[7] F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. 43 (1937), pp. 671–677.
[8] I. Juhász, *Cardinal Functions in Topology*, Amsterdam 1971.
[9] K. Kunen, *Products of S-spaces*, preprint.
[10] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[11] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), pp. 375–376.
[12] E. Michael, *Paracompactness and the Lindelöf property in finite and countable cartesian products*, Compositio Math. 23 (1971), pp. 199–214.
[13] K. Morita, *Products of normal spaces with metric spaces II*, Sci. Rep. Tokyo Kyoiku Daigaku 8 (1963), pp. 87–92.
[14] K. Nagami, *A note on the normality of inverse limits and products*, Proc. Intern. Symp. Topology Appl., Herceg-Novi 1968, Beograd 1969, pp. 261–264.
[15] T. Przymusiński, *A Lindelöf space X such that X^2 is normal but not paracompact*, Fund. Math. 78 (1973), pp. 291–296.
[16] — *Collectionwise Hausdorff property in product spaces*, Colloq. Math. 36 (1976), pp. 49–56.
[17] — *Normality and paracompactness in subsets of product spaces*, Fund. Math. 91 (1976), pp. 161–165.
[18] — *On the notion of n-cardinality*, Proc. Amer. Math. Soc. 69 (1978), pp. 333–338.
[19] M. E. Rudin, *Lectures on Set-Theoretic Topology*, Providence 1975.
[20] — and M. Starbird, *Products with a metric factor*, Gen. Topology Appl. 5 (1975), pp. 235–248.
[21] F. D. Tall, *Problem Session*, Colloq. Math. Soc. János Bolyai 8, Topics in Topology, North Holland, Amsterdam 1974.
[22] H. Tamano, *Note on paracompactness*, J. Math. Kyoto Univ. 3 (1) (1963), pp. 137–143.
[23] — *Normality and product spaces*, Gen. Topology and its relations to modern analysis and algebra II, Academic Press 1967.
[24] J. Worrell and H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.
[25] P. Zenor, *Countable paracompactness in product spaces*, Proc. Amer. Math. Soc. 30 (1971), pp. 199–201.

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Almost every tree function is independent

by

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Abstract. Points of the Cantor set C may be represented as branches of an infinite dyadic tree. Nodes of the tree may be randomly labeled with 0's and 1's. A tree function is a mapping from C to $[0,1]$ determined by assigning to each branch the real number having binary representation as the labeling of the branch. A tree function f is independent over a relation $R \subseteq [0,1]^n$ if for every sequence $x_1, \ldots, x_n$ of distinct elements of C we have $(f(x_1), \ldots, f(x_n)) \notin R$. We define a Borel probability measure on the set of tree functions and show that if R is null with respect to a special Hausdorff measure on $[0,1]^n$ then almost every tree function is independent over R.

1. Introduction. A generalized notion of independence was introduced by Marczewski in [2] and extended by Mycielski to relational structures in [4]. Following [5] and [6] we consider *relational structures* of the form $\langle M, R_k \rangle_{k<\omega}$ where M is a non-empty, complete metric space, $R_k \subseteq M^{r(k)}$ and $1 \leqslant r(k) < \omega$ for all $k < \omega$. For any set X a function $f\colon X \to M$ is *independent over* the R_k's if for every k and every sequence $x_1, \ldots, x_{r(k)}$ of *distinct* elements of X we have $(f(x_1), \ldots, f(x_{r(k)})) \notin R_k$.

The Cantor set is denoted by the symbol C and is understood to be the discontinuum $\{0,1\}^\omega$ under the usual totally disconnected metrization. M^C is the space of all continuous functions $f\colon C \to M$ with the usual uniform convergence topology.

The main result of this paper is a theorem analogous to the main theorems of [5] and [6]. In [6] Mycielski proves that if each R_k is meagre in $M^{r(k)}$ then the set of functions $f \in M^C$ independent over all R_k's is comeager in the space M^C. In [5] he lets M be Euclidean n-space and shows that if the R_k's are of Lebesgue $r(k)$-dimensional measure zero then there exist independent functions $f \in M^C$. We let $M = [0,1]$ and prove that if each R_k is h_0-null (see below) in $[0,1]^{r(k)}$ then almost every tree function (see below) is also independent over the R_k's (see Remark 1, Section 5).

