Let V_k be the union of the P-manifold components of M_k. Inductively we define V_{n+1} $(n \geqslant k)$ by taking all the P-manifold components of M_{n+1} which are not contained in the union of the previous V_j. The Cantor set $X' = \bigcap_{n=k}^{\infty}(M_n - \bigcup_{i=k}^{n} V_i)$ can be slipped off the $(n-2)$-skeleton of any triangulation of E^n by property 4 of the generalized Daverman–Edwards construction. Hence, X' is tame [13] and there is a $\frac{1}{2}\varepsilon$-homeomorphism h of E^n onto itself which takes X' plus the union of all the pinchpoints of the V_i off R. Therefore, for some n, $h(R) \cap (M_n - \bigcup_{i=k}^{n} V_i) = \emptyset$. Let g be a homeomorphism which is fixed outside $\bigcup_{i=k}^{n} V_i$ and moves $M_i \cap V_i$ off $h(R)$. This is accomplished by moving the knotted strand inside each component of V_i off $h(R)$, fixing the boundary, and then pulling the intersection of M_i with that component close to the image of the knotted strand. We now have $g(X) \cap h(R) = \emptyset$, and $h^{-1} \circ g$ is the desired homeomorphism.

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A fixed point principle for locally expansive multifunctions

by

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Abstract. Let (X, d) be a well-chained metric space and F a uniformly open multifunction in $X \times X$ with complete graph so that there exist $\alpha > 0$ and an isotone φ: $[0, \alpha) \to [0, \infty)$ such that $\varphi(t) > t$ for $0 < t < \alpha$ and $d(u, v) \geqslant \varphi[d(x, y)]$ whenever $d(x, y) < \alpha$, $u \in F(x)$, $v \in F(y)$. Then $p \in F(p)$ for some p. In particular, every locally expansive, open multifunction with closed graph on a compact, connected metric space has a fixed point.

1. Introduction. Let (X, d) be a metric space and f: $X \to X$. F is *expansive* on a set B if

(1) $$d(fx, fy) > d(x, y) \quad \text{for all } x, y \text{ in } B \text{ with } x \neq y.$$

f is *contractive* on B if (1) holds with the inequality reversed. f is a *local expansion* (*local contraction*) if every point in X has a neighborhood B on which f is expansive (resp., contractive).

We seek here a fixed point principle that will provide a common base for the following pair of dual theorems: Let (X, d) be a compact, connected metric space. (i) Every continuous, open, local expansion f on X has a fixed point. (ii) Every local contraction g on X has a fixed point. Theorem (i) generalizes a theorem of Rosenholtz [3] who proved (i) for local expansions with the condition $d(fx, fy) \geqslant \lambda d(x, y)$ for some $\lambda > 1$ replacing the less stringent inequality in (1). Theorem (ii) is a variant of a theorem of Edelstein [1].

We can unite (i) and (ii) in a single theorem if we formulate it in terms of multifunctions (i.e. binary relations).

Let F be a subset of $X \times X$. Let $F(x)$ be the set of all y such that $(y, x) \in F$. Let $F(B)$ be the set of all y such that $(y, x) \in F$ for some x in B. F is *expansive* on B if $d(u, v) > d(x, y)$ whenever $x, y \in B$, $x \neq y$, $u \in F(x)$, and $v \in F(y)$.

The definition of local expansion is then the same as for single-valued mappings.

With $F = f$ in (i) and $F = g^{-1}$ in (ii), both (i) and (ii) are subsumed by

Theorem 1 below. A simple compactness argument left to the reader shows that if g is a local contraction on a compact space, then g^{-1} is a local expansion.

THEOREM 1. *Let (X, d) be a compact, connected metric space. Let F be a closed, nonempty subset of $X\times X$ such that*

(a) *F is a local expansion,*

(b) *F is an open mapping: $F(B)$ is open whenever B is open in X.*

Then $p\in F(p)$ for some p in X.

Theorem 1 will be proved at the end of the paper as a special case of a fixed point principle (Theorem 5) which does not require compactness.

2. Uniform α-local expansions. Let (X, d) be a metric space, F a subset of $X\times X$, and $0<\alpha\leqslant\infty$. F is an *α-local expansion* if

(2) $\quad d(u, v)>d(x, y)$ whenever $0<d(x, y)<\alpha,\ u\in F(x),\ v\in F(y)$.

