Table des matières du tome CVI, fascicule 1

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Chaque volume paraît en 3 fascicules

Adresse de la Rédaction:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Pologne)

Adresse de l'Échange:
INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
Śniadeckich 8, 00-950 Warszawa (Pologne)

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ARS POLONA, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)

ISBN 83-01-01399-0 ISSN 0016-2736

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Weakly chainable circle-like continua

by

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Abstract. This paper investigates the problem of ascertaining which circle-like continua are continuous images of chainable continua. In the second section, the notion of the "revolving number" of a map from S^1 onto S^1 is introduced and used to classify the planar, non-chainable, circle-like continua by structure: "self-entwined" (a concept introduced in Section 2); decomposable; indecomposable, non-self-entwined. The main theorem in Section 3 is a characterization of weakly chainable circle-like continua; the classification scheme of Section 2 is used to prove this result.

Section 1. Suppose that for each positive integer i, X_i is a compact metric space and f_i^{i+1} is a map from X_{i+1} onto X_i. Let M be the subset of the Cartesian product space $\prod_{i=1}^{\infty} X_i$ consisting of the set of all sequences p such that for each i, p_i is in X_i and $f_i^{i+1}(p_{i+1}) = p_i$. Then M, with the relative topology from $\prod_{i=1}^{\infty} X_i$, is called the *inverse limit of the inverse system* (X_i, f_i^{i+1}), and denoted $\varprojlim(X_i, f_i^{i+1})$. If $m>n$, f_n^m will denote the composition of the maps $f_n^{n+1}, f_{n+1}^{n+2}, \ldots, f_{m-1}^m$; f_m^m will denote the identity function on X_m. For each positive integer i, PR_i will denote the natural projection of M onto X_i.

Definition (see [7]). Suppose each of A and B is a metric space and each of u and v is a map from A into B. Suppose $c>0$. The statement that $u \underset{c}{=} v$ means that for each point x in A, $\mathrm{dist}_B(u(x), v(x)) < c$.

The following theorem, a corollary to Theorem 3 of [7], will be used several times:

Theorem A. *Let* $M = \varprojlim(X_i, f_i^{i+1})$ *and* $K = \varprojlim(Y_i, g_i^{i+1})$. *Suppose* e *is a decreasing sequence of positive numbers with sequential limit* 0. *Suppose* h *is a sequence of maps such that*

(1) *for each positive integer* i, h_{2i} *is a map from* Y_{2i} *onto* X_{2i} *and* h_{2i-1} *is a map from* X_{2i-1} *onto* Y_{2i-1};

(2) *for each triple* (i, j, k) *of positive integers with* $i<j$ *and* $k \leqslant 2i-1$,

$$g_k^{2i-1} \circ h_{2i-1} \circ f_{2i-1}^{2j-1} \underset{e_{2i-1}}{=} g_k^{2j-1} \circ h_{2j-1}$$

and

$$g_k^{2i-1}\circ h_{2i-1}\circ f_{2i-1}^{2j-2}\circ h_{2j-2}\underset{e_{2i-1}}{=}g_k^{2j-2};$$

(3) *for each triple* (i,j,k) *of positive integers with* $i<j$ *and* $k\leqslant 2i$,

$$f_k^{2i}\circ h_{2i}\circ g_{2i}^{2j}\underset{e_{2i}}{=}f_k^{2j}\circ h_{2j}$$

and

$$f_k^{2i}\circ h_{2i}\circ g_{2i}^{2j-1}\circ h_{2j-1}\underset{e_{2i}}{=}f_k^{2j-1}.$$

Then M is homeomorphic to K. In case $X_i=Y_i$ *and* h_i *is the identity map for each* i, *it suffices that for each ordered triple* (i,j,k) *of positive integers with* $k\leqslant i<j$,

$$g_k^i\circ f_i^j\underset{e_i}{=}g_k^j\quad\text{and}\quad f_k^i\circ g_i^j\underset{e_i}{=}f_k^j$$

for M to be homeomorphic to K.

Section 2. In [1], Bing characterized the class of non-planar circle-like continua, and in [3], Ingram characterized the chainable circle-like continua. In this chapter, the class of non-chainable, planar, circle-like continua is subdivided into three subclasses: the decomposable; the self-entwined (a concept to be introduced in this chapter); the indecomposable, non-self-entwined. This classification scheme is used to prove the main result of Section 3.

The "circle", S^1, is the unit circle on the complex plane. If P and Q are two non-antipodal points of the circle, and L the length (in the usual metric) of the minor arc between them, then the distance from P to Q, denoted $|P-Q|$, is defined as $L/2\pi$. The distance between antipodal points is $\frac{1}{2}$. The "wrapping function", denoted φ, is the map from the real line onto S^1 which sends the number x to $e^{2\pi ix}$. Let S^1 be oriented so that φ is order-preserving. If A and B are points of S^1, then the arc $[A,B]$ of S^1 is the φ-image of an interval $[a,b]$, $b-a<1$, with $\varphi(a)=A$ and $\varphi(b)=B$. If C is a point of S^1, then we write $A<C<B$ in case there is a number c, $a<c<b$, with $\varphi(c)=C$.

The next two definitions are modifications of concepts developed by J. T. Rogers in [9], approached here from a homotopy-theoretic rather than combinatorial point of view.

Suppose f is a map from S^1 onto S^1, and $\deg f\geqslant 0$.

DEFINITION. Suppose T is an arc in S^1. Let u be a lift of $f|T$, i.e., u is a map from T into the real line, and $f|T=\varphi\circ u$. Then $\deg(T,f)$ is defined as $\operatorname{diam}u(T)$; this number is independent of which lift map is taken.

In case $\deg(T,f)$ is an integer, $\deg(T,f)$ is the number of times the arc T is "wrapped around" the circle by f.

