Almost every tree function is independent

by

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Abstract. Points of the Cantor set \( C \) may be represented as branches of an infinite dyadic tree. Nodes of the tree may be randomly labeled with 0's and 1's. A tree function is a mapping from \( C \) to \([0,1]\) determined by assigning to each branch the real number having binary representation as the labeling of the branch. A tree function \( f \) is independent over a relation \( R \subseteq [0,1]^n \) if for every sequence \( x_0, \ldots, x_n \) of distinct elements of \( C \) we have \((f(x_0), \ldots, f(x_n)) \notin R \). We define a Borel probability measure on the set of tree functions and show that if \( R \) is null with respect to a special Hausdorff measure on \([0,1]^n \) then almost every tree function is independent over \( R \).

1. Introduction. A generalized notion of independence was introduced by Marczewski in [2] and extended by Mycielski to relational structures in [4]. Following [5] and [6] we consider relational structures of the form \( (M, R_K) \), where \( M \) is a non-empty, complete metric space, \( R_K \subseteq M^{(k)} \) and \( 1 \leq r(k) < \infty \) for all \( k < \infty \). For any set \( X \) a function \( f : X \to M \) is independent over the \( R_K \) if for every \( k \) and every sequence \( x_1, \ldots, x_{r(k)} \) of distinct elements of \( X \) we have \((f(x_1), \ldots, f(x_{r(k)})) \notin R_K \).

The Cantor set is denoted by the symbol \( C \) and is understood to be the discontinuum \([0,1]^n \) under the usual totally disconnected metrization. \( M^C \) is the space of all continuous functions \( f : C \to M \) with the usual uniform convergence topology.

The main result of this paper is a theorem analogous to the main theorems of [5] and [6]. In [6] Mycielski proves that if each \( R_k \) is meagre in \( M^{(k)} \) then the set of functions \( f \in M^C \) independent over all \( R_k \) is meagre in the space \( M^C \). In [5] he lets \( M \) be Euclidean \( n \)-space and shows that if the \( R_k \)'s are of Lebesgue \( n \)-dimensional measure zero then there exist independent functions \( f \in M^C \). If \( M = [0,1] \) and prove that if each \( R_k \) is \( h \)-null (see below) in \([0,1]^n \) then almost every tree function (see below) is also independent over the \( R_k \)'s (see Remark 1, Section 5).

In Section 2 we define randomly labeled trees and tree functions. We also construct a probability measure over the set of all tree functions and estimate the measure of certain useful subsets. In Section 3 we prove that two interesting properties are true for almost all tree functions. In Section 4 we define \( h \)-null sets and compare

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them with sets of measure zero under Hausdorff $h$-measure. We also show that sets of measure zero under a product Hausdorff measure are $h$-null. In Section 5 we prove the main result of this paper.

We wish to thank Andrzej Ehrenfeucht for the main idea used in the proof of Theorem 3.1.

The major results of this paper were announced in [1].

2. Tools and lemmas. A tree is a partially ordered set with the property that the set of predecessors of any element is well-ordered. For ordinals $\alpha$ the $\alpha$th-level of a tree is the set of all elements whose predecessors are order isomorphic to $\alpha$. A branch of a tree is a linearly ordered subset which intersects every level of the tree.

We let $(x_1,...,x_n) = (x_1,...,x_n)$ and denote by $T$ the infinite dyadic tree consisting of all finite sequences of 0s and 1s under the partial ordering $(a_1,...,a_n) \leq (b_1,...,b_n)$. We denote by $2^T$ the set of all $[0,1]$-labelings of $T$, i.e., functions $x: T \rightarrow [0,1]$. Clearly each $x \in 2^T$ is a tree under the partial ordering $(a, x(a)) \leq (b, x(b))$ if and only if $a \leq b$. Also $2^T$ carries the natural product topology and the product measure for which $P((x \in 2^T: x(i) = \delta) = \frac{1}{2}$ for all $i$ in $T$ and $\delta \in [0,1]$. For these reasons elements of $2^T$ will be referred to as randomly labeled trees.

