) = G''(x, 1) = G''(f''(x, t), e_{0_0} = e_{0_0} = 0(x)) \]

We have shown the following.

**proposition 6.1.** \( L[w * w] = \Gamma[w] + L[w''] \). (**w'** denotes composition of homotopy classes; i.e. \( [f'] + [g'] = [f' + g'] \)).

**Theorem 6.2.** Let \( p : E \rightarrow B \) be an s-fibration and let \( b_0, b_1 \) lie in the same path component of \( B \). Then there exists a homotopy equivalence \( p^{-1}(b_0) \rightarrow p^{-1}(b_1) \).

**Theorem 6.3.** Let \( p : E \rightarrow B \) and \( p' : E' \rightarrow B' \) be s-fibrations, let \( k : B \rightarrow B' \) be a continuous map and let \( K : E \rightarrow E' \) be a (\( q, p, k \))-preserving tower of maps. If \( K \) is a homotopy equivalence, then \( a \) through \( b \) which lie in \( E \) is connected by \( b_n \), \( K \mid p^{-1}(b) = p^{-1}(k(b)) \) is a homotopy equivalence, then for all \( b \) which lie in the path component of \( B \) which contains \( b_n \), \( K \mid p^{-1}(b) = p^{-1}(k(b)) \) is a homotopy equivalence. In particular, if \( B \) is path-connected, then \( K \) is a bundle map.

**Proof.** Let \( w : [0, 1] \rightarrow B \) be a path such that \( w(0) = b_0 \) and \( w(1) = b_1 \). Construct \( E, G \) and \( f \) as above. Then \( k \) is a path in \( B' \) such that \( w(0) = b_0 \) and \( w(1) = b_1 \); construct the corresponding \( H' \), \( G' \), and \( f' \) and \( f'' \) as above. Let \( K = K \mid p^{-1}(b_0) \) and \( K = K \mid p^{-1}(b_1) \) and \( g : p^{-1}(k(b_1)) \rightarrow p^{-1}(k(b)) \) the homotopy inverse of \( G \).

Let \( G'' = K \mid G \mid (x) \mid (g \times id) : p^{-1}(k(b_0)) \times [0, 1] \rightarrow E' \) and let \( f'' : p^{-1}(k(b)) \rightarrow p^{-1}(k(b_1)) \) be the composition

\[ f'' \mid k(b_0) \ni f'' \mid k(b_1) \ni \text{id} \ni f'' \mid k(b_1) \ni \]

Note that

\[ G'' = G E \mid (x) \mid (g \times id) = k \mid \text{id} \mid (g \times id) = h \mid (g \times id) = p' G \]

and

\[ G'' \mid k^{-1}(b_0) \ni \mid (x) \ni K \mid (x) \mid k^{-1}(b_1) \ni (x) \]

which is homotopic to the homotopy class map \( p^{-1}(k(b_0)) \times [0, 1] \). By Proposition 5.5, this homotopy can be extended to a bundle homotopy between \( G' \) and \( G'' \) which covers \( G' \). In particular, \( f'' \) and \( f'' \) are homotopic; hence \( f'' \) is a homotopy equivalence. And since \( f'' = k \mid f'' \ni g \) and since both \( f'' \) and \( g \) are homotopy equivalences, \( K = \mid f'' \ni g \) is a homotopy equivalence.

**7. Path lifting property.** W. Hurewicz defined the concept of path lifting property of a map and showed its equivalence with the covering homotopy property [9]. We develop an analogous theory in this section which will be useful in the next two sections.

If \( E = \{ E, q, f, \Gamma \} \) is a tower of spaces, then \( E' \) is the collection of paths in \( E \) with the compact-open topology and \( \text{ps}(\phi)(t) = \text{ps}(\phi)(t) \).

**Proposition 7.3.** Let \( p : E \rightarrow B \) be an s-fibration and let \( \lambda : E \rightarrow B' \) be a lifting function for \( p \). Define \( \lambda : E' \rightarrow E' \) by

\[ \lambda(w) = \lambda(w(0), P_{\lambda(w)} * w) \]