Using the uniform continuity of f, one establishes

LEMMA 1. *Suppose D is the number set to which a number r belongs if and only if there is an arc Q in* S^1 *such that* $r=\deg(Q,f)$. *Then D is bounded above.*

DEFINITION. Suppose D is as in the hypothesis of Lemma 1. The revolving number of f, denoted $R(f)$, is $\sup D$.

LEMMA 2. *Suppose P and Q are points of* S^1. *Let T be a point sequence with each value in the interior of the arc* $[Q,P]$, *and T converges to P. Let u be a sequence of maps such that for each positive integer i,* u_i *is a lift of* $f|[P,T_i]$, *and* $u_i(P)=u_1(P)=Z$. *Then* $\lim\limits_{i\to\infty}u_i(T_i)=Z+\deg f$.

Proof. Suppose $\deg f=n$. Let v be a lift of f/I^n. Then $f=I^n\cdot(\varphi\circ v)$. Let m be a positive integer such that $|T_m-P|<\frac{1}{2}$. Let $h=\varphi^{-1}|[P,T_m]$. Then

$$\varphi\circ u_m=f|[P,T_m]=\varphi\circ(nh+v),$$

and

$$\begin{aligned}u_m(T_m)-u_m(P)&=nh(T_m)+v(T_m)-nh(P)-v(P)\\&=n(1-|P-T_m|)+v(T_m)-v(P).\end{aligned}$$

Since $T\to P$ and $v(T)\to v(P)$, $u(T)\to u_1(P)+n=Z+n$.

Lemma 2 yields immediately $R(f)\geqslant\deg f$.

DEFINITION. If A is an arc in S^1, and t is a lift of $f|A$, then there is a subarc B of A such that the map t sends the endpoints of B to the endpoints of the interval $t(A)$. An arc with this property of B will be called *type* 1.

LEMMA 3. *If* $R(f)>\deg f$, *then there is an arc D in* S^1 *such that* $\deg(D,f)=R(f)$.

Proof. Let A be a sequence of arcs in S^1 such that (1) each value of A is of type 1; (2) for each i, $\deg(A_i,f)\leqslant\deg(A_{i+1},f)$; (3) $\deg(A,f)$ converges to $R(f)$; (4) letting $A_k=[P_k,Q_k]$, the point sequence P converges to a point c, and Q converges to a point d. Let L be the limiting set of A. Then L is the arc $[c,d]$, and $\deg(L,f)=R(f)$.

DEFINITION. If P is an arc in S^1, P is of type 1, and $\deg(P,f)=R(f)$, then P is called a *defining arc for* $R(f)$.

LEMMA 4. *Suppose each of f and g is a map from* S^1 *into* S^1; $\deg f=\deg g=1$; $\frac{1}{2}>e>0$; *d is a positive number such that if* $|x-y|<d$, *then* $|g(x)-g(y)|<e$; $R(f)>2-d$. *Then* $R(g\circ f)>2-e$.

Proof. Suppose A is a defining arc for $R(f)$, and that $R(f)\geqslant 2$. Suppose B is a subarc of A, with $\deg(B,f)=2$. Then f wraps B twice around S^1. Lemma 2 yields $R(g\circ f)\geqslant 2$. Appealing to the uniform continuity of g and to the local isometry of φ yields Lemma 4.

Using the results of Ingram in [3] and of McCord (page 29 of [6]), we have

THEOREM B. *If C is a circle-like continuum, then C is planar and nonchainable if and only if C is homeomorphic to* $\varprojlim(X_i,f_i^{i+1})$, *in which each* X_i *is* S^1, *and* $\deg f_i^{i+1}=1$ *for each i.*

Notation. "p.n.c.c.l." will mean "planar, non-chainable, circle-like".

We are ready to prove the main result of this section.

DEFINITION. Suppose M is a p.n.c.c.l. continuum as in Theorem B. Then M is said to be in *class* 1 if, for each positive integer i, there exists a number Z_i, $1 \leqslant Z_i < 2$, such that for each positive integer j, $R(f_i^{i+j}) \leqslant Z_i$. We say that M is in *class* 2 if for each i, and each number y, $1 \leqslant y < 2$, there is j such that $R(f_i^{i+1}) > y$. Similarly, M is in *class A* if, for each i, there exists Z_i, $1 \leqslant Z_i < 3$, such that for each positive integer j, $R(f_i^{i+1}) \leqslant Z_i$; also, M is in *class B* if for each i, and each y, $1 \leqslant y < 3$, there is j such that $R(f_i^{i+j}) > y$.

THEOREM 1. *Suppose M is a p.n.c.c.l. continuum. Then either M is a member of class 2 or M is homeomorphic to a member of class 1. Furthermore, either M is a member of class B or M is homeomorphic to a member of class A.*

Proof. Let $M = \underleftarrow{\mathrm{Lim}}(X_i, f_i^{i+1})$ as in Theorem B. Suppose M is not in class 2. Then there is a number Z, $1 \leqslant Z < 2$, and there is a positive integer i such that for each j, $R(f_i^{i+j}) \leqslant Z$. Let D be the set of all ordered pairs (p, y) such that p is a positive integer, y is a number, $1 \leqslant y < 2$, and for each positive integer j, $R(f_p^{p+j}) \leqslant y$.

Case (1). The domain of the relation D is bounded. Let K be the greatest element in the domain of D, and let (K, t) be an element of D. Let $e = \min(\frac{1}{4}, 2-t)$. Since $K+1$ is not in the domain of D, by Lemma 4 there is an integer n such that $R(f_K^n) = R(f_K^{K+1} \circ f_{K+1}^n) > 2 - e \geqslant t$, a contradiction.