To each $x \in 2^T$ we may associate a continuous mapping $f_x: C \rightarrow [0,1]$ by putting

$$f_x(s) = \sum_{a \in x} x(s_0, ..., s_{n-1})/2^n$$

for each $x = (s_0, s_1, ..., s_0) \in C$. The $f_x$'s are called tree functions. The set of tree functions is identified with $2^T$ under the mapping $x \mapsto f_x$ and inherits the topology and measure of $2^T$.

For any $\delta = (\delta_1, ..., \delta_2) \in (0,1)^n$ a dyadic interval of $C$ has the set

$$C(\delta) = \{x \in C: (x_0, ..., x_{n-1}) = \delta\}$$

and a dyadic interval of $[0,1]$ is the set

$$I(\delta) = \{y \in [0,1]: \sum_{i=0}^n \delta_i 2^i/2 \leq y < \sum_{i=0}^n \delta_i 2^i + 1/2\}.$$

For dyadic intervals of $C$ and $[0,1]$ we define

$$H(0, n) = \{x \in 2^T: f_x(C) \cap I(\delta_1, ..., \delta_n) \neq \emptyset\}$$

and

$$H(m, n) = \{x \in 2^T: f_x(C(\delta_1, ..., \delta_n)) \cap I(\delta_1, ..., \delta_n) \neq \emptyset\}.$$

By symmetry it is clear that the $P$-measure of $H(m, n)$ depends only on $m$ and $n$, and not on the $\delta_1$ and $\delta_n$ involved. If $n \leq m$ it is easily shown that all

$$P(H(m, n)) = 1/2^n.$$

To see this we let $C(\delta_1, ..., \delta_n)$ and $I(\delta_1, ..., \delta_n)$ be dyadic intervals defining $H(m, n)$ and observe that every $x \in H(m, n)$ satisfies $x(\delta_1, ..., \delta_n) = 1/2^k$ for $k = 1, ..., n$.
It follows that \( f(k) < g(k) \) for all positive integers \( k \). The desired result follows from
Lemma 2.1.2. Q.E.D.

Remark. It can also be shown that for large \( k \), \( P(H(0, k)) \) approaches \( 4(4k + k) \)
asymptotically.

3. Properties true for almost every tree function. The following results show that
almost all tree functions are not one-to-one but retain an important property of one-to-one functions in that they map \( C \) onto a perfect subset of \([0, 1]\).

Theorem 3.1 (Ehrenfeucht). Let \( T_1 \) and \( T_2 \) be disjoint substructures of \( T \) randomly
labeled with \( 0's \) and \( 1's \), each having a unique smallest element. With probability \( \geq \frac{1}{2} \)
there will be a branch of \( T_1 \) labeled in the same way as a branch of \( T_2 \).

Proof. For \( t \in T \) the set \( D(t) = \{ x \in T : t \models x \} \) characterizes substructures of \( T \) having
a unique smallest element. For \( x \in 2^T \) and branches \( B = (b_t) \) of \( D(t) \) we put \( a(B) = (a(b_t)) \). For \( t \) and \( x \) two incomparable elements of \( T \), i.e., \( t \models x \) and \( x \not\models t \), we denote by \( A \) the set of \( a \in 2^T \) for which there exist branches \( B_t \) of \( D(t) \) and \( B_x \) of \( D(x) \) satisfying \( a(B_t) = a(B_x) \). It suffices to show that \( P(A) \geq \frac{1}{2} \).

For \( t_1, \ldots, t_m \) pairwise incomparable elements of \( T \) the set \( D_{t_1}(t_2, \ldots, t_m) = \{ x \in T : t_i \models x \) for some \( i \) and \( \text{Card}\{x < y \mid y < x \} = n \} \) is a subtree of height \( n \) with \( m \) roots. If \( m = j + k \) we denote by \( M(j, k) \) the set of \( a \in 2^T \) satisfying the property that for all branches \( B_j \) of \( D(t_1, \ldots, t_j) \) and \( B_k \) of \( D(t_{j+1}, \ldots, t_m) \), \( a(B_j) \neq a(B_k) \). We put \( g_j(k) = P(M(j, k)) \) and observe that \( g_j(k) \) does not depend on the choice of the underlying \( D_{t_1}(t_2, \ldots, t_m)\). We partition \( D_{t_1}(t_2) \) into \( \{0(0), D(0, 0), D(0, 1)\} \) and \( D_{t_1}(t_2) \) into \( \{1(1), D(1, 0), D(0, 1)\} \). It is then not difficult to verify the equation

\[
q_{e_1}(1, 1) = \frac{1}{2} + q_1(2, 2)
\]