In Section 2 we define randomly labeled trees and tree functions. We also construct a probability measure over the set of all tree functions and estimate the measure of certain useful subsets. In Section 3 we prove that two interesting properties are true for almost all tree functions. In Section 4 we define h-null sets and compare

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them with sets of measure zero under Hausdorff h-measure. We also show that sets of measure zero under a product Hausdorff measure are h-null. In Section 5 we prove the main result of this paper.

We wish to thank Andrzej Ehrenfeucht for the main idea used in the proof of Theorem 3.1.

The major results of this paper were announced in [1].

2. Tools and lemmas. A *tree* is a partially ordered set with the property that the set of predecessors of any element is well-ordered. For ordinals n the nth-*level* of a tree is the set of all elements whose predecessors are order isomorphic to n. A *branch* of a tree is a linearly ordered subset which intersects every level of the tree.

We let $(x_1, \ldots, x_n) = \{\langle 1, x_1\rangle, \ldots, \langle n, x_n\rangle\}$ and denote by T the infinite dyadic tree consisting of all finite sequences of 0's and 1's under the partial ordering $(t_1, \ldots, t_m) \subseteq (s_1, \ldots, s_n)$. We denote by 2^T the set of all $\{0, 1\}$-labelings of T, i.e., functions $\alpha\colon T \to \{0, 1\}$. Clearly each $\alpha \in 2^T$ is a tree under the partial ordering $(t, \alpha(t)) \leqslant (s, \alpha(s))$ if and only if $t \subseteq s$. Also 2^T carries the natural product topology and probability product measure for which $P(\{\alpha \in 2^T\colon \alpha(t) = \delta\}) = \frac{1}{2}$ for all $t \in T$ and $\delta \in \{0, 1\}$. For these reasons elements of 2^T will be referred to as *randomly labeled trees*.

To each $\alpha \in 2^T$ we may associate a continuous mapping $f_\alpha\colon C \to [0, 1]$ by putting

$$f_\alpha(x) = \sum_{n=1}^{\infty} \alpha(x_0, \ldots, x_{n-1})/2^n$$

for each $x = (x_0, x_1, \ldots) \in C$. The f_α's are called *tree functions*. The set of tree functions is identified with 2^T under the mapping $\alpha \Leftrightarrow f_\alpha$ and inherits the topology and measure of 2^T.

For any $\delta = (\delta_1, \ldots, \delta_n) \in \{0, 1\}^n$ a *dyadic interval of* C is the set

$$C(\delta) = \{x \in C\colon (x_0, \ldots, x_{n-1}) = \delta\}$$

and a *dyadic interval of* $[0, 1]$ is the set

$$I(\delta) = \{y \in [0, 1]\colon \sum_{i=1}^{n} \delta_i/2^i \leqslant y < \sum_{i=1}^{n} \delta_i/2^i + 1/2^n\}.$$

For dyadic intervals of C and $[0, 1]$ we define

$$H(0, n) = \{\alpha \in 2^T\colon f_\alpha(C) \cap I(\delta'_1, \ldots, \delta'_n) \neq \emptyset\},$$
$$H(m, n) = \{\alpha \in 2^T\colon f_\alpha(C(\delta_1, \ldots, \delta_m)) \cap I(\delta'_1, \ldots, \delta'_n) \neq \emptyset\}.$$

By symmetry it is clear that the P-measure of $H(m, n)$ depends only on m and n, and not on the δ_i and δ'_i involved. If $n \leqslant m$ it is easily shown that all

$$P(H(m, n)) = 1/2^n.$$

To see this we let $C(\delta_1, \ldots, \delta_m)$ and $I(\delta'_1, \ldots, \delta'_n)$ be dyadic intervals defining $H(m, n)$ and observe that every $\alpha \in H(m, n)$ satisfies $\alpha(\delta_1, \ldots, \delta_k) = \delta'_k$ for $k = 1, \ldots, n$.

An upper bound for the P-measure of sets $H(m, n)$ with $n > m$ is needed in Section 5. The following Lemma 2.2 specifies the desired bound.

LEMMA 2.1.

1) $P(H(0, k+1)) = P(H(0, k)) - P(H(0, k))^2/4$,

2) $P(H(m, m+k)) = P(H(0, k))/2^m$.