Case (2). The domain of D is not bounded. Let $(n_1, n_2, n_3, \ldots)$ be an increasing sequence of positive integers whose range is the domain of D. Let h be a function whose domain is the domain of D, and h is a subset of D. Let $C = \underleftarrow{\mathrm{Lim}}(X_{n_i}, f_{n_i}^{n_{i+1}})$. Then C is in class 1. For: if i is a positive integer, then $h(n_i)$ is a number, $1 \leqslant h(n_i) < 2$, such that for each j, $R(f_{n_i}^{n_{i+j}}) \leqslant h(n_i)$. We have M homeomorphic to C. The second assertion of Theorem 1 is proved similarly.

From now on, class $1'(A')$ will denote the class of all p.n.c.c.l. continua homeomorphic to a member of class $1(A)$.

Trivially, class B is a subset of class 2. The collection of all p.n.c.c.l. continua is class 1 $\cup$ class B $\cup$ (class 2$\setminus$class B). We will see that if M is a p.n.c.c.l. continuum, then M is indecomposable if and only if M is a member of class 2.

DEFINITION. The continua which belong to class B will be called *self-entwined* (this notion is also a modified version of an idea in [9]).

We will see that the self-entwined continua have some of the properties of non-planar circle-like continua (e.g., Corollary to Lemma 9; Theorem 5).

Application of Lemma 2 several times yields

LEMMA 5. *Suppose f is a map from S^1 onto S^1, and* $\deg f \geqslant 1$. *Suppose $[a, b]$ is a defining arc for $R(f)$. Let t be a lift of $f\,|\,[a, b]$. Then $t(a) < t(b)$, and* $\deg([b, a], f) = R(f) - \deg f$.

The conclusion of Lemma 5 implies that the arc $[b, a]$ is of type 1: $v([b, a]) = [v(a), v(b)]$, when v is a lift of $f\,|\,[b, a]$.

THEOREM 2. *If M is a p.n.c.c.l. continuum, then M is indecomposable if and only if M is a member of class 2.*

Proof. Let $M = \underleftarrow{\mathrm{Lim}}(X_i, f_i^{i+1})$ as in Theorem B. Suppose M is in class 2. By a result of D. Kuykendall ([5, Theorem 2]), M is indecomposable if and only if for each positive integer n, and each number $e > 0$, there are a positive integer j and three points of X_{n+j} such that if K is a subcontinuum of X_{n+j} containing two of them, then $\mathrm{dist}_n(x, f_n^{n+j}(K)) < e$, for each point x in X_n. Suppose n is a positive integer and $\frac{1}{2} > e > 0$. Let j be such that $R(f_n^{n+j}) > 2 - e$. Let $[A, B]$ be a defining arc for $R(f_n^{n+j})$, and t a lift for $f_n^{n+j}\,|\,[A, B]$. Then $t([A, B]) = [t(A), t(B)] = [a, b]$, with $b > a+1$. Let C be a point in $[A, B]$, with $t(C) = a+1$. Then $[a, a+1] \subseteq t([A, C])$, and $[a+1, b] \subseteq t([C, B])$. By Lemma 5, letting v be a lift of $f_n^{n+j}\,|\,[B, A]$ such that $v(B) = t(B)$, we have $v([B, A]) = [a+1, b]$. Now, $\varphi([a, a+1]) = S^1$, and $\varphi([a+1, b])$ is either S^1 or an arc of length greater than $1-e$. For each point x of S^1,

$$|x - f_n^{n+j}([A, C])| = |x - f_n^{n+j}([A, B])| = 0,$$

$$|x - f_n^{n+j}([C, A])| \leqslant |x - f_n^{n+j}([C, B])| < e,$$

$$|x - f_n^{n+j}([B, C])| \leqslant |x - f_n^{n+j}([B, A])| < e.$$

By Kuykendall's theorem, M is indecomposable.

To prove the converse, suppose M is indecomposable. Then for each integer $K \geqslant 2$, there are K points of M such that M is irreducible between each two of them. A corollary of [5, Theorem 2] is that M being indecomposable implies for each triple (n, p, e), n a positive integer, p an integer, $p \geqslant 2$, and $e > 0$, there are a positive integer j and p points of X_{n+j} such that if L is a subcontinuum of X_{n+j} containing two of them, then $\mathrm{dist}_n(x, f_n^{n+j}(L)) < e$, for each point x in X_n. Suppose n is a positive integer and $1 > e > 0$. Let N be an integer such that $N-2 > 1/e$. Let j be a positive integer and W be a set of N points of S^1 such that if A is an arc containing two of them, then $\deg(A, f_n^{n+j}) > 1 - e/(N-1)$ (similar to the previous paragraph). Let $(p_1, p_2, \ldots, p_N)$ be a reversible sequence of points of S^1, ordered by the orientation of S^1, whose range is the set W. Let v be a lift of $f_n^{n+j}\,|\,[p_1, p_N]$. Let, for $1 \leqslant i \leqslant N-1$, $[a_i, b_i]$ be a subarc of $[p_i, p_{i+1}]$ of type 1. If, for some i, $v([a_i, b_i]) = [v(b_i), v(a_i)]$, then since $v(a_i) - v(b_i) > 1 - e/(N-1)$, by Lemma 2, $R(f_n^{n+j}) > 2 - e/(N-1) > 2 - e$. Similarly, if, for some i, $v(b_i) - v(a_{i+1}) > 1 - e$, then $R(f_n^{n+j}) > 2 - e$. Assume that for $1 \leqslant i \leqslant N-1$, $v(b_i) - v(a_i) > 1 - e/(N-1)$, and for $1 \leqslant i \leqslant N-2$, $v(b_i) - v(a_{i+1}) \leqslant 1 - e$; then $v(a_{i+1}) - v(a_i) > e - e/(N-1)$. Therefore

$$v(a_{N-1}) - v(a_1) = \sum_{i=1}^{N-2} \big(v(a_{i+1}) - v(a_i)\big) > (N-2)\left(e - \frac{e}{N-1}\right).$$

But

$$v(b_{N-1})-v(a_{N-1})>1-\frac{e}{N-1} \quad \text{and} \quad v(b_{N-1})-v(a_1)>1+(N-2)e-e\,.$$

Since $(N-2)e>1$, we have $R(f_n^{n+j})\geqslant v(b_{N-1})-v(a_1)>2-e$, whence M is in class 2.