Similarly we partition \( D_{e_1}(0(0), 0(1)) \) and \( D_{e_1}(1(0), 1(1)) \) into points and subtrees of height \( n \) and arrive at equation

\[
q_{e_1}(2, 2) = \frac{1}{2} + q_2(2, 2) + q_4(2, 4)
\]

It can be seen that \( g_{e_1}(2, 2) > g_2(2, 2) > g_4(2, 4) \) if we substitute \( g_2(2, 2) \) for the other quantities in equation (2) we get \( g_2(2, 2) = \frac{1}{2} + g_4(2, 4) \geq 1 > 0 \). It follows that \( g_2(2, 2) < 1 \) so substitution into (3) gives \( g_2(1, 1) = \frac{1}{2} \) for all \( n \). Put \( A_n = 2^{A_n} \) for all \( A_n \in 2^{M(1, 1)} \) where \( M(1, 1) \) is determined by \( D(1) \) and \( D(2) \). Since \( P(M(1, 1)) < \frac{1}{2} \) we see that \( P(A_n) \geq \frac{1}{n} \) for \( n > 1 \). Also the sets \( A_n \) are monotone decreasing and \( A = \bigcap A_n \). Thus \( P(A) \geq \frac{1}{2} \) since \( P \) is continuous from above. Q.E.D.

Corollary 3.2. Almost every tree function is not one-to-one.

Proof. Let \( B \) be the set of \( a \in 2^T \) for which the tree function \( f_a \) is one-to-one. It suffices to show that \( P(B) < \epsilon \) for arbitrary \( \epsilon > 0 \). Choose \( n \) sufficiently large so that \( \epsilon = \frac{1}{2n} \). Let \( t \) be any element of the nth level of \( T \), and denote by \( M(t) \) the set of all \( a \in 2^T \) for which there does not exist a branch of \( D(t, 0) \) (see proof of Theorem) with \( a \)-labeling the same as some branch of \( D(t, 1) \). Clearly \( M(t) \) is the complement of some \( A \) as \( A = \bigcap M(t) \) so \( P(M(t)) \geq \frac{1}{2n} \) for all \( t \). Also \( B \subseteq \bigcap M(t) \) so \( P(B) \leq P(\bigcap M(t)) < \frac{1}{2n} \). Q.E.D.

Theorem 3.3. Almost every tree function maps the Cantor set onto a perfect subset
of \([0, 1]\).

Proof. For each positive integer \( n \) and each \( t \in T \) we denote by \( B_0(t) \) the set of \( a \in 2^T \) for which \( a(t, u) = a(t, v) \) for all \( u, v \in \{0, 1\}^* \). Each \( a \in B_0(t) \) maps all elements of the nth level of \( D(a(t)) \) (see proof of 3.1) into some binary digit, so \( P(B_0(t)) = \left( \frac{1}{2} \right)^{n-1} \). Thus the set \( B = \bigcap \bigcap B_0(t) \) has \( P \)-measure zero.

It suffices to show that \( f_a(C) \) is perfect in \([0, 1]\) for all \( a \in 2^T \). All \( f_a \)'s are continuous so \( f_a(C) \) is closed in \([0, 1]\). Let \( z \in f_a(C) \) be arbitrary and choose \( x \in C \) such that \( x = f_a(x) \). Let \( G \) be any open neighborhood of \( x \) and choose \( n \) sufficiently large so that the interval of radius \( 1/2^n \) about \( x \) is contained in \( G \). Since \( \epsilon \not\subseteq B \) there exists \( y \in C (x_0, \ldots, x_{n-1}) \) with \( f_a(y) = f_a(x) \). Also \( f_a(x) - f_a(y) < 1/2^n \) and \( f_a(y) \in G \). Thus \( z \) is a limit point and \( f_a(C) \) is perfect. Q.E.D.