where \( \phi \in E' \). Then there exists a homotopy \( \Phi : E' \times [0, 1] \rightarrow E' \) such that \( \phi \mid E' \times [0, 1] \ni \text{id} \Phi \mid E' \times [0, 1] \ni \lambda \) and \( \Phi \) covers the projection \( E' \times [0, 1] \rightarrow E' \).
Proof. Let \( \varphi \) be the \( t \)-function for \( p \) and let \( w \in E_{0 \alpha}^{[1]} \). Define \( \sigma = \rho_{0\alpha} \circ w \) and \( \sigma^{t-1} \in B^{[1]} \) by
\[
\sigma^{t-1}(t) = \begin{cases} 
\sigma(t+1) & \text{for } t \in [0, 1) \\
\sigma(0) & \text{for } t \in [1, \infty]
\end{cases}
\]
Define \( H_{\alpha} : E_{0\alpha}^{[1]} \times [0, 1] \to B^{[1]} \) by
\[
H_{\alpha}(w, s) = \begin{cases} 
\rho_{0\alpha}(w(t)) & \text{for } t \in [0, s] \\
\rho_{0\alpha}(\sigma(t)) & \text{for } t \in [s, 1]
\end{cases}
\]
By Proposition 5.1, \( \varphi = \{ H_{\alpha} : E_{0\alpha}^{[1]} \times [0, 1] \to B^{[1]} \) is a tower of maps; it is easily checked that \( \varphi \) has the desired properties.

8. Pullbacks. Let \( \xi = (p, E, B) \) be an \( s \)-fibration and let \( f : B' \to B \) be a continuous map. For \( x \in \Gamma \), let \( E'_x = \{ (b, x) \in B' \times E \mid f(b) = p_x(x) \} \) with the subspace topology.

If \( b \neq x \), define
\[
(i) \quad c'_x : E'_x \to E_x \quad \text{by} \quad c'_b(b, x) = (b, e_x(b)),
\]
\[
(ii) \quad F_x : E'_x \to E_x \quad \text{by} \quad F_x(b, x) = x,
\]
\[
(iii) \quad F : \Gamma \to \Gamma \quad \text{by} \quad F(\alpha) = \alpha,
\]
\[
(iv) \quad p'_x : E'_x \to B' \quad \text{by} \quad p'_b(b, x) = b.
\]
It is easily checked that \( \xi' = \{ p'_x : E' \to B' \} \) is a map.

THEOREM 8.1. \( \xi' : E' \to B' \) is an \( s \)-fibration with the same \( s \)-function as \( p \) and \( (E, F) \) is a bundle map.

Proof. Given \( x \in \Gamma \), choose \( \beta = \varphi(x) \) where \( \varphi \) is the \( t \)-function for \( p \). Let \( G : X \to B' \) be a homotopy for \( p \). Let \( g : X \times [0, 1] \to E \) be a map such that \( p_g \circ G = H \). There exist functions \( e_x(p_g \circ G)(x) = G(x, 0) \). Define \( \xi' \) as the \( s \)-fibration. Since \( F' \) is a bundle map, \( (E', f) \) is a bundle map.

THEOREM 9.1. Let \( \xi = (p, E, B) \) be an \( s \)-fibration and let \( f_0, f_1 : X \to B \) be homotopic maps. Then \( f_0^* \xi \) and \( f_1^* \xi \) are \( s \)-homotopic.

Proof. (see [16], p. 102). Let \( f_0^* \xi = (p^0, E^0, X) \) and \( f_1^* \xi = (p^1, E^1, X) \) and let \( (E^0, f_0^* \xi) = (p^0, E^0, X) \) and \( (E^1, f_1^* \xi) = (p^1, E^1, X) \) be the \( s \)-structures constructed above. Let \( F : X \times [0, 1] \to B \) be a homotopy such that \( F_0 = f_0 \), \( F_1 = f_1 \), and \( F(t) = f(t) \). By Theorem 5.3, there exist towers of maps \( G^0 : E^0 \times [0, 1] \to E \) and \( G^1 : E^1 \times [0, 1] \to E \) such that
\[
\rho_{\alpha_0} G^0(x, t) = \rho_{\alpha_0} G^1(x, t), \quad t = 0, 1,
\]
\[
G^0(x, 0) = e_{\alpha_0} \rho_{\alpha_0} G^0, \quad G^1(x, 1) = e_{\alpha_0} \rho_{\alpha_0} G^1.
\]
Define \( g^0 : E^0 \to E \) and \( g^1 : E^1 \to E \) by
\[
g^0(b, x) = (b, G^0(b, x, 1)) \quad \text{for} \quad (b, x) \in E^0_{\alpha_0},
\]
\[
g^1(b, x) = (b, G^1(b, x, 0)) \quad \text{for} \quad (b, x) \in E^1_{\alpha_0}.
\]
Note that
\[
G^0 g^0 = G^1 g^1 = E \quad \text{and} \quad g^0 G^0 = G^1 g^1 = E \cdot 1
\]
and
\[
p g^0 = p g^1 = (p g^0)_{\alpha_0} = (p g^1)_{\alpha_0} = p_{\alpha_0}.
\]
By Proposition 5.5, \( G^0 \) and \( G^1 \) are bundle homotopy. Similarly, \( F^0 \) and \( F^1 \) are bundle homotopy. Since \( G^0 g^0 = G^1 g^1 \), the restriction of this bundle homotopy to \( E_1 \) induces a bundle homotopy \( E_1 g^0 = E_1 g^1 \) which covers \( f_1 \). This, in turn, induces a bundle homotopy \( g^0 g^1 = g^1 g^0 \) covering \( g \). Similarly, \( g^0 g^1 \).