DEFINITION. Suppose g is a map from a continuum X onto a continuum Y. Then g is said to be *weakly confluent* if, for each subcontinuum K of Y, there is a component C of $g^{-1}(K)$ such that $g(C)=K$.

LEMMA 6. *If g is a map from a continuum X onto S^1, and g is essential, then g is weakly confluent.*

Proof. Suppose g is a map from X onto S^1, and g is not weakly confluent. Let $[p,q]$ be an arc in S^1 such that no component of g^{-1} of it maps onto it under g. We may assume that $[p,q]$ is properly contained in a semi-circle. Let $W=g^{-1}([p,q])$; $Y=$ the set of all components of W; $Y_1=$ the set of components of W which contain a point of $g^{-1}(p)$; $Y_2=$ the set of components of W which contain a point of $g^{-1}(q)$. Then $Y=Y_1\cup Y_2$; $W=Y_1^*\cup Y_2^*$, with Y_1^* and Y_2^* mutually exclusive, closed point sets.

Let r be a function from X into S^1 such that $r=g$ on the set $X\setminus W$; $r(Y_1^*)=(p)$; $r(Y_2^*)=(q)$. Then r is continuous, and $r(X)\neq S^1$, thus r is inessential. Since $r\underset{\frac{1}{2}}{=}g$, g is homotopic to r, and g is inessential.

LEMMA 7. *Suppose each of f and g is a map from S^1 onto S^1, and* $\deg g\geqslant 0$, $\deg f\geqslant 1$. *Then* $R(g\circ f)\geqslant R(g)$.

Proof. In case $R(g)=\deg g$, we have $R(g\circ f)\geqslant \deg(g\circ f)=(\deg g)(\deg f)\geqslant \deg g=R(g)$. Suppose $R(g)>\deg g$. Since f is essential, thus weakly confluent, there is an arc in S^1 whose f-image is a defining arc for $R(g)$. This yields $R(g\circ f)\geqslant R(g)$.

LEMMA 8. *Suppose f is a map from S^1 onto S^1, $\deg f=1$, e is a number, $0<e<\frac{1}{2}$, and $R(f)>2-e$. Then there is a map g from S^1 onto S^1 such that $\deg g=1$, $R(g)\geqslant 2$, and $f\underset{e}{=}g$.*

Indication of proof. If $R(f)<2$, and T is a defining arc for $R(f)$, we may "stretch" the map f on T by letting v be a lift of $f|T$, p a homeomorphism from $v(T)$ onto an interval of length 2, and $g=\varphi\circ p\circ v$. Similarly we may "stretch" f on the complimentary arc of T. Taking p to be such that $p\underset{e}{=}\mathrm{Id}$, we have the lemma.

The following theorem, whose proof is technically complicated, is intuitively an obvious consequence of Mioduszewski's theorem.

THEOREM 3. *If M is a p.n.c.c.l. continuum, and M is in class 2, then M is homeomorphic to $\varprojlim(Y_i, g_i^{i+1})$ such that each Y_i is S^1, $\deg g_i^{i+1}=1$, and $R(g_i^j)\geqslant 2$, for each pair of positive integers i and j, $i<j$.*

Proof. Let $M=\varprojlim(X_i, f_i^{i+1})$, each $X_i=S^1$, and M is in class 2. Let e be the number sequence $(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots)$. Let $p_1=1$. Let p_2 be the first positive integer j such that $R(f_1^j)>2-\frac{1}{2}$. Let $F_1^2=f_{p_1}^{p_2}$. Let g_1^2 be a map from S^1 onto S^1 such that $g_1^2\underset{\frac{1}{2}}{=}F_1^2$, and $R(g_1^2)\geqslant 2$.

We proceed by induction. Suppose $p_1,p_2,\ldots,p_n,p_{n+1}$ are defined; $F_1^2,F_2^3,\ldots,F_n^{n+1}$ are defined, with $F_i^{i+1}=f_{p_i}^{p_{i+1}}$ for $1\leqslant i\leqslant n$; $g_1^2, g_2^3,\ldots,g_n^{n+1}$ are defined, with $R(g_i^{i+1})\geqslant 2$, for $1\leqslant i\leqslant n$; for each triple (k,i,j) of positive integers with $k\leqslant i<j\leqslant n+1$,

$$g_k^i\circ F_i^j \underset{\left(1-\frac{1}{2^{j-i}}\right)e_i}{=} g_k^j \quad \text{and} \quad F_k^i\circ g_i^j \underset{\left(1-\frac{1}{2^{j-i}}\right)e_i}{=} F_k^j\,.$$

Using the uniform continuity of the maps from S^1 into S^1, let $a>0$ such that if x and y are points of S^1 and $x-y<a$, then for $1\leqslant k\leqslant i<j\leqslant n+1$,

$$|g_k^i\circ F_i^j(x)-g_k^i\circ F_i^j(y)|<\frac{e_i}{2^{n+2-i}} \quad \text{and} \quad |F_k^i\circ g_i^j(x)-F_k^i\circ g_i^j(y)|<\frac{e_i}{2^{n+2-i}}\,.$$