4. Hausdorff h-measure and h-nullity. In this section we consider a special collection of subsets of Euclidean \( n \)-space \( R^n \). These sets, which we call \( h \)-null (Definition 4.1), are defined in a manner similar to sets of measure zero under Hausdorff \( h \)-measure (see [7]). In Theorems 4.3 and 4.4 we show that every \( N \subseteq \mathbb{R}^n \) which has measure zero with respect to Hausdorff \( h \)-measure or has measure zero with respect to a product Hausdorff measure, is \( h \)-null.

Let \( h \) be a real-valued, monotonic increasing function defined for \( t \geq 0 \), positive for \( t > 0 \), continuous on the right, and with \( h(0) = 0 \). For any set \( X \subseteq \mathbb{R}^n \) we denote by \( d(X) \) the diameter of \( X \).

Definition 4.1. A set \( N \subseteq \mathbb{R}^n \) is said to be \( h \)-null if for every \( \epsilon > 0 \) there exists a collection \( \{ G(j, k) : j = 1, \ldots, n; k = 1, 2, \ldots, \} \) of bounded open sets of \( R \) such that

\[
N \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} G(j, k) \quad \text{and} \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} d(G(j, k)) < \epsilon.
\]

It is easily seen that a countable union of \( h \)-null sets is \( h \)-null. We also observe that vertical and horizontal lines in \( R^2 \) are \( h \)-null, but that for some \( h \)'s (e.g. \( h(t) = \sqrt{t} \)) the diagonal in \( R^2 \) is not \( h \)-null. Thus \( h \)-null sets are not invariant under congruences. Also, with \( h(t) = t^{1/t} \) vertical and horizontal lines in \( R^2 \) are not of measure zero under Hausdorff \( h \)-measure ([7]) p. 79 so \( h \)-null sets need not be of measure zero under \( h \)-measure.

Lemma 4.2. If \( \{ N(j) : j = 1, \ldots, n \} \) is a finite collection of subsets of \( R \) such that at least one \( N(j) \) is \( h \)-null, then \( N = \bigcup_{j=1}^{\infty} N(j) \) is \( h \)-null in \( \mathbb{R}^n \).

Proof. Since \( h \) is continuous on the right and positive for \( t > 0 \) there exists \( \delta > 0 \) such that \( 0 < h(x) < 1 \) whenever \( 0 < x < \delta \). We consider \( R \) as a countable union of
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Clearly each set \( G(j, k) \) can be covered by at most three dyadic intervals of \([0, 1]\) with diameters less than or equal to \( d(G(j, k)) \). Thus

\[
G(j, k) \subseteq I(j, k, 1) \cup I(j, k, 2) \cup I(j, k, 3)
\]

and

\[
N = \bigcup_{k=1}^{3} \bigcup_{i_1=1}^{2} \bigcup_{i_2=1}^{2} \bigcup_{i_3=1}^{2} P(I(j, k, i))
\]

where \( d(I(j, k, i_1)) \leq d(G(j, k)) \) for all \( i_1 \). Also each \( I(j, k, i) = I(\delta) \) for some \( \delta \in [0, 1]^{d(A, 0)} \) so

\[
h_0(d(I(j, k, i))) = -1/((\log(1/2d(A, j))) = 1/p(j, k, i)\log 2)
\]

Thus

\[
\sum_{j=1}^{n} \frac{3}{2} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} \prod_{k=1}^{3} h_0(d(I(j, k, i)))
\]

\[
\leq (\log 2)^3 \sum_{j=1}^{n} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} h_0(d(G(j, k)))
\]

\[
\leq 3^n (\log 2)^n \sum_{j=1}^{n} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} h_0(d(G(j, k))) < \epsilon. \text{ Q.E.D.}
\]

Let \( \{I_j; j = 1, \ldots, n\} \) be a collection of distinct elements of \([0, 1]^n\) and let \( \{I_j; j = 1, \ldots, n\} \) be a collection of dyadic intervals of \([0, 1]\) each of diameter less than \(1/2^n\). Put \( H_j = \{x \in 2^n : f(x) \cap I_j \neq \emptyset\} \) and \( H = \bigcap_{j=1}^{n} H_j \).