Proof. For positive integers n and $t = \emptyset$ or $t \in T$ the set

$$D_n(t) = \{x \in T\colon t \subseteq x, t \neq x, \operatorname{Card}(x \setminus t) \leqslant n\}$$

is a subtree of T of height n. For branches $B = (b_1, \ldots, b_n)$ of $D_n(t)$ we put $\alpha(B) = (\alpha(b_1), \ldots, \alpha(b_n))$ and for $\delta \in \{0, 1\}^n$ put $M(D_n(t), \delta) = \{\alpha \in 2^T\colon \alpha(B) \neq \delta$ for any branch B of $D_n(t)\}$. It is clear that the P-measure of $M(D_n(t), \delta)$ depends only on n and not on the choice of t or δ, so we denote by p_n the P-measure of such sets.

Let $I(\delta_1, \ldots, \delta_k)$ be a dyadic interval of $[0, 1]$ that determines one of the sets $H(0, k)$. For every $\alpha \in H(0, k)$ there exists an $x \in C$ with $f_\alpha(x) \in I(\delta_1, \ldots, \delta_k)$ so it follows that $H(0, k) = 2^T \setminus M(D_k(\emptyset), \delta)$ where $\delta = (\delta_1, \ldots, \delta_k)$. Thus $P(H(0, k)) = 1 - p_k$ for all positive integers k. Next let $\delta = (\delta_1, \ldots, \delta_{k+1}) \in \{0, 1\}^{k+1}$ and for $\delta^* = (\delta_2, \ldots, \delta_{k+1})$ put $M = M(D_{k+1}(\emptyset), \delta)$, $M_1 = M(D_k((0)), \delta^*)$, and $M_2 = M(D_k((1)), \delta^*)$. The set M may be partitioned into four disjoint sets as follows:

$$S_1 = \{\alpha\colon \alpha((0)) = \alpha((1)) \neq \delta_1\},$$
$$S_2 = \{\alpha\colon \alpha((0)) \neq \alpha((1)) = \delta_1 \text{ and } \alpha \in M_2\},$$
$$S_3 = \{\alpha\colon \delta_1 = \alpha((0)) \neq \alpha((1)) \text{ and } \alpha \in M_1\},$$
$$S_4 = \{\alpha\colon \alpha((0)) = \alpha((1)) = \delta_1 \text{ and } \alpha \in M_1 \text{ and } \alpha \in M_2\}.$$

Thus $P(M) = \frac{1}{2}\cdot\frac{1}{2}(1 + P(M_2) + P(M_1) + P(M_1)P(M_2))$ and $p_{k+1} = \frac{1}{4}(1 + 2p_k + p_k^2) = \frac{1}{4}(1 + p_k)^2$. The substitution $p_k = 1 - P(H(0, k))$ gives (1).

To prove (2) we let $C(\delta_1, \ldots, \delta_m)$ and $I(\delta'_1, \ldots, \delta'_{m+k})$ be dyadic intervals that determine one of the sets $H(m, m+k)$. Let $M = M(D_k((\delta_1, \ldots, \delta_m)), (\delta'_{m+1}, \ldots, \delta'_{m+k}))$. It is clear that every $\alpha \in H(m, m+k)$ satisfies $\alpha(\delta_1, \ldots, \delta_n) = \delta'_n$ for $n = 1, \ldots, m$ and that $\alpha \notin M$. Thus $P(H(m, m+k)) = (\frac{1}{2})^m P(2^T \setminus M) = (1 - p_k)/2^m = P(H(0, k)/2^m$. Q.E.D.

LEMMA 2.2. $P(H(m, m+k)) \leqslant \left(\dfrac{4}{4+k}\right)/2^m$.

Proof. Let f and g be defined over the positive integers so that $f(k) = P(H(0, k))$ and $g(k) = 4/(4+k)$. By direct observation we see that $f(1) = \frac{3}{4}$. It is also easily seen that $g(k)(1 - \frac{1}{4}g(k)) \leqslant g(k+1)$. In addition, for real numbers a and b satisfying $0 \leqslant a \leqslant b \leqslant 1$, the inequality $a(1 - \frac{1}{4}a) \leqslant b(1 - \frac{1}{4}b)$ is valid. Clearly $f(1) = \frac{3}{4} < \frac{4}{5} = g(1)$. Proceeding by induction we assume that $0 \leqslant f(k) \leqslant g(k) \leqslant 1$ has been shown. From Lemma 2.1.1 and the above we see that

$$0 \leqslant f(k+1) = f(k)(1 - \tfrac{1}{4}f(k)) \leqslant g(k)(1 - \tfrac{1}{4}g(k)) \leqslant g(k+1) \leqslant 1.$$

It follows that $f(k) \leqslant g(k)$ for all positive integers k. The desired result follows from Lemma 2.1.2. Q.E.D.