Let $d=\min(a,\frac{1}{2}e_n)$. Let p_{n+2} be the first positive integer j such that $R(f_{p_{n+1}}^j)>2-\frac{1}{2}d$. Let $F_{n+1}^{n+2}=f_{p_{n+1}}^{p_{n+2}}$. Let g_{n+1}^{n+2} be a map from S^1 onto S^1 such that $g_{n+1}^{n+2}\underset{\frac{1}{2}d}{=}F_{n+1}^{n+2}$ and $R(g_{n+1}^{n+2})\geqslant 2$. Let x be a point of S^1. Suppose $1\leqslant k\leqslant i<n+1$. Then

$$(*) \qquad \left|g_k^i\circ F_i^{n+1}(F_{n+1}^{n+2}(x))-g_k^i\circ F_i^{n+1}(g_{n+1}^{n+2}(x))\right|<\frac{e_i}{2^{n+2-i}}\,.$$

Also,

$$\left|g_k^i\circ F_i^{n+1}(g_{n+1}^{n+2}(x))-g_k^{n+1}(g_{n+1}^{n+2}(x))\right|<\left(1-\frac{1}{2^{n+1-i}}\right)e_i\,.$$

Hence

$$\left|g_k^i\circ F_i^{n+1}(F_{n+1}^{n+2}(x))-g_k^{n+1}(g_{n+1}^{n+2}(x))\right|<\left(1-\frac{1}{2^{n+2-i}}\right)e_i\,.$$

The last inequality implies

$$g_k^i\circ F_i^{n+2} \underset{\left(1-\frac{1}{2^{n+2-i}}\right)e_i}{=} g_k^{n+2}\,.$$

Similarly,

$$F_k^i\circ g_i^{n+2} \underset{\left(1-\frac{1}{2^{n+2-i}}\right)e_i}{=} F_k^{n+2}\,.$$

Now, since $\frac{1}{2}d\leqslant\frac{1}{2}e_{n+1}$, $g_{n+1}^{n+2}\underset{\frac{1}{2}e_{n+1}}{=}F_{n+1}^{n+2}$. Also, since $\dfrac{e_k}{2^{n+2-k}}=\dfrac{e_{n+1}}{2}$, the inequality $(*)$ yields

$$F_k^{n+1}\circ g_{n+1}^{n+2} \underset{\frac{1}{2}e_{n+1}}{=} F_k^{n+2}\,.$$

Similarly,

$$g_k^{n+1}\circ F_{n+1}^{n+2} \underset{\frac{1}{2}e_{n+1}}{=} g_k^{n+2}\,.$$

Thus, for each triple (k, i, j) of positive integers, with $k \leqslant i < j \leqslant n+2$,

$$g_k^i \circ F_i^j \underset{\left(1-\frac{1}{2^{j-i}}\right)e_i}{=} g_k^j \quad \text{and} \quad F_k^i \circ g_i^j \underset{\left(1-\frac{1}{2^{j-i}}\right)e_i}{=} F_k^j .$$

Recursively, there a exists sequence $(F_1^2, F_2^3, F_3^4, \ldots)$ of maps, a sequence $(g_1^2, g_2^3, g_3^4, \ldots)$ of maps, with $R(g_i^{i+1}) \geqslant 2$, and a decreasing sequence e of positive numbers with sequential limit 0, such that for each triple (k, i, j) of positive integers, with $k \leqslant i < j$,

$$g_k^i \circ F_i^j \underset{e_i}{=} g_k^j \quad \text{and} \quad F_k^i \circ g_i^j \underset{e_i}{=} F_k^j .$$

Let $K = \varprojlim(X_i, g_i^{i+1})$. By Theorem A, K is homeomorphic to $\varprojlim(X_i, F_i^{i+1})$, which is homeomorphic to M. Since $R(g_i^{i+1}) \geqslant 2$, for each i, Lemma 7 yields $R(g_i^j) \geqslant 2$ for $i < j$. This completes the proof.

A similar pattern of argument yields

THEOREM 4. *If M is a p.n.c.c.l. continuum in class B, then M is homeomorphic to $\varprojlim(Y_i, g_i^{i+1})$ such that each $Y_i = S^1$, $\deg g_i^{i+1} = 1$, and $R(g_i^j) \geqslant 3$ for each pair of positive integers i and j, with $i < j$.*

D. R. Read proved in [8, Theorem 10] that each map from a continuum onto an arc is weakly confluent.

LEMMA 9. *Suppose each of f and g is a map from S^1 onto S^1; $R(f) > \deg f \geqslant 1$; $R(g) > \deg g \geqslant 1$; $R(f) \geqslant 2$. Then $R(g \circ f) \geqslant ([R(f)]-2)\deg g + R(g)$, in which $[R(f)]$ is the greatest integer not exceeding $R(f)$.*

Proof. Let $[a, b]$ be a defining arc for $R(f)$, $[c, d]$ a defining arc for $R(g)$, v a lift for $f|[a, b]$, u a lift for $g|[c, d]$. Now, $v(b)-v(a) \geqslant 2$. Let c' be the least number x, $v(a) \leqslant x$, such that $\varphi(x) = c$, and d' be the greatest number y, $y \leqslant v(b)$, such that $\varphi(y) = d$. Let c'' be the greatest number x, $x < d'$, such that $\varphi(x) = c$. Let z be a lift of $g \circ \varphi|[c', d']$. Then $z(d')-z(c'') = u(d)-u(c) = R(g)$, and by Lemma 2, $z(c'')-z(c') = (c''-c')\deg g \geqslant ([R(f)]-2)\deg g$. We have $z(d')-z(c') \geqslant R(g)+([R(f)]-2)\deg g$. Since v is weakly confluent, let A be a subarc of $[a, b]$ with $v(A) = [c', d']$. Then

$$R(g \circ f) \geqslant \deg(A, g \circ f) = z(d')-z(c') \geqslant R(g)+([R(f)]-2)\deg g .$$