COROLLARY 9.3. Let \( \xi = (p, E, B) \) be an \( s \)-fibration and let \( B \) be contractible; then \( \xi \) is bundle equivalent to the trivial \( s \)-fibration \( p \) (\( B_{\alpha_0} \times \alpha_0 \to B \) for any \( b_0 \in B \).

9. Homotopy extension theorem. The following is the main result of this section.

The analogous theorem for Hurewicz fibrations was proved by Fadell [5]; this proof is very similar.

THEOREM 9.1. Let \( p : E \to B \) and \( g : E' \to B \) be \( s \)-fibrations where \( B \) is a polyhedron and let \( A \) be a subpolyhedron of \( B \). Let \( H : X \times [0, 1] \to B \times [0, 1] \) and \( T = (p \times g)^{-1}(X) \) where \( p \times g : E \times [0, 1] \to B \times [0, 1] \). Let \( \varphi : T \to E \) be a bundle map covering the map \( X \to B \) defined by \( (y, t) \to y \). Then there exists a bundle map \( \Phi : E \times [0, 1] \to E \) covering \( \varphi \) and \( \Phi \) is a bundle homotopy.

Proof. Let \( B \) be an open set in \( B \times [0, 1] \) such that \( X \subseteq U \) and \( X \) is a strong deformation retract of \( U \); let \( H : U \times [0, 1] \to U \) be a homotopy such that \( H(x, 0) = x, H(x, 1) \in X \) and \( H(x, t) = x \) for all \( x \in X, t \in [0, 1] \). Let \( \Phi : E \times [0, 1] \to E \) be defined by \( \Phi(x)(t) = H(x, t) \). Let \( \varphi_{\alpha_0} = (p_{\alpha_0} \times g_{\alpha_0})^{-1} \) be the projection. Consider the projections \( \pi : B \times [0, 1] \to B \), \( \pi : E \times [0, 1] \to E \), \( e : B \times [0, 1] \to [0, 1] \).

Let \( \xi = (p, E, B) \) be induced by \( \pi \) and define \( \psi_{\alpha_0} : V_{\alpha_0} \to B_{\alpha_0} \) by \( \psi_{\alpha_0} = \pi \circ \pi_{\alpha_0} = (p_{\alpha_0} \times g_{\alpha_0})^{-1} \) and define \( \Phi_{\alpha_0} : E_{\alpha_0} \to E_{\alpha_0} \) by \( \Phi_{\alpha_0}(x)(t) = \psi_{\alpha_0}(x)(t) \).

Let \( \xi \) and \( \xi' \) be regular lifting functions for \( p \) and \( p' \), respectively (cf. Corollary 7.2). Define \( \Phi' : \Gamma \to \Gamma \) by \( \Phi'(x)(t) = \psi_{\alpha_0}(x)(t) \).

We now define the towers of maps \( G^0 : E^0 \times [0, 1] \to E \) and \( G^1 : E^1 \times [0, 1] \to E \) such that
\[
G^0(x, 0) = e_{\alpha_0} \rho_{\alpha_0} G^0, \quad G^1(x, 1) = e_{\alpha_0} \rho_{\alpha_0} G^1.
\]
Define \( g^0 : E^0 \to E \) and \( g^1 : E^1 \to E \) by
\[
g^0(b, x) = (b, G^0(b, x, 1)) \quad \text{for} \quad (b, x) \in E^0_{\alpha_0},
\]
\[
g^1(b, x) = (b, G^1(b, x, 0)) \quad \text{for} \quad (b, x) \in E^1_{\alpha_0}.
\]
Let $g : E \to D$ be an $s$-fibration where $D^s$ is the $n$-cell and let $S = (D^s \times [0, 1]) \cup (D^s \times [0]) \cup (D^s \times [1])$. Let $D = (p \times \text{id})^{-1}(S)$. If $f : D \to D$ is a bundle equivalence such that $f|_{E \times \{0\}} = \text{id}$, then there is a bundle equivalence $g : D \to D$ such that $g|_{(p \times \text{id})^{-1}(D^s)} = (p \times \text{id})^{-1}(S)$ and the bundle map $g : f : D \to D$ admits an extension $g : E \times [0, 1] \to (E \times [0, 1])$ which is also a bundle equivalence.