Remark. It can also be shown that for large k, $P(H(0, k))$ approaches $4/(4+k)$ asymptotically.

3. Properties true for almost every tree function. The following results show that almost all tree functions are not one-to-one but retain an important property of one-to-one functions in that they map C onto a perfect subset of $[0, 1]$.

THEOREM 3.1 (Ehrenfeucht). *Let T_1 and T_2 be disjoint subtrees of T randomly labeled with 0's and 1's, each having a unique smallest element. With probability $\geqslant \frac{1}{4}$ there will be a branch of T_1 labeled in the same way as a branch of T_2.*

Proof. For $t \in T$ the set $D(t) = \{x \in T: t \subseteq x\}$ characterizes subtrees of T having a unique smallest element. For $\alpha \in 2^T$ and branches $B = (b_i)$ of $D(t)$ we put $\alpha(B) = (\alpha(b_i))$. For t and s two incomparable elements of T, i.e., $t \not\subseteq s$ and $s \not\subseteq t$, we denote by A the set of $\alpha \in 2^T$ for which there exist branches B_1 of $D(t)$ and B_2 of $D(s)$ satisfying $\alpha(B_1) = \alpha(B_2)$. It suffices to show that $P(A) \geqslant \frac{1}{4}$.

For $t_1, \ldots, t_m$ pairwise incomparable elements of T the set $D_n(t_1, \ldots, t_m) = \{x \in T: t_i \subseteq x$ for some i and $\mathrm{Card}(x \setminus t_i) \leqslant n-1\}$ is a subtree of height n with m roots. If $m = j+k$ we denote by $M_n(j, k)$ the set of $\alpha \in 2^T$ satisfying the property that for all branches B_1 of $D_n(t_1, \ldots, t_j)$ and B_2 of $D_n(t_{j+1}, \ldots, t_{j+k})$, $\alpha(B_1) \neq \alpha(B_2)$. We put $q_n(j, k) = P(M_n(j, k))$ and observe that $q_n(j, k)$ does not depend on the choice of the underlying D_n's. We partition $D_{n+1}((0))$ into $\{(0), D_n((0, 0), (0, 1))\}$ and $D_{n+1}((1))$ into $\{(1), D_n((1, 0), (1, 1))\}$. It is then not difficult to verify the equation

$$q_{n+1}(1, 1) = \tfrac{1}{2} + \tfrac{1}{2} q_n(2, 2)\,. \tag{1}$$

Similarily we partition $D_{n+1}((0, 0), (0, 1))$ and $D_{n+1}((1, 0), (1, 1))$ into points and subtrees of height n and arrive at equation

$$q_{n+1}(2, 2) = \tfrac{1}{16}\big(2 + 8q_n(4, 2) + 4q_n(2, 2)^2 + 2q_n(4, 4)\big)\,. \tag{2}$$

It is clear that $q_{n+1}(2, 2) > q_n(2, 2) > q_n(4, 2) > q_n(4, 4)$ so if we substitute $q_n(2, 2)$ for the other quantities in equation (2) we get $2q_n(2, 2)^2 - 3q_n(2, 2) + 1 > 0$. It follows that $q_n(2, 2) < \frac{1}{2}$ so substitution into (1) gives $q_{n+1}(1, 1) < \frac{3}{4}$ for all n. Put $A_n = 2^T \setminus M_n(1, 1)$ where $M_n(1, 1)$ is determined by $D_n(t)$ and $D_n(s)$. Since $P(M_n(1, 1)) < \frac{3}{4}$ we see that $P(A_n) > \frac{1}{4}$ for $n > 1$. Also the sets A_n are monotone decreasing and $A = \bigcap_{n=1}^{\infty} A_n$. Thus $P(A) \geqslant \frac{1}{4}$ since P is continuous from above. Q.E.D.

COROLLARY 3.2. *Almost every tree function is not one-to-one.*

Proof. Let B be the set of $\alpha \in 2^T$ for which the tree function f_α is one-to-one. It suffices to show that $P(B) < \varepsilon$ for arbitrary $\varepsilon > 0$. Choose n sufficiently large so that $(\frac{3}{4})^{2^n} < \varepsilon$. Let t be any element of the nth level of T, and denote by $M(t)$ the set of all $\alpha \in 2^T$ for which there does not exist a branch of $D((t, 0))$ (see proof of Theorem) with α-labeling the same as some branch of $D((t, 1))$. Clearly $M(t)$ is the complement of some A as in the Theorem so $P(M(t)) \leqslant \frac{3}{4}$ for all t. Also $B \subseteq \bigcap_t M(t)$ so $P(B) \leqslant P(\bigcap_t M(t)) \leqslant (\frac{3}{4})^{2^n} < \varepsilon$. Q.E.D.