COROLLARY. *Suppose M is a p.n.c.c.l. continuum, $M = \varprojlim(S^1, f_i^{i+1})$, such that for each i, $\deg f_i^{i+1} = 1$ and $R(f_i^{i+1}) \geqslant 3$. Then for each positive integer j, the sequence $(R(f_j^{j+1}), R(f_j^{j+2}), R(f_j^{j+3}), \ldots)$ increases without bound.*

Section 3. In [2], Henderson proved that no non-planar circle-like continuum is the continuous image of a continuum contractible with respect to the circle (c.r. S^1). In [9], Rogers proved that no chainable continuum can be mapped onto a circle-like continuum which is "self-entwined" (in his sense). In this chapter, Henderson's

result is extended to include the circle-like continua which are self-entwined (in my sense). Also, two theorems are proved, each of which states necessary and sufficient conditions for a circle-like continuum to be the continuous image of a chainable continuum.

Using Read's theorem and Lemma 6, one easily shows

LEMMA 10. *If X is a continuum, and f a map from X onto S^1, and A an arc in S^1, and B the complementary arc of A, then either there is a subcontinuum H of X such that $f(H) = A$ or there is a subcontinuum K of X such that $f(K) = B$.*

THEOREM 5. *Suppose M is a self-entwined p.n.c.c.l. continuum. Then M is not the continuous image of a continuum c.r. S^1.*

Proof. Suppose M is self-entwined, and X is a continuum c.r. S^1. We may assume, by Theorem 4, that $M = \varprojlim(S^1, f_i^{i+1})$, with $\deg f_i^{i+1} = 1$, and $R(f_i^{i+1}) \geqslant 3$, for each i. Suppose g is a map from X onto M. Then $PR_1 \circ g$ is inessential; let u be a lift of $PR_1 \circ g$. Let, by the corollary to Lemma 9, n be a positive integer such that $R(f_1^n) > (\operatorname{diam} u(X))+1$. Let $[a, b]$ be a defining arc for $R(f_1^n)$; t a lift of $f_1^n|[a, b]$; v a lift of $f_1^n|[b, a]$. Suppose H is a subcontinuum of X such that $PR_n \circ g(H) = [b, a]$. Then $\varphi \circ u|H = PR_1 \circ g|H = f_1^n \circ PR_n \circ g|H = \varphi \circ v \circ PR_n \circ g|H$. By Lemma 5,

$$\operatorname{diam} u(H) = \operatorname{diam} v\big(PR_n(g(H))\big) = \operatorname{diam} v([b, a]) = R(f_1^n)-1 > \operatorname{diam} u(X),$$

a contradiction. Similarly, if K is a subcontinuum of X such that $PR_n \circ g(K) = [a, b]$, then

$$\operatorname{diam} u(K) = \operatorname{diam} t([a, b]) = R(f_1^n) > \operatorname{diam} u(X),$$

a contradiction.

To prove Theorem 6, the main result, a technical lemma is required.

LEMMA 11. *Suppose each of f and g is a map from S^1 onto S^1 such that $\deg f = \deg g = 1$, $R(g) \geqslant 2$, and d is a number, $0 < d < 1$, such that $R(f \circ g) \leqslant 2+d$ and $R(f) \leqslant 2+d$. Let $[a, b]$ be a defining arc for $R(g)$ and w be a lift of $g|[a, b]$. Let $w([a, b]) = [p-1, q]$. The map $f \circ \varphi|[p, p+1]$ is inessential; let t be a lift of it. Then $\operatorname{diam} t([p, p+1]) \leqslant 1+d$.*

Proof. Let $t([p, p+1]) = [A, B]$. Suppose $B-A > 1+d$. For each number x between p and $p+1$ there is a lift z of $f|[\varphi(p), \varphi(x)]$ such that $z \circ \varphi|[p, x] = t|[p, x]$. Let u be a map from the ray $[\varphi(p), \varphi(p+1))$ such that $u \circ \varphi|[p, p+1) = t|[p, p+1)$. By Lemma 2, $\lim\limits_{x \to p+1} u(\varphi(x)) = u(\varphi(p))+1$.

There is a proper subinterval Y of $[p, p+1]$ such that $t(Y) = [A, B]$. For: $t(p+1) = \lim\limits_{x \to p+1} t(x) = \lim\limits_{x \to p+1} u(\varphi(x)) = u(\varphi(p))+1 = t(p)+1$. Since $B-A > 1$, it is not true that both endpoints of $[p, p+1]$ are mapped by t to the endpoints of $[A, B]$. Let $[e, r]$ be a proper subinterval of $[p, p+1]$ whose endpoints are mapped by t to the endpoints of $[A, B]$. In case $r = p+1$, there is a map u' from $[\varphi(e), \varphi(r)]$ such that $t|[e, r] = u' \circ \varphi|[e, r]$. Relabel $u = u'$ if necessary. Either $u(\varphi(e)) = A$ and $u(\varphi(r)) = B$ or $u(\varphi(e)) = B$ and $u(\varphi(r)) = A$.

Suppose $u(\varphi(e)) = B$. Let v be a map from the ray $[\varphi(e), \varphi(e+1))$ into the numbers, v an extension of u, such that $f|[\varphi(e), \varphi(e+1)) = \varphi \circ v$. By Lemma 2, $\operatorname{Lim}_{x\to e+1} v(\varphi(x)) = v(\varphi(e))+1 = B+1$. Also $v(\varphi(r)) = u(\varphi(r)) = A$. Thus

$$\deg([\varphi(r), \varphi(e)], f) \geqslant B+1-A > 2+d,$$

contradicting $R(f)\leqslant 2+d$. Therefore $u(\varphi(e)) = A$ and $u(\varphi(r)) = B$.