Proof. Let $E = p^{-1}(x_0)$ be the fibre of $p$ where $x_0$ is a vertex of $D^s$. By Corollary 9.2, there exists a bundle equivalence $\tilde{g} : E \to E^s$ such that $\tilde{g}|_{E \times \{0\}} = \text{id} \times \{x_0\}$. We may choose $\tilde{g}^{-1}$ so that $\tilde{g}^{-1} : E \times \{x_0\} \to (E \times \{x_0\})$ is (the shift of) the projection along the first factor. $\tilde{g}$ induces bundle equivalences $\tilde{g} : E \times \{0\} \to E \times \{0\}$ and $g : D \to E \times S$ where $g = \tilde{g} \times \text{id}$.

Consider $f' = \tilde{g}^{-1} \circ f \circ \tilde{g} : E \times S \to E \times S$. If we choose $(x_0, 0)$ to be a base point for $S$, then $f'$ represents an element of $\pi_n(\mathbb{F}(E), \text{id})$. Let $g' : E \times S \to E \times S$ represent the inverse of the class of $f'$; we may assume that $g'|_{E \times \{(E \times \{0\}) \cup (D^s \times [0, 1]) \cup (D^s \times [1])\}} = \text{id}$.

Let $g = g'^{-1} \circ g' \circ \tilde{g}$. Since $\mathbb{F}(E)$ is a simplicial group and by the choice of $g'$, there exists an extension $g' : E \times D^s \to E \times D^s$ of $g' \circ f'$. Let $g = (p \times \text{id})^{-1}(S)$ be a $s$-fibration. $\tilde{g} = (p \times \text{id})^{-1}(S)$ is a bundle homotopy to $g' \circ f' \circ \tilde{g} = g$, and $g$ is a bundle homotopy to $\tilde{g}$. Since $g \circ f$ is bundle homotopy to $\tilde{g} \circ f$, the existence of the extension $g$ follows from a third application of Theorem 9.1.

LEMMA 10.2. Let $g : E \to D^s$ be an s-fibration and let $H : E \to E'$ be a bundle map covering the identity. Define $g' = (p \times \text{id})^{-1}(S)$ be a bundle map covering the identity such that $g \circ H|_{(p \times \text{id})^{-1}(D^s)}$ is bundle homotopic to $g|_{(p \times \text{id})^{-1}(D^s)}$. Then there exists a bundle map $g' : E' \to E$ covering the identity such that $g \circ H$ is bundle homotopic to $\tilde{g}$ by a homotopy $\Phi$ which extends $\tilde{g}$ and covers the projection $D^s \times [0, 1] \to D^s$.

Sublemma. Let $x_0$ be a vertex of $D^s$. Define $C : D^s \to (D^s)^0$ by $D^s \times [0, 1] \cup (D^s \times [0]) \cup (D^s \times [1])$. Then $\tilde{g} = (p \times \text{id})^{-1}(S)$ and $g'$ be a homotopy inverse for $H^0$. Define $\tilde{H} : E \to E'$ and $\tilde{g}' : E' \to E$ by
There is no loss of generality in assuming that \( \sigma(a) \gg \lambda(a) \) and \( \sigma'(b) \gg \lambda'(b) \) for all \( a \) and \( b \). Note that

\[
\mathcal{R}(1, 0) = \lambda \{K_{w_{\mathcal{H}}}, \mathcal{L}(w_{\mathcal{H}}), \mathcal{C}(p_{\mathcal{H}w_{\mathcal{H}}}(\mathcal{H}))\}(0, 1), \mathcal{C}(p_{\mathcal{H}w_{\mathcal{H}}}(\mathcal{H}))\}(1)
\]

(1)

\[
\mathcal{R}(1, 1) = \lambda \{K_{w_{\mathcal{H}}}, \mathcal{L}(w_{\mathcal{H}}), \mathcal{C}(p_{\mathcal{H}w_{\mathcal{H}}}(\mathcal{H}))\}(0, 1), \mathcal{C}(p_{\mathcal{H}w_{\mathcal{H}}}(\mathcal{H}))\}(1)
\]

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Define $h: D \to \Gamma$ to be an increasing function such that

\[ h(\alpha) = \begin{cases} 
    0, & \text{if } \alpha \in \mathcal{H}G' \circ \mathcal{O}(\alpha), \\
    \phi(\alpha), & \text{for all } \alpha \in \Gamma; \\
    \phi(\alpha) \text{ is the } t \text{-function of } \phi. 
\end{cases} \]

\[ h_\delta: D_{\delta(\alpha)} \to D_{\delta} \]

by

\[ \begin{aligned}
    L_{\delta(\alpha)}(t, \alpha) &= \begin{cases} 
        \delta(\alpha), & t = 1, \delta(\alpha) \in \delta^*, \\
        \phi(\alpha), & t = [0, 1], \\
        \phi(\alpha), & t = 0.
    \end{cases}
\end{aligned} \]

It is easily checked that $h = (h_\delta)$ is a bundle equivalence of $D$.