THEOREM 3.3. *Almost every tree function maps the Cantor set onto a perfect subset of* $[0, 1]$.

Proof. For each positive integer n and each $t \in T$ we denote by $B_n(t)$ the set of $\alpha \in 2^T$ for which $\alpha((t, u)) = \alpha((t, v))$ for all $u, v \in \{0, 1\}^n$. Each $\alpha \in B_n(t)$ maps all elements of the nth level of $D(t) \setminus \{t\}$ (see proof of 3.1) into the same binary digit, so $P(B_n(t)) = (\frac{1}{2})^{2^n - 1}$. Thus the set $B = \bigcup_{t \in T} \bigcap_{n=1}^{\infty} B_n(t)$ has P-measure zero. It sufices to show that $f_\alpha(C)$ is perfect in $[0, 1]$ for all $\alpha \in 2^T \setminus B$. All f_α's are continuous so $f_\alpha(C)$ is closed in $[0, 1]$. Let $z \in f_\alpha(C)$ be arbitrary and choose $x \in C$ such that $z = f_\alpha(x)$. Let G be any open neighborhood of z and choose n sufficiently large so that the interval of radius $1/2^n$ about z is contained in G. Since $\alpha \notin B$ there exists $y \in C(x_0, \ldots, x_{n-1})$ with $f_\alpha(x) \neq f_\alpha(y)$. Also $|f_\alpha(x) - f_\alpha(y)| < 1/2^n$ and $f_\alpha(y) \in G$. Thus z is a limit point and $f_\alpha(C)$ is perfect. Q.E.D.

4. Hausdorff h-measure and h-nullity. In this section we consider a special collection of subsets of Euclidean n-space $\boldsymbol{R}^n$. These sets, which we call h-null (Definition 4.1), are defined in a manner similar to sets of measure zero under Hausdorff h-measure (see [7]). In Theorems 4.3 and 4.4 we show that every $N \subseteq \boldsymbol{R}^n$ which has measure zero with respect to Hausdorff h-measure or has measure zero with respect to a product Hausdorff measure, is h-null.

Let h be a real-valued, monotonic increasing function defined for $t \geqslant 0$, positive for $t > 0$, continuous on the right, and with $h(0) = 0$. For any set $X \subseteq \boldsymbol{R}^n$ we denote by $d(X)$ the diameter of X.

DEFINITION 4.1. A set $N \subseteq \boldsymbol{R}^n$ is said to be *h-null* if for every $\varepsilon > 0$ there exists a collection $\{G(j, k): j = 1, \ldots, n;\ k = 1, 2, \ldots\}$ of bounded open sets of $\boldsymbol{R}$ such that

$$N \subseteq \bigcup_{k=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} G(j, k) \quad \text{and} \quad \sum_{k=1}^{\infty} \prod_{j=1}^{n} h\big(d(G(j, k))\big) < \varepsilon.$$

It is easily seen that a countable union of h-null sets is h-null. We also observe that vertical and horizontal lines in $\boldsymbol{R}^2$ are h-null, but that for some h's (eg. $h(t) = \sqrt{t}$) the diagonal in $\boldsymbol{R}^2$ is not h-null. Thus h-null sets are not invariant under congruences. Also, with $h(t) = \sqrt{t}$, vertical and horizontal lines in $\boldsymbol{R}^2$ are not of measure zero under Hausdorff h-measure ([7] p. 79) so h-null sets need not be of measure zero under h-measure.