F is a *uniform α-local expansion* if there exists $\varphi\colon [0, \alpha)\to[0, \infty)$ such that

1° φ is isotone: $\varphi(s)\leqslant\varphi(t)$ for $s\leqslant t$,

2° $\varphi(t)>t$ for $0<t<\alpha$,

3° $d(u, v)\geqslant\varphi[d(x, y)]$ if $d(x, y)<\alpha$, $u\in F(x)$, $v\in F(y)$.

A *uniform expansion* is a uniform ∞-local expansion.

We call F *complete* if given (x_n, y_n) in F for $n = 1, 2, \dots$ with both $\langle x_n\rangle$ and $\langle y_n\rangle$ Cauchy, there exists (x, y) in F such that $x_n\to x$ and $y_n\to y$. If (X, d) is a complete metric space, then F is complete if and only if F is closed in $X\times X$ (i.e. F has closed graph).

Let F be a uniform α-local expansion. Some properties of F are given in Lemmas 1 through 6.

LEMMA 1. *F is an α-local expansion.*

Proof. Apply 3° and 2° with $t = d(x, y)$.

LEMMA 2. *Let $\langle y_k\rangle$ and $\langle z_k\rangle$ be sequences in X such that $d(y_k, z_k)<\alpha$, $(y_k, y_{k+1})\in F$, and $(z_k, z_{k+1})\in F$ for all k. Then $d(y_k, z_k)\downarrow 0$.*

Proof. Since (2) holds by Lemma 1, there exists t in $[0, \alpha)$ such that $d(y_k, z_k)\downarrow t$. Hence, $t\leqslant d(y_k, z_k)<\alpha$ for all k. So by 3° and 1°, $d(y_k, z_k)\geqslant\varphi[d(y_{k+1}, z_{k+1})]\geqslant\varphi(t)$. As $k\to\infty$ this gives $t\geqslant\varphi(t)$ which by 2° implies $t = 0$.

LEMMA 3. *Let $\langle y_k\rangle$ be a sequence in X such that*

(3) $\quad d(y_k, y_{k+1})<\alpha$ *and* $(y_k, y_{k+1})\in F$ *for all* k.

Then $d(y_k, y_{k+1})\downarrow 0$.

Proof. Apply Lemma 2 with $z_k = y_{k+1}$.

LEMMA 4. *Let $0<\varepsilon<\alpha$. Let $\delta>0$ be less than $\alpha-\varepsilon$ and $\varphi(\varepsilon)-\varepsilon$. Let $y_k\in X$ for $k = 0, 1, \dots$ such that $d(y_0, y_1)<\delta$ and (3) holds. Then for all integers $n>0$*

(4) $$d(y_1, y_n)<\varepsilon.$$

Proof. We prove (4) by induction. (4) is trivial for $n = 1$. Given (4) for some $n>0$ we contend

(5) $$d(y_1, y_{n+1})<\varepsilon.$$

By Lemma 3, $d(y_n, y_{n+1})\leqslant d(y_0, y_1)<\delta$ which together with (4) gives

$$d(y_1, y_{n+1})\leqslant d(y_1, y_n)+d(y_n, y_{n+1})<\varepsilon+\delta<\alpha.$$

So by 3°, since (y_0, y_1) and (y_n, y_{n+1}) are in F,

(6) $$\varphi[d(y_1, y_{n+1})]\leqslant d(y_0, y_n).$$

Using (4) again we get

(7) $$d(y_0, y_n)\leqslant d(y_0, y_1)+d(y_1, y_n)<\delta+\varepsilon<\varphi(\varepsilon).$$

From (6) and (7) we conclude $\varphi[d(y_1, y_{n+1})]<\varphi(\varepsilon)$ which by 1° implies (5).

LEMMA 5. *Let $\langle y_k\rangle$ be a sequence in X such that (3) holds. Then $\langle y_k\rangle$ is Cauchy.*

Proof. Given $0<\varepsilon<\alpha$ choose δ as in Lemma 4. By Lemma 3, $d(y_{N-1}, y_N)<\delta$ for some $N>1$. Apply Lemma 4 to $\langle y_{N-1+k}\rangle$ for $k = 0, 1, \dots$ to conclude by (4) that $d(y_N, y_{N+m})<\varepsilon$ for all positive integers m.