By Lemma 10.1, there is a bundle map $\gamma: D \to D$ such that

\[ \gamma : \{(\phi \circ \phi^{-1}(\delta^*) \times \{0, 1\}) \cup (\delta \times [0, 1]) \} = \text{id} \]

and $\gamma$ can be extended to a bundle map $\gamma': E \times [0, 1] \to E \times [0, 1]$. Let $\gamma'$ be the composition

\[ E \times [0, 1] \to E \times [0, 1] \]

where $\pi$ is projection. $\gamma' = \gamma'$ and $\gamma = \pi \gamma'$ are the desired bundle maps.

**Theorem 10.3.** Let $p: E \to B$ and $\phi: E' \to B$ be s-fibrations such that $B$ is a finite connected polyhedron $B$. If $h: E \to B$ is a bundle covering the identity, then $h$ is a bundle equivalence.

The proof is by induction on the number of simplexes of a triangulation of $B$ and by the use of Lemma 10.2 (cf. [5]).

**11. Classification theorem.** Let $s\mathcal{F}(X, E)$ denote the set of equivalence classes of s-fibrations over $X$ whose fibre is homotopy equivalent to $E$ where the equivalence relation is bundle equivalence.

Consider the n-sphere $S^n = B^n \cup B^n$ where $B^n$ and $B^n$ are n-cells with $B^n \cap B^n = S^{n-1}$, an $(n-1)$-sphere. Suppose that $x_0 \in S^{n-1}$. Let $p: E \to S^n$ be an s-fibration such that $p^{-1}(x_0)$ is homotopy equivalent to the tower of spaces $E$. By Corollary 9.2, there exist equivalences $H^+: E^+ \to p^{-1}(x_0) \times B^n$ and $H^+: E^+ \to p^{-1}(x_0) \times B^n$ where $H^+(X^{-i}(x_0)) = x_0 \times \{0\}, \phi^+: E^+ \to B^n$ and $\phi^+: E^+ \to B^n$ are restrictions of $p$ to $B^n$ and $B^n$, respectively, Note that $L = H^+(x_0) \times \{0\}$. By assumption, there exists a homotopy equivalence $G: p^{-1}(x_0) \to E$. Define $\mu: s\mathcal{F}(S^n, E) \to \pi_{n+1}(\mathcal{F}(E))$ by sending the equivalence class of $p: E \to S^n$ to the homotopy class represented by

\[ (G \times \text{id}) \circ L^*(G^{-1} \times \text{id}) : E \times S^{n-1} \to E \times S^{n-1}. \]

Note that this map represents an element of $\pi_{n+1}(\mathcal{F}(E))$; however, in order to show that $\mu$ is well-defined, we have to pass to $\pi_{n+1}(\mathcal{F}(E))$. The main result of this section is the following.

**Theorem 11.1.** $\mu: s\mathcal{F}(S^n, E) \to \pi_{n+1}(\mathcal{F}(E))$ is a bijection.

**Proposition 11.2.** $\mu$ is independent of the choices of the equivalences $H^+, H^-$ and $G$.

**Proof.** Let $\tilde{G}: S^n$ be an equivalence; then

\[ (G \times \text{id}) \circ L^*(G^{-1} \times \text{id}) = (G \times \text{id}) \circ L^*(G^{-1} \times \text{id}) \]

which, by using the action of $\pi_{n}(\mathcal{F}(E))$ on $\pi_{n+1}(\mathcal{F}(E))$, is equivalent to

\[ (G \times \text{id}) \circ L^*(G^{-1} \times \text{id}) \]

The independence of the choices of $H^+$ and $H^-$ follows by a similar, but much simpler argument since the restriction of two equivalences $E^+ \to E^{-1}(x_0) \times B^n$ to the s-fibration over $S^n$ is homotopic.