LEMMA 4.2. *If $\{N(j): j = 1, \ldots, n\}$ is a finite collection of subsets of $\boldsymbol{R}$ such that at least one $N(j)$ is h-null, then $N = \mathop{\boldsymbol{P}}_{j=1}^{n} N(j)$ is h-null in $\boldsymbol{R}^n$.*

Proof. Since h is continuous on the right and positive for $t > 0$ there exists $\delta > 0$ such that $0 < h(x) < 1$ whenever $0 < x < \delta$. We consider $\boldsymbol{R}$ as a countable union of intervals of diameter less than δ. Since a countable union of h-null sets is h-null we may assume without loss of generality that $N \subseteq F_1 \times \ldots \times F_n$ where $F_i \subseteq \boldsymbol{R}$ and $d(F_i) < \delta$. Let s be chosen so that $N(s)$ is h-null. For arbitrary $\varepsilon > 0$ there exist sets $G(k)$ open in $\boldsymbol{R}$ such that

$$N(s) \subseteq \bigcup_{k=1}^{\infty} G(k) \quad \text{and} \quad \sum_{k=1}^{\infty} h\big(d(G(k))\big) < \varepsilon .$$

By putting $G(j, k) = G(k)$ if $j = s$ and $G(j, k) = F_j$ otherwise, we observe that

$$N \subseteq \bigcup_{k=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} G(j, k) \quad \text{and} \quad \sum_{k=1}^{\infty} \prod_{j=1}^{n} h\big(d(G(j, k))\big) \leqslant h\big(d(G(k))\big) < \varepsilon . \quad \text{Q.E.D.}$$

THEOREM 4.3. *Let h be as above and put $h_n(t) = \big(h(t)\big)^n$. If $N \subseteq \boldsymbol{R}^n$ has Hausdorff h_n-measure zero then N is h-null.*

Proof. For $\varepsilon > 0$ it follows by [7], Def. 16, that there exists a collection $\{N(k)\colon k = 1, 2, \ldots\}$ of open sets of $\boldsymbol{R}^n$ such that

$$N \subseteq \bigcup_{k=1}^{\infty} N(k) \quad \text{and} \quad \sum_{k=1}^{\infty} h_n\big(d(N(k))\big) = \sum_{k=1}^{\infty} \big(h(d(N(k)))\big)^n < \varepsilon .$$

We define $S(j, k) = \{x_j\colon (x_1, \ldots, x_j, \ldots, x_n) \in N(k)\}$ and put $a(j, k) = \inf S(j, k)$ and $b(j, k) = \sup S(j, k)$. Let $G(j, k) = \big(a(j, k), b(j, k)\big)$ so $d\big(G(j, k)\big) \leqslant d\big(N(k)\big)$ and $N(k) \subseteq \mathop{\boldsymbol{P}}_{j=1}^{n} G(j, k)$ for all k. The open intervals $G(j, k)$ satisfy Definition 4.1. Q.E.D.

Remark. We recall that Hausdorff h-measure depends only on the germ of h at 0 and that if $h \leqslant h^*$ then Hausdorff h-measure $\leqslant$ Hausdorff h^*-measure ([7], Theorem 40). Thus if $N \subseteq \boldsymbol{R}^n$ has Hausdorff h_m-measure zero for $1 \leqslant m \leqslant n$ it has Hausdorff h_n-measure zero and is h-null.

Let π be Hausdorff h-measure on $\boldsymbol{R}$ and let π^n be the corresponding product measure on $\boldsymbol{R}^n$ (see e.g. [3], p. 90). Recall that $\pi^n(X_1 \times \ldots \times X_n) = \prod_{i=1}^{n} \pi(X_i)$ where multiplication is extended by assuming $0 \cdot \infty = \infty \cdot 0 = 0$.

THEOREM 4.4. *If $\pi^n(N) = 0$ then N is h-null in $\boldsymbol{R}^n$.*

Proof. Let $\varepsilon > 0$ be given and put $\varepsilon^* = \varepsilon/2^n$. Since $\pi^n(N) = 0$ there exists sets $R(j, k) \subseteq \boldsymbol{R}$ such that

$$N \subseteq \bigcup_{k=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} R(j, k) \quad \text{and} \quad \sum_{k=1}^{\infty} \prod_{j=1}^{n} \pi\big(R(j, k)\big) < \varepsilon^* .$$

If $\pi\big(R(j, k)\big) = 0$ for any k then $\mathop{\boldsymbol{P}}_{j=1}^{n} R(j, k)$ is h-null by Lemma 4.2 since h-measure zero and h-null are identical on $\boldsymbol{R}$. Thus without loss of generality we may assume

that $\pi\big(R(j, k)\big) < \infty$ for all j and k. Then by the definition of π there exist open sets $G(j, k, r)$ in E such that

$$R(j, k) \subseteq \bigcup_{r=1}^{\infty} G(j, k, r) \quad \text{and} \quad \sum_{r=1}^{\infty} h\big(d(G(j, k, r))\big) < 2\pi\big(R(j, k)\big) .$$