LEMMA 6. *Let F be complete and $\langle y_k\rangle$ satisfy (3). Then there exists p such that $y_k\to p$ and $p\in F(p)$.*

Proof. $\langle y_k\rangle$, and hence $\langle y_{k+1}\rangle$, is Cauchy by Lemma 5. Also $(y_k, y_{k+1})\in F$ and F is complete. Hence there exists (q, p) in F such that $y_k\to q$ and $y_{k+1}\to p$. So $q = p$.

THEOREM 2. *Let F be a complete, uniform α-local expansion in (X, d). Let $\langle x_j\rangle$ be a sequence in X, and N be a positive integer such that for all j, $(x_j, x_{j+N})\in F$ and $d(x_j, x_{j+1})<\alpha$. Then there exists p such that $x_j\to p$ and $p\in F(p)$.*

Proof. For $r = 0, 1, \dots$ let $y_k(r) = x_{kN+r}$. Apply Lemma 2 with $y_k = y_k(r)$ and $z_k = y_k(r+1)$ to conclude that the sequences $\langle y_k(r)\rangle$ for $r = 0, 1, \dots, N$ are all equivalent. Now for $y_k = y_k(0)$ we have $y_{k+1} = y_k(N)$. So by equivalence $d(y_k, y_{k+1})\to 0$. Hence $d(y_k, y_{k+1})<\alpha$ ultimately. Also $(y_k, y_{k+1})\in F$. So Lemma 6 gives $y_k\to p$ where $p\in F(p)$. By equivalence, $y_k(r)\to p$ for $r = 0, 1, \dots, N$. For any positive integer j we have the representation $j = kN+r$ with $0\leqslant r<N$. Hence, since $k\to\infty$ as $j\to\infty$ and $r<N$, $x_j = y_k(r)\to p$.

3. The fixed point principle. Given a metric space (X, d) let $U_\alpha = d^{-1}[0, \alpha)$, the set of all (x, y) in $X\times X$ with $d(x, y)<\alpha$. For F and G subsets of $X\times X$ define the composition $F\circ G$ to be the set of all (z, x) for which there exists y with (z, y) in F and (y, x) in G. In these terms F is a uniform α-local expansion if and only if given $\varepsilon>0$ there exists $\delta>0$ such that $U_\alpha\cap(F^{-1}\circ U_{\varepsilon+\delta}\circ F)\subseteq U_\varepsilon$. (X, d) is *α-chained* if $\bigcup_{n=1}^{\infty} U_\alpha^n = X\times X$, where the superscript n denotes n-fold composition. (X, d) is *well-chained* if it is α-chained for all $\alpha>0$.

THEOREM 3. *Let (X, d) be an α-chained metric space. Let F be a nonempty, complete, uniform α-local expansion in (X, d) such that*

$$U_\alpha \circ F \subseteq F \circ U_\alpha . \tag{8}$$

Then there exists p such that $p \in F(p)$.

Proof. Choose (x, y) in the nonempty set F. Since X is α-chained there exist $x_0, x_1, \ldots, x_N$ such that $x_0 = x$, $x_N = y$, and $(x_{i+1}, x_i) \in U_\alpha$ for $0 \leqslant i < N$. We shall extend this finite sequence inductively to an infinite sequence to which Theorem 2 applies.

Given $x_0, \ldots, x_{N+k}$ for some $k \geqslant 0$ with

$$(x_{i+1}, x_i) \in U_\alpha \text{ for } 0 \leqslant i < N+k \quad \text{and} \quad (x_k, x_{N+k}) \in F \tag{9}$$

we choose x_{N+k+1} as follows. By (9) with $i = k$, $(x_{k+1}, x_{N+k}) = (x_{k+1}, x_k) \circ (x_k, x_{N+k}) \in U_\alpha \circ F$. Hence, (8) implies $(x_{k+1}, x_{N+k}) \in F \circ U_\alpha$. That is, $(x_{k+1}, y) \in F$ and $(y, x_{N+k}) \in U_\alpha$ for some y. Pick such a y to be x_{N+k+1}. Then we have $x_0, \ldots, x_{N+k+1}$ for which (9) holds with k replaced by $k+1$.