**Proposition 11.3.** $\mu$ is well-defined.

**Proof.** Let $p: E \to S^n$ be an s-fibration such that there exists an equivalence $K: E \to E$. Let $G = K \times \gamma^{-1}(x_0), H^+ = (G^{-1} \times \gamma \times \gamma^{-1}) \times L^*(B^n), H^+ = (G \times \gamma \times \gamma^{-1}) \times L^*(B^n)$ and $L = L^*(B^n)$. Then

\[ (G \times \gamma \times \gamma^{-1} \times \text{id}) \]

Now suppose that $E$ is a singleton set $F$ where $F$ is an ANR. We shall show that $\mu$ is a bijection by constructing its inverse $\lambda: \pi_{n+1}(\mathcal{F}(E)) \to \pi_{n+1}(\mathcal{F}(E), E)$. Let $p: E \to S^n$ be an s-fibration which represents an element of $\pi_{n+1}(\mathcal{F}(E))$ such that $h(x_0 \times \{0\})$ is homotopic to $\alpha$. Let $h = \{h\}$, then let $E$ be the quotient space obtained from $(F \times B^n)$ by identifying $x \in F \times S^{n-1}$ with $h(x)$. It is at this point that the hypothesis that $F$ is a singleton is used; the author is unable to perform a similar construction to obtain $E$ otherwise.

Define $p: E \to S^n$ by $p(x, y) = y$, where $x \in F$ and $y \in S^n$.

**Proposition 11.4.** $E \to S^n$ is an approximate fibration.

**Proof.** It is easily checked that $p$ is a complete mapping and, hence, by [3] (see also [6]), $p$ is an approximate fibration.

Let $p: E \to S^n$ be an s-fibration associated to $p$ as in Proposition 3.3. Recall that the fibre of $p$ is homotopy equivalent to $E$.

Define $\lambda: \pi_{n+1}(\mathcal{F}(E)) \to \pi_{n+1}(\mathcal{F}(E))$ by sending the class of $h$ to the class of $p: E \to S^n$.

**Proposition 11.5.** $\lambda$ is well-defined.
Proof. Let us first consider the case when $h$ is homotopic to $y'$ by a homotopy $H': X \times S^{n-1} \times [0,1] \rightarrow X \times S^{n-1} \times [0,1]$ where $H'[x \times S^{n-1} \times 0] = h[x \times 0]$ and $H'[x \times S^{n-1} \times 1] = y'[x \times 1]$. Let $D$ be the quotient space obtained from $(F \times X) \times [0,1] \cup (F \times X) \times [0,1]$ by identifying $x \in F \times S^{n-1} \times [0,1]$ with $H(x)$. Define $\delta : D \rightarrow S^{n} \times [0,1]$ by $\delta(x,y) = y$ where $x \in F$ and $y \in S^{n} \times [0,1]$. As in Proposition 11.5, $\delta$ is an approximation of the fibration. Let $D : D \rightarrow S^{n} \times [0,1]$ be the associated $s$-fibration as given in Proposition 3.3.

By Theorems 3.5 and 10.3, the $s$-fibrations $p : E \rightarrow S^{n}$ and

$$p : E \rightarrow S^{n} \times [0,1]$$

are equivalent (we identify $S^{n}$ with $S^{n} \times [0,1]$). Let $\varphi : E \rightarrow D$ be the composition $E \rightarrow D : S^{n} \times [0,1] \rightarrow D$ such that the first map is an equivalence of given $E$. Note that $\varphi$ is a bundle map such that $\delta \varphi = p$. By Theorem 3.5, there exists a tower of maps $\varphi : E \rightarrow S^{n} \times [0,1] \rightarrow D$ such that $\varphi \delta \varphi = \varphi \varphi = p \times \text{id}$ and $\varphi = (q \times \text{id}, \delta, \text{id})$-preserving. Since $\varphi$ is a bundle map, by Theorem 6.3, $\varphi$ is a bundle map covering $\varphi$ and hence, by Theorem 10.3, $\varphi$ is a bundle equivalence. In particular, $q \times \text{id}$ and $\delta \varphi (S^{n} \times [1])$ are equivalent; again by using Theorems 3.5 and 10.3, these two $s$-fibrations are equivalent to $p : E \rightarrow S^{n}$ and $p' : E' \rightarrow S^{n}$, respectively, where $p'$ and $E'$ are defined analogous to $p$ and $E$, for $h'$. Thus $p$ and $p'$ are equivalent.

Let $g : E \rightarrow E$ be a homotopy equivalence; to complete the proof of the proposition, we must show that the $s$-fibrations associated to $h$ and $h = (q \times \text{id}) \ast h \ast (q \times \text{id})$

are equivalent.