Now, $[e-1, r]\subset[p-1, p+1]\subset w([a, b])$. By an argument similar to that for Lemma 9, there is an arc M lying in $[a, b]$ such that $\deg(M, f\circ g)\geqslant B-(A-1)>2+d$, a contradiction. This completes the proof.

The following lemma is easily verified.

LEMMA 12. *If u is a map from a continuum A onto a continuum B, and v is a map from B onto a continuum C, and $v\circ u$ is weakly confluent, then v is weakly confluent.*

DEFINITION. By *class W* we shall mean the class of all continua Y such that if X is a continuum, and f a map from X onto Y, then f is weakly confluent.

Theorems 10 and 11 of [8] assert that arcs and arc-like continua are in class W.

THEOREM 6. *If C is a circle-like continuum then C is the continuous image of a chainable continuum if and only if either C is chainable or C is not in class W.*

Proof. Suppose C is a circle-like continuum not in class W. Let $C = \underleftarrow{\operatorname{Lim}}(S^1, f_i^{i+1})$, and let g be a non-weakly confluent map from a continuum X onto C. Suppose that for all but finitely many positive integers i, $PR_i\circ g$ is essential. Then for almost all i, $PR_i\circ g$ is weakly confluent. The argument for [8, Theorem 11] implies that g is weakly confluent, a contradiction. Hence for infinitely many, and therefore all, positive integers i, $PR_i\circ g$ is inessential. The argument for Theorem 4.2 and Corollary 4.3 of [4] implies that C is the continuous image of a chainable continuum.

Suppose that C is the continuous image of a chainable continuum X under the map g, and C is not chainable. By [2], C is planar, and by Theorem 5, C is not self-entwined. Let $C = \underleftarrow{\operatorname{Lim}}(S^1, f_i^{i+1})$, with $\deg f_i^{i+1} = 1$, for each i. Let, for each positive integer j, t_j be a lift of $PR_j\circ g$. Now, there exist a sequence $(d_1, d_2, \ldots)$ of numbers, with $0\leqslant d_i<1$, and a sequence $(V_1, V_2, \ldots)$ of intervals, with $V_i\subset t_i(X)$ and $\operatorname{diam} V_i = 1$ for each i, such that if i and j are positive integers with $i<j$, and p is a lift of $f_i^j\circ\varphi|V_j$, then $\operatorname{diam} p(V_j)\leqslant 1+d_i$. The proof of this assertion involves two cases.

Case 1. Suppose C is decomposable. By Theorem 2, C is homeomorphic to a member of class 1. Let, for each positive integer i, d_i be a number, $0\leqslant d_i<1$ such that for $k>i$, $R(f_i^k)\leqslant 1+d_i$. Let, for each positive integer p, V_p be any subinterval of $t_p(X)$ with length 1. Suppose i and j are positive integers, $i<j$. For any proper subinterval U of V_j, $\varphi(U)$ is an arc in S^1, and $\deg(\varphi(U), f_i^j)\leqslant R(f_i^j)\leqslant 1+d_i$; thus if p is a lift of $f_i^j\circ\varphi|V_j$, $\operatorname{diam} p(U)\leqslant 1+d_i$. Since this holds for each such U, $\operatorname{diam} p(V_j)\leqslant 1+d_i$.

Case 2. Suppose C is indecomposable. By Theorem 2 and 3, we may assume that for each i and j, $i<j$, $R(f_i^j)\geqslant 2$. Since C is not self-entwined, let, for each i, d_i be a number, $0\leqslant d_i<1$, such that for $k>i$, $R(f_i^k)\leqslant 2+d_i$. Suppose j is a positive integer. By Theorem 4.1 of [4] let u be a lift of $f_j^{j+1}\circ\varphi|t_{j+1}(X)$ such that $u(t_{j+1}(X)) = t_j(X)$. Let $[a, b]$ be a defining arc for $R(f_j^{j+1})$. Let A be the least number in $\varphi^{-1}(a)\cap t_{j+1}(X)$, and let B be the least number in $\varphi^{-1}(b)\cap t_{j+1}(X)$. Let r be a lift of $f_j^{j+1}|[a, b]$ such that $r(b) = r(\varphi(B)) = u(B)$. Let y be a lift of $f_j^{j+1}|[b, a]$ such that $y(b) = r(b)$. If $A<B$, then $u([A, B]) = r([a, b]) = [r(a), r(b)]$, and $[r(a)+1, r(a)+2]\subset t_j(X)$. If $B<A$, then $u([B, A]) = y([b, a]) = [r(a)+1, r(b)]$ by Lemma 5, and $[r(a)+1, r(a)+2]\subset t_j(X)$. Let $V_j = [r(a)+1, r(a)+2]$. Suppose i and j are positive integers, $i<j$. By Lemma 11, if p is a lift of $f_i^j\circ\varphi|V_j$, then $\operatorname{diam} p(V_j)\leqslant 1+d_i$.

Let $(d_1, d_2, \ldots)$ be a sequence of numbers and $(V_1, V_2, \ldots)$ a sequence of intervals as described. Since each map t_i is weakly confluent, let, for each positive integer j, K_j be a subcontinuum of X such that $t_j(K_j) = V_j$. Let $(K_{i_1}, K_{i_2}, K_{i_3}, \ldots)$ be a subsequence of K with a sequential limiting set M. Then M is a continuum.