Induction thus produces a sequence $\langle x_j \rangle$ which satisfies the hypotheses of Theorem 2. Apply Theorem 2 to get p.

Remark. If F is a uniform 2α-local expansion with the other conditions as they stand in Theorem 3, then the extension of $x_0, \ldots, x_N$ is unique.

For the case $\alpha = \infty$ Theorem 3 reduces to the following.

THEOREM 4. *Let F be a complete, uniform expansion in (X, d) such that $F(X) = X$. Then there exists a unique p such that $p \in F(p)$. Moreover, $y_k \to p$ for every sequence $\langle y_k \rangle$ with y_k in $F(y_{k+1})$ for all k.*

Proof. Every metric space in ∞-chained since $U_\infty = X \times X$. Moreover, $F \circ U_\infty = F(X) \times X = X \times X$ if $F(X) = X$. So (8) holds for $\alpha = \infty$. Hence, Theorem 3 yields p. Uniqueness of p and the last statement in Theorem 4 follow from Lemma 2 with $\alpha = \infty$ and $z_k = p$ for all k.

Meir and Keeler [2] called a mapping $g\colon X \to X$ on a metric space (X, d) a *weakly uniformly strict contraction* if for every $\varepsilon > 0$ there exists $\delta > 0$ with $d(gx, gy) < \varepsilon$ for all x, y such that $d(x, y) < \varepsilon + \delta$. That is, $g \circ U_{\varepsilon+\delta} \circ g^{-1} \subseteq U_\varepsilon$. It is easy to see that a mapping g on X is a weakly uniformly strict contraction if and only if g^{-1} is a uniform expansion in our sense. Since such a contraction g is continuous, its graph is closed, hence complete in a complete space. So Theorem 4 implies the Meir–Keeler generalization of the Banach contraction principle: Every weakly uniformly strict contraction g on a complete metric space X has a unique fixed point p and $g^n(x) \to p$ for all x in X [2].

We now present our main theorem.

THEOREM 5. *Let (X, d) be a well-chained metric space. Let F be a nonempty, complete, uniform α-local expansion in $X \times X$ for some $\alpha > 0$. Let F also be a uniform-open mapping: given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$U_\delta \circ F \subseteq F \circ U_\varepsilon . \tag{10}$$

Then there exists p such that $p \in F(p)$.

Theorem 5 follows from Theorem 3 with $\alpha = \delta$ by the following lemma and Lemma 1.

LEMMA 7. *Let F be an α-local expansion which is also a uniform-open mapping. Then for all sufficiently small $\delta > 0$,*

$$U_\delta \circ F \subseteq F \circ U_\delta . \tag{11}$$

Proof. Consider δ small enough for (10) to hold with $\varepsilon = \alpha$. Consider any x in X. By (10) the F-image of $U_\varepsilon(x)$ covers $U_\delta \circ F(x)$. Since F is ε-locally expansive, F maps $U_\varepsilon(x) - U_\delta(x)$ into the complement of $U_\delta \circ F(x)$. Hence, the F-image of $U_\delta(x)$ must cover $U_\delta \circ F(x)$. Since this holds for all x in X, we get (11).

4. Application to compact spaces. To get Theorem 1 from Theorem 5 we need two more lemmas.

LEMMA 8. *Let (X, d) be compact. Let F be a local expansion, closed in $X \times X$. Then F is a uniform α-local expansion for some $\alpha > 0$.*

Proof. Choose a finite open covering $\{B_1, \ldots, B_n\}$ of X such that F is expansive on each B_i. By Lebesgue's covering lemma there exists β such that every subset of X whose diameter is less than β is contained in some B_i. So F is a β-local expansion. Pick α in $(0, \beta)$. Pick K greater than both α and the diameter of X. For t in $[0, \alpha]$ let $C_t = F \circ d^{-1}[t, \alpha] \circ F^{-1}$. Since a composition of closed relations in a compact space is closed, C_t is compact. If C_t is nonempty define $\varphi(t)$ to be the minimum of d on C_t. If C_t is empty take $\varphi(t) = K$. Then 1°, 2°, 3° can be readily verified.