**Lemma 5.2:** \( P(H) = 2^n \prod_{j=1}^{n} p(H_j) \) for some \( K < m^{2n} \).

**Proof:** We put \( I_j = I(\delta_{j_1}, \ldots, \delta_{j_m}) \) and observe that \( n(j) > m \) for all \( j \). We put \( D_j = D_{(j_1, \ldots, j_m)} \) as defined in the proof of Lemma 2.1. Clearly each \( H_j \) is identical to some \( H((n, j)) \) as defined in Section 2 so by Lemma 2.1 \( P(H_j) = 1/2^n P(H_j^*) \) where \( H_j^* \) is the set of all \( \alpha \in 2^n \) for which there exists a branch of \( D_j \) whose \( \alpha \)-labeling is identical to \( (\delta_{j_{m+1}}, \ldots, \delta_{j_m}) \). Let \( t_j = (t_{j_1}, \ldots, t_{j_m}) \) and put

\[
X_j = \{x \in 2^n : \alpha(t_{j_1}, \ldots, t_{j_m}) = \delta_j \text{ for } k = 1, \ldots, m \}.
\]

Each \( X_j \) is determined by fixing the image of \( m \) elements of \( T \) and thus \( X_j \) is determined by fixing at most \( mn \) elements of \( T \). It follows that \( P(\bigcap_{j=1}^{n} X_j) = 1/2^{m-n} \) where \( K \) is the number of redundant labelings. Clearly \( H_j = X_j \cap H_j^* \) so

\[
H = \bigcap_{j=1}^{n} X_j \cap H_j^*
\]

and

\[
P(H) = P(\bigcap_{j=1}^{n} X_j) P(\bigcap_{j=1}^{n} H_j^*) = (1/2^{m-n}) \prod_{j=1}^{n} P(H_j) = 2^n \prod_{j=1}^{n} P(H_j).
\]

It remains only to determine \( K \). Let \( l \in \{1, \ldots, m\} \) be given and for \( \delta \in \{0, 1\}^{d(A, 0)} \) put \( B(\delta) = \{x : (x_{j_1}, \ldots, x_{j_m}) = \delta\} \). Define \( \text{REDUN}(B(\delta)) = \max\{0, \text{Card}(B(\delta)) - 1\} \). It follows that

\[
K = \sum_{\delta \in 0, 1}^{n} \text{REDUN}(B(\delta)).
\]

The \( t_j \)'s are all distinct so \( \text{Card}(B(\delta)) \leq 2^{m-n} \) regardless of the value of \( n \). Thus

\[
K \leq \sum_{\delta \in 0, 1}^{n} \text{REDUN}(B(\delta)) = \sum_{\delta \in 0, 1}^{n} (2^{m-n} - 1) \leq m \cdot 2^{m-n} \leq m^{2m}. \text{ Q.E.D.}
\]

**Theorem 5.3:** If \( R = h_0 \text{-null in } [0, 1]^n \), then almost every tree function is independent over \( R \).

**Proof:** Let \( m \) be a positive integer such that \( 2^m \geq n \) and consider a sequence \( L = (t_1, \ldots, t_n) \) of distinct elements of \([0, 1]^n\) and put

\[
W(R, m, L) = \{x \in 2^n : \bigcap_{j=1}^{n} f_j(C(x_j)) \cap L \neq \emptyset\}.
\]

To show that \( P(W(R, m, L)) = 0 \) we let \( \epsilon > 0 \) be given and put \( M = m^{2m} \) and \( \epsilon^* = \epsilon(4M)^n \). Let \( \{I_j; j = 1, \ldots, n; k = 1, 2, \ldots\} \) be a collection of dyadic intervals of \([0, 1]\) covering \( R \) as in Lemma 5.1 with \( \epsilon = \epsilon^* \). Without loss of generality we choose all \( I(j, k) \) so that \( p(j, k) > m \). Putting

\[
B(j, k) = \{x \in 2^n : f_j(C(x_j)) \cap I(j, k) \neq \emptyset\}
\]

we see that \( W(R, m, L) \subseteq \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} B(j, k) \) and that

\[
P(W(R, m, L)) = \sum_{j=1}^{n} \sum_{k=1}^{m} P(B(j, k)) \leq 2^n \sum_{k=1}^{m} P(B(j, k)) \text{ by Lemma 5.2}
\]

\[
\leq 2^m \sum_{k=1}^{m} \prod_{j=1}^{n} P(B(j, k)) \text{ by Lemma 2.2}
\]

\[
\leq 2^m \prod_{j=1}^{n} \prod_{k=1}^{m} P(B(j, k)) \leq 2^m \cdot 2^n \text{ by definition of } B(j, k)
\]