The function $p' : F \rightarrow F \times Q$ is homotopic to an embedding $\lambda_{1} : F \rightarrow F \times Q$ and there exists $\lambda_{2} : F \times Q \rightarrow F$ such that $\lambda_{2} = \lambda_{1} \circ \text{id}$. Define $\lambda_{3}(x,y,z) = (\lambda_{2}(x,y),z)$ and $\lambda_{4}(x,y,z) = (\lambda_{2}(x,y),z)$ $\ast (q \times \text{id})$ for $(x,y,z) \in F \times Q \times S^{n-1}$. Note that $q \times \text{id} : F \times Q \times S^{n-1} \rightarrow E \times S^{n-1}$ to $\lambda_{2}$ and $\lambda_{4}$ respectively.

Define $\lambda_{3} : F \times Q \times S^{n-1} \rightarrow E \times Q \times S^{n-1}$ and $\lambda_{4} : F \times Q \times S^{n-1} \rightarrow E \times Q \times S^{n-1}$ by

$$\lambda_{3}((x,y,z)) = \lambda_{2}(x,y), \lambda_{4}((x,y,z)) = (q \times \text{id})(h(\lambda_{2}(x,y),z))$$

where $p : F \times Q \rightarrow Q$ is projection and $\tau : F \times S^{n-1} \rightarrow \tau \times Q \times S^{n-1}$ is given by $(x,y,z) = (x,y,z)$. Form $g' : E' \rightarrow S^{n}$ and $g'' : E'' \rightarrow S^{n}$ corresponding to $h'$ and $h''$, respectively. Since $h'$ is bundle homotopic to $h''$ (over $S^{n-1}$), the $s$-fibrations $(g', E', S^{n})$ and $(g'', E'', S^{n})$ are bundle equivalent by the first part of this proof.

Define $q : E' \rightarrow E''$ by $q(x,y) = (x,0,0)$ where $(x,y) \in F \times (B^{+} \cup B^{-})$. By Theorems 3.5 and 10.3, $q$ induces a bundle equivalence $q : E' \rightarrow E''$. Define $\xi : E' \rightarrow E''$ by $\xi(x,y) = (\lambda_{3}(x,y),y)$ where $(x,y) \in F \times (B^{+} \cup B^{-})$. Again by Theorems 3.5 and 10.3, $\xi$ induces a bundle equivalence $\xi : E' \rightarrow E''$. Hence, $E'$ and $E''$ are bundle equivalent and the proof of 11.6 is completed.

**PROPOSITION 11.7.** $\mu \ast = \text{id}$.

Proof. Let $\beta : E \times S^{n-1} \rightarrow E \times S^{n-1}$ represent an element of $\pi_{n-1}(S^{n}, x)$.

Let $\beta : E \rightarrow S^{n}$ be the $s$-fibration associated to $h$ by $h$ as constructed above. Let $\varphi : E \times S^{n} \rightarrow E \times S^{n}$ be the inclusion map and define $\varphi' : E \times S^{n} \rightarrow E \times S^{n}$ by $\varphi'((x,y)) = (\varphi(x), y)$. Note that $\varphi'_{*} : E \times S^{n} \rightarrow E \times S^{n}$ is a tower of maps which by Theorems 3.5 and 10.3 is a bundle equivalence. Define $g_{\beta} : E \times S^{n} \rightarrow E \times S^{n}$ similarly; note that $g_{\beta}^{*} h = g_{\beta}^{*}$. $g_{\beta} = (g_{\beta}^{*})^{-1} E \times S^{n-1}$ is a bundle equivalence which represents the class associated to $\beta$ by $\mu$. Note that $g_{\beta}^{*} h = g_{\beta}^{*}$ is bundle homotopic to $g_{\beta}^{*} E \times S^{n-1}$ and since $g_{\beta}^{*} h = g_{\beta}^{*}$, $g_{\beta}$ is bundle homotopic to $h$.

**PROPOSITION 11.8.** $\mu \ast = \text{id}$.

Proof. Let $\rho : E \rightarrow S^{n}$ be an $s$-fibration; choose equivalences $H' : E' \times B^{n} \rightarrow E'' \times B^{n}$, $H'' : E' \times B^{n} \rightarrow E'' \times B^{n}$ and $G : E' \rightarrow E''$ as above. Let $h = (q \times \text{id}) \ast H' \ast (H'')^{-1}$ and $(q \times \text{id}) \ast G$. $h$ is an equivalence of $F \times S^{n-1} \rightarrow E$ as constructed above. Define $g_{\rho} : E' \times B^{n} \rightarrow E''$ and $g_{\rho} : E' \times B^{n} \rightarrow E''$ as in the proof of Proposition 11.7.