LEMMA 9. *Let (X, d) be compact. Let F be a closed subset of $X \times X$ such that F is an open mapping. Then F is a uniform-open mapping.*

Proof. Suppose the conclusion were false. Then there would exist $\varepsilon > 0$, $\delta(n) \to 0$, and (y_n, x_n) such that

$$(y_n, x_n) \in U_{\delta(n)} \circ F \tag{12}$$

and

$$(y_n, x_n) \notin F \circ U_\varepsilon . \tag{13}$$

Since X is compact we may assume $x_n \to x$ and $y_n \to y$ for some x, y in X, since we can replace the original sequences by appropriate subsequences. By (12) there exists z_n such that

$$(y_n, z_n) \in U_{\delta(n)} \tag{14}$$

and

$$(z_n, x_n) \in F . \tag{15}$$

Since $y_n \to y$, (14) implies $z_n \to y$. Hence, since F is closed in $X \times X$, (15) implies $(y, x) \in F$. Pick δ in $(0, \varepsilon)$. Then for arbitrary z, $d(z, x) < \delta$ implies

$$d(z, x_n) \leqslant d(z, x) + d(x, x_n) < \delta + d(x, x_n) < \varepsilon \quad \text{for} \quad d(x, x_n) < \varepsilon - \delta .$$

Hence, since $x_n \to x$, $U_\delta(x) \subseteq U_\varepsilon(x_n)$ ultimately as $n \to \infty$. Applying F we get

(16) $$F \circ U_\delta(x) \subseteq F \circ U_\varepsilon(x_n) \text{ ultimately}.$$

Since $(y, x) \in F$ and $(x, x) \in U_\delta$, $y \in F(x) \subseteq F \circ U_\delta(x)$. Since $U_\delta(x)$ is open, $F \circ U_\delta(x)$ is open by hypothesis. Therefore, since $y_n \to y$ and $y \in F \circ U_\delta(x)$, $y_n \in F \circ U_\delta(x)$ ultimately. Hence by (16), $y_n \in F \circ U_\varepsilon(x_n)$ ultimately, which contradicts (13).

Finally, Theorem 1 follows from Theorem 5 under Lemmas 8 and 9 since every connected metric space is well-chained.

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s-Fibrations

by

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Abstract. The concept of s-fibration is introduced which generalized the notions of Hurewicz fibrations and approximate fibrations. Many results about Hurewicz fibrations which are not true for approximate fibrations are proved for s-fibrations. For example, a homotopy classification theorem for s-fibrations over then n-sphere is proved.

1. Introduction. A mapping $f\colon E \to B$ between compact metric spaces is an *approximate fibration* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $h\colon X \to E$ and $H\colon X \times [0, 1] \to B$ are maps with $d(H(x, 0), fh(x)) < \delta$, then there exists $G\colon X \times [0, 1] \to E$ such that $G(x, 0) = h(x)$ and $d(H(x, t), fG(x, t)) < \varepsilon$ for all $x \in X$ and $t \in [0, 1]$. Coram and Duvall [2] introduced approximate fibrations as a generalization of cell-like mappings [10] and showed that the uniform limit of a sequence of Hurewicz fibrations is an approximate fibration. By using shape theoretic concepts, they also showed that approximate fibrations possessed many properties shared by Hurewicz fibrations.

One notable exception is that the pullback of an approximate fibration need not be an approximate fibration. In this work, we define the concept of s-fibrations which we show generalizes the concepts of approximate fibrations and Hurewicz fibrations. Pullbacks behave properly and many other results about Hurewicz fibrations carry over. For example, a homotopy classification theorem for s-fibrations over the n-sphere is proved (Theorem 11.1). As a consequence, information about cell-like decompositions of ANR's is obtained (Theorem 12.1).

T. B. Rushing has informed the authors that S. Mardešić and he [11] have also generalized the theory of approximate fibrations but that the overlap between these works is little. R. Goad [6] has also a generalization of approximate fibration but, again, there is no overlap with this work.

2. Definitions. We shall assume that the reader is familiar with [12]. Since our results are valid for a larger category of spaces than that considered in [12], our definitions will sometimes differ.