Now, $g(M) = C$. For: Let y be an element of C. Since, for each j, $PR_j\circ g(K_j) = \varphi\circ t_j(K_j) = \varphi(V_j) = S^1$, let, for each n, x_n be a point of K_{i_n} with $PR_{i_n}\circ g(x_n) = y_{i_n}$. Let z be a cluster point of x, z in M. Suppose $g(z)\neq y$. Let n be a positive integer such that $PR_{i_n}\circ g(z)\neq y_{i_n}$. Let U and D be disjoint open sets in S^1 such that $PR_{i_n}(g(z))$ is in U and y_{i_n} is in D. Let $Q = (PR_{i_n}\circ g)^{-1}(U)$. Then Q is open in X, and z is in Q. Hence there exists $m>n$ with x_m in Q. Therefore

$$y_{i_n} = f_{i_n}^{i_m}(y_{i_m}) = f_{i_n}^{i_m}\big(PR_{i_m}(g(x_m))\big) = PR_{i_n}\circ g(x_m)$$

which is in U, since x_m is in Q. This involves a contradiction.

Now, for each j, $\operatorname{diam} t_j(M)\leqslant 1+d_j$. For: Suppose n is a positive integer such that $\operatorname{diam} t_n(M)>1+d_n$. Let $t_n(M) = [p, q]$. Let p' and q' be points of M such that $t_n(p') = p$, and $t_n(q') = q$. Let a be a number such that $0<a<\frac{1}{2}(q-p-1-d_n)$. Let b be a positive number such that if z is a point of X, with $\operatorname{dist}_X(p', z)<b$, then $|t_n(p')-t_n(z)|<a$, and if z is a point of X, with $\operatorname{dist}_X(q', z)<b$, then $|t_n(q')-t_n(z)|<a$. Let m be an integer, $m\geqslant n$, such that if $j\geqslant m$, then there are points x_j and y_j in K_{i_j} such that $\operatorname{dist}_X(p', x_j)<b$ and $\operatorname{dist}_X(q', y_j)<b$.

Consider $t_{i_m}(K_{i_m}) = V_{i_m}$. Let u be a lift of $f_n^{i_m}\circ\varphi|V_{i_m}$. Then

$$\varphi\circ t_n|K_{i_m} = PR_n\circ g|K_{i_m} = f_n^{i_m}\circ PR_{i_m}\circ g|K_{i_m} = f_n^{i_m}\circ\varphi\circ t_{i_m}|K_{i_m} = \varphi\circ u\circ t_{i_m}|K_{i_m}.$$

Hence $\operatorname{diam} t_n(K_{i_m}) = \operatorname{diam} u\big(t_{i_m}(K_{i_m})\big)\leqslant 1+d_n$. Let x_m and y_m be points of K_{i_m} such that $\operatorname{dist}_X(p', x_m)<b$ and $\operatorname{dist}_X(q', y_m)<b$. Then $|p-t_n(x_m)| = |t_n(p')-t_n(x_m)|<a$ and $|q-t_n(y_m)|<a$. We have

$$|t_n(x_m)-t_n(y_m)|\geqslant(q-p)-|p-t_n(x_m)|-|q-t_n(y_m)|>q-p-2a>1+d_n.$$

Thus $\operatorname{diam} t_n(K_{i_m})>1+d_n$, a contradiction.

Suppose j is a positive integer. Then $1\leqslant\operatorname{diam} t_j(M)\leqslant 1+d_j<2$, and $\varphi|t_j(M)$ is not weakly confluent. Hence $PR_j\circ g|M = \varphi\circ t_j|M$ is not weakly confluent by

Lemma 12. Since $\deg f_i^{i+1} = 1$ for each i, PR_j is an essential map from C onto S^1, thus PR_j is weakly confluent. If $g|M$ were weakly confluent, then $PR_j \circ g|M$ would be weakly confluent. Therefore $g(M)$ is C and $g|M$ is not weakly confluent, implying that C is not in class W. This completes the proof.

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Accepté par la Rédaction le 14. 6. 1976

Extending a partial equivalence to a congruence and relative embeddings in universal algebras

by

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Abstract. i) The partial equivalences which extend to congruences on arbitrary finitary universal algebras are characterized (as in [5] but with an additional particularization) freeing SC of [3] from the requirement that the equivalence have an initial generating domain, and yielding ii) the characterization of "admissible" subsets for semigroups developed in [1] as well as iii) a characterization of the partial algebras relatively embeddable in the full agebras of an equational class, which specialized to iv) a characterization of partial S-sets relatively embeddable in full ones, leads to v) that exactly for the subsemigroups T right unitary in S can T-sets be relatively embedded in S-sets.

Let ϱ be a *partial equivalence* ([1], p. 43), i.e. a symmetric transitive relation, on a finitary universal algebra A: we ask when the classes of ϱ are (in their totality) those of a single congruence on A. It is clear that if this is so for any congruence, then it will be so for the congruence θ generated by (i.e. the smallest congruence containing) ϱ; hence we investigate when strengthening ϱ to θ does not enlarge its classes.

The generation of θ from ϱ may be effected in stages. First, one extends ϱ to the smallest containing equivalence on A. Initially ϱ may only be defined on a proper subset D of A: it suffices to make it reflexive by augmenting it with the diagonal on the complement D' of D in A; in this process it loses neither its symmetry nor its transitivity and so becomes an equivalence on A. Its individual classes do not become enlarged; the new classes are just the singletons of the complement D'.

The next stage is to strengthen the relation to one having the substitution property for each of the operations which define the algebraic structure of A. This means that whenever an argument is replaced by an element modϱ-related to it, the value of the operation is to change (at most) to an element modϱ-related to the value. We must thus strengthen the equivalence to include the relation which holds between (possibly inequivalent) operation values for equivalent arguments — and then iteratively for arguments related in the so strengthened way. This strengthened relation turns out to be still reflexive and symmetric but may fail to be transitive; however its transitive closure is the desired congruence; and since the passage to this closure