Let $\xi_{\ast} = g_{\rho}^{*} G \times \text{id} \ast H' \ast (H'')^{-1}$ and $\xi_{\ast} = g_{\rho}^{*} G \times \text{id} \ast H' \ast (H'')^{-1}$ and $\xi_{\ast} = g_{\rho}^{*} G \times \text{id} \ast H' \ast (H'')^{-1}$. We would like to define $\xi : E \rightarrow E$ by using $\xi_{\ast}$ and $\xi_{\ast}$, unfortunately they do not agree on $E \times S^{n-1}$. Note that $\xi_{\ast}^{*} | E \times S^{n-1} = g_{\rho}^{*} G \times \text{id} \ast H' \ast (H'')^{-1}$ and $\xi_{\ast}^{*} | E \times S^{n-1} = g_{\rho}^{*} G \times \text{id} \ast H' \ast (H'')^{-1}$. By Theorem 9.1, this bundle homotopy can be extended to a bundle homotopy of $E' \times B^{n}$. By using the end of this bundle homotopy $\xi_{\ast}$ can be extended to $\xi : E \rightarrow E$. Since $\xi_{\ast}$ and $\xi_{\ast}$ are equivalences, $\xi$ is a bundle map by Theorem 6.3 and hence, by Theorem 10.3, $\xi$ is an equivalence.

**12. An application.** Let $E$ and $E'$ be compact ANR's such that there exist cell-like maps $p : E \rightarrow S^{n}$ and $q : E' \rightarrow S^{n}$, i.e., for each $x \in S^{n}$, $p^{-1}(x)$ and $q^{-1}(x)$ have the shape of a point [13]. By [10], $p$ and $q$ are approximate fibrations and hence, by Theorem 3.4, the associated maps $p : E \rightarrow S^{n}$ and $q : E' \rightarrow S^{n}$ are $s$-fibrations. As in the proof of Theorem 3.5, the fibres $p^{-1}(x)$ and $q^{-1}(x)$ are homotopy equivalent to the inverse system consisting of a single point $[x]$. Note that $\pi_{n-1}(S^{n}, x)$ is the trivial group. By Theorem 11.5, $p : E' \rightarrow S^{n}$ and $q' : E' \rightarrow S^{n}$ are bundle equivalent. The following result is a consequence of Theorem 3.6.
THEOREM 12.1. Let $p : E \to S^n$ and $q : E' \to S^n$ be cell-like mappings of compact ANR's onto $S^n$. Then for each $\varepsilon > 0$ there exist mappings $h : E \to E'$ and $g : E' \to E$ such that $d(\varepsilon, p) < \varepsilon$, $d(\varepsilon, q) < \varepsilon$ and the composites $hg$ and $gh$ are homotopic to the identity.

Theorem 12.1 follows also from [1] and [15] in the case when $E = E' = S^n$, $n \neq 4$.

References


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The Bing–Borsuk conjecture is stronger than the Poincaré conjecture

by

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Abstract. It is shown that the existence of a fake 3-cell implies the existence of a 3-dimensional homogeneous compact ANR-space which is not a manifold.

We say that the space $X$ is homogeneous, if for every pair of points $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. We are concerned with the following conjecture:

CONJECTURE 1 (Bing, Borsuk [4]). Every $n$-dimensional homogeneous compact ANR-space is an $n$-dimensional manifold.

In dimensions 1 and 2 this conjecture was proved by Bing and Borsuk in [4]. Here we prove that in dimension 3 Conjecture 1 is stronger than the Poincaré conjecture.

CONJECTURE 2 (Poincaré). Every homotopy 3-sphere is homeomorphic to a 3-sphere.

By a homotopy 3-sphere we mean a closed 3-dimensional manifold which has a homotopy type of 3-sphere. We shall use the term fake 3-cell for a compact contractible 3-manifold which is not homeomorphic to a 3-cell. It is known ([6], p. 26) that (2) is equivalent to the statement that there are no fake 3-cells. Our main goal may be formulated as follows:

THEOREM 3. If there exists a fake 3-cell $F$, then there exists a 3-dimensional homogeneous compact ANR-space $K$ which is not a manifold.

The proof of Theorem 3 consists of several parts: first we shall construct the space $K$ (assuming the existence of the fake 3-cell), then we shall prove that $K \in$ ANR, that $K$ is homogeneous, that $K$ is not a manifold, and finally that $\dim K = 3$. All the time we shall assume the existence of a fixed fake 3-cell $F$ with a given triangulation (by [3] $F$ can be triangulated) and with a fixed orientation. Moreover, we can assume that there exists a homotopy 3-sphere $H$ such that $F$ is obtained from $H$ by removing from it a single open 3-simplex, in particular that the boundary $\partial F$ is equal to the boundary of a 3-simplex [see [6], p. 26].