It then follows that

$$N \subseteq \bigcup_{k=1}^{\infty} \bigcup_{s_1=1}^{\infty} \ldots \bigcup_{s_n=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} G(j, k, s_j)$$

and

$$\sum_{k=1}^{\infty} \sum_{s_1=1}^{\infty} \ldots \sum_{s_n=1}^{\infty} \prod_{j=1}^{n} h\big(d(G(j, k, s_j))\big) = \sum_{k=1}^{\infty} \prod_{j=1}^{n} \sum_{r=1}^{\infty} h\big(d(G(j, k, r))\big)$$

$$< \sum_{k=1}^{\infty} \prod_{j=1}^{n} 2\pi\big(R(j, k)\big) < 2^n \varepsilon^* = \varepsilon . \quad \text{Q.E.D.}$$

5. Independent tree functions. In this section we put $h_0(t) = -1/\log t$ for $0 < t < 1$ and restrict our attention to h_0-null sets in n-cubes $[0, 1]^n$. We then prove Theorem 5.3, the main result of this paper. The choice of the function h_0 is suggested by Lemma 2.2 since we desire that $h_0(1/2^k)$ should be not lesser in magnitude than $P\big(H(0, k)\big)$.

It is not immediately clear that the collection of h_0-null sets is non-trivial. But we can show that in every n-cube there are comeager h_0-null sets. We let $(r_i\colon i = 1, 2, \ldots)$ be any ordering of the points of $[0, 1]^n$ having all components rational in $[0, 1]$. We put $\delta(m) = 1/e^{2^m}$ and let $N(i, m)$ denote an open sphere of diameter $\delta(m)$ containing the point r_i. Then the set $G = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} N(i, i+j)$ is comeager in $[0, 1]^n$ because it is the countable intersection of open dense sets. To show that G is also h_0-null in $[0, 1]^n$ we let $\varepsilon > 0$ be given and choose a positive integer j so that $1/2^j < \varepsilon$. Then

$$G \subseteq \bigcup_{i=1}^{\infty} N(i, i+j) \quad \text{and} \quad \sum_{i=1}^{\infty} h_0\big(d(N(i, i+j))\big) = 1/2^j < \varepsilon$$

so Definition 4.1 is satisfied.

LEMMA 5.1. *If $N \subseteq [0, 1]^n$ is h_0-null, then for every $\varepsilon > 0$ there exists a collection $I(j, k)$ of dyadic intervals of $[0, 1]$ such that $N \subseteq \bigcup_{k=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} I(j, k)$ and $\sum_{k=1}^{\infty} \prod_{j=1}^{n} 1/p(j, k) < \varepsilon$ where $I(j, k) = I(\delta)$ for some $\delta \in \{0, 1\}^{p(j,k)}$.*

Proof. Put $\varepsilon^* = \varepsilon/\big(3^n(\log 2)^n\big)$. By Definition 4.1 there exists a collection $G(j, k)$ of open sets of $[0, 1]$ such that

$$N \subseteq \bigcup_{k=1}^{\infty} \mathop{\boldsymbol{P}}_{j=1}^{n} G(j, k) \quad \text{and} \quad \sum_{k=1}^{\infty} \prod_{j=1}^{n} h_0\big(d(G(j, k))\big) < \varepsilon^* .$$

Clearly each set $G(j,k)$ can be covered by at most three dyadic intervals of $[0,1]$ with diameters less than or equal to $d(G(j,k))$. Thus

$$G(j,k)\subseteq I(j,k,1)\cup I(j,k,2)\cup I(j,k,3)$$

and

$$N\subseteq\bigcup_{k=1}^{\infty}\bigcup_{i_1=1}^{3}\dots\bigcup_{i_n=1}^{3}\mathop{\boldsymbol{P}}_{j=1}^{n} I(j,k,i_t)$$

where $d(I(j,k,i_t))\leqslant d(G(j,k))$ for all i_t. Also each $I(j,k,i)=I(\delta)$ for some $\delta\in\{0,1\}^{p(j,k,i)}$ so