Since \( \epsilon > 0 \) was arbitrary it follows that \( P(W(R, m, L)) = 0 \). We then put \( W = \bigcup_{j=1}^{m} W(R, m, L) \) and observe that \( W \) has \( \mathcal{F} \)-measure zero. We claim that every \( f_x \) for which \( x \notin W \) is independent. This follows because for any set \( \{x_1, \ldots, x_n\} \) of distinct elements of \( C \) with \( (f_{x_1}, \ldots, f_{x_n}) \in R \) there exists an \( m \geq n \) and an \( L \) such that \( x \in W(R, m, L) \). Q.E.D.
Remarks 1) To get a theorem similar to Mycielski's (see introduction) we can extend Theorem 5.3 to the case where the number of relations is countable. This follows by the countable additivity of the measure $P$ in $2^\mathbb{N}$.

2) Combining Theorems 3.3 and 5.3 it follows that almost every tree function is independent with perfect range in $[0, 1]$.

3) Theorem 4.3 ensures that Theorem 5.3 remains valid if the Hausdorff $(-1/\log y)^N$-measure of $R$ is zero.

4) Theorem 4.4 ensures that Theorem 5.3 remains valid if $R$ is of measure zero with respect to the product measure ($\delta_0$-measure).

5) The mapping $x \rightarrow f_x$ from $2^\mathbb{N}$ to the space $[0, 1]^2$ is continuous: For any Borel set $B \subseteq [0, 1]^2$ we define $\mu(B) = P(\{z \in 2^\mathbb{N} : f_z \in B\})$. Under this Borel measure, and for any $h$-null $R \subseteq [0, 1]^2$, almost every $f \in [0, 1]^2$ is independent over $R$.

References


Sequence of iterates of generalized contractions

by

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Abstract. The main purpose of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if $T$ is a generalized contraction mapping of closed, bounded and convex subset of a uniformly convex Banach space into itself with nonempty fixed points set, then the mapping $T_n$ defined by $T_n = \lambda T + (1-\lambda) T$, for any $\lambda$ such that $0 < \lambda < 1$ is asymptotically regular. As a corollary of this we get the result of Schaefer (ibid. Deutsch. Math. Veren. 1957), pp. 131-140. In Section 2, we prove for Hilbert spaces the mapping $T_n$ as defined above is a reasonable wanderer. As a corollary of this we obtain the result of Browder and Petryshyn (J. Math. Anal. and Appl. 20 (1967), pp. 197-228). Finally in Sections 3 and 4, we have obtained some results for the weak and strong convergence of sequence of iterates for mappings of this type.

Introduction. The main aim of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if $T$ is a generalized contraction mapping of a closed, bounded and convex subset of a uniformly convex Banach space into itself with non-empty fixed point set, then the mapping $T_n$ defined by $T_n = \lambda T + (1-\lambda) T$, for any $\lambda$ such that $0 < \lambda < 1$ is asymptotically regular. In section, it is shown that if $T$ is a generalized contraction self mapping of a closed, convex subset of Hilbert space with non-empty fixed point set, then the mapping $T_n$ defined as above is a reasonable wanderer with the same fixed point as $T$. Finally in Sections 3 and 4 we have obtained some results for the weak and strong convergence of sequence of iterates of such kind of mappings.

Definition 1.1. Let $C$ be a closed, bounded and convex subset of a Banach space $X$. A mapping $T: C \rightarrow C$ is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||$$

for all $x, y$ in $C$.

Definition 1.2. A mapping $T: C \rightarrow C$ is said to be quasi-nonexpansive if

$$||Tx - Ty|| \leq a||x - y|| + b||x - Tx|| + c||y - Ty||$$

for all $x, y$ in $C$, $a \geq 0$, $b \geq 0$, $c \geq 0$ and $a + b + c \leq 1$.

The following example shows that there are quasi-nonexpansive mappings which are not nonexpansive.