$$h_0\big(d(I(j,k,i))\big)=-1/(\log(1/2^{p(j,k,i)}))=1/(p(j,k,i)\log 2)$$

Thus

$$\sum_{k=1}^{\infty}\sum_{i_1=1}^{3}\dots\sum_{i_n=1}^{3}\prod_{j=1}^{n}1/p(j,k,i_t)=\sum_{k=1}^{\infty}\sum_{i_1=1}^{3}\dots\sum_{i_n=1}^{3}\prod_{j=1}^{n}(\log 2)h_0\big(d(I(j,k,i_t))\big)$$
$$\leqslant(\log 2)^n\sum_{k=1}^{\infty}\sum_{i_1=1}^{3}\dots\sum_{i_n=1}^{3}\prod_{j=1}^{n}h_0\big(d(G(j,k))\big)$$
$$\leqslant 3^n(\log 2)^n\sum_{k=1}^{\infty}\prod_{j=1}^{n}h_0\big(d(G(j,k))\big)<\varepsilon.\quad \text{Q.E.D.}$$

Let $\{t_j\colon j=1,\dots,n\}$ be a collection of distinct elements of $\{0,1\}^m$ and let $\{I_j\colon j=1,\dots,n\}$ be a collection of dyadic intervals of $[0,1]$ each of diameter less than $1/2^m$. Put $H_j=\{\alpha\in 2^T\colon f_\alpha(C(t_j))\cap I_j\neq\emptyset\}$ and $H=\bigcap_{j=1}^{n}H_j$.

LEMMA 5.2. *$P(H)=2^K\prod_{j=1}^{n}P(H_j)$ for some $K<m2^m$.*

Proof. We put $I_j=I(\delta_{j1},\dots,\delta_{jn(j)})$ and observe that $n(j)>m$ for all j. We put $D_j=D_{n(j)-m}(t_j)$ as defined in the proof of Lemma 2.1. Clearly each H_j is identical to some $H(m,n(j))$ as defined in Section 2 so by Lemma 2.1.2 $P(H_j)=1/2^m P(H_j^*)$ where H_j^* is the set of all $\alpha\in 2^T$ for which there exists a branch of D_j whose α-labeling is identical to $(\delta_{j,m+1},\dots,\delta_{jn(j)})$. We let $t_j=(t_{j1},\dots,t_{jm})$ and put

$$X_j=\{\alpha\in 2^T\colon \alpha(t_{j1},\dots,t_{jk})=\delta_{jk}\ \text{for}\ k=1,\dots,m\}.$$

Each X_j is determined by fixing the image of m elements of T and thus $\bigcap_{j=1}^{n}X_j$ is determined by fixing at most mn elements of T. It follows that $P(\bigcap_{j=1}^{n}X_j)=1/2^{mn-K}$ where K is the number of redundant labelings. Clearly $H_j=X_j\cap H_j^*$ so

$$H=\bigcap_{j=1}^{n}X_j\bigcap_{j=1}^{n}H_j^*$$

and

$$P(H)=P(\bigcap_{j=1}^{n}X_j)P(\bigcap_{j=1}^{n}H_j^*)=(1/2^{mn-K})\prod_{j=1}^{n}P(H_j^*)=2^K\prod_{j=1}^{n}P(H_j).$$

It remains only to determine K. Let $i\in\{1,\dots,m\}$ be given and for $\delta\in\{0,1\}^i$ put $B_i(\delta)=\{t_j\colon (t_{j1},\dots,t_{ji})=\delta\}$. Define $\mathrm{REDUN}(B_i(\delta))=\max\{0,\mathrm{Card}(B_i(\delta))-1\}$. It follows that

$$K=\sum_{i=1}^{m}\sum_{\delta\in\{0,1\}^i}\mathrm{REDUN}(B_i(\delta)).$$

The t_j's are all distinct so $\mathrm{Card}(B_i(\delta))\leqslant 2^{m-i}$ regardless of the value of n. Thus

$$K\leqslant\sum_{i=1}^{m}\sum_{\delta\in\{0,1\}^i}(2^{m-i}-1)=\sum_{i=1}^{m}(2^m-2^i)<m2^m.\quad \text{Q.E.D.}$$

THEOREM 5.3. *If R is h_0-null in $[0,1]^n$, then almost every tree function is independent over R.*

Proof. Let m be a positive integer such that $2^m\geqslant n$. Consider a sequence $L=(t_1,\dots,t_n)$ of distinct elements of $\{0,1\}^m$ and put

$$W(R,m,L)=\{\alpha\in 2^T\colon \mathop{\boldsymbol{P}}_{j=1}^{n}f_\alpha(C(t_j))\cap R\neq\emptyset\}.$$

To show that $P(W(R,m,L))=0$ we let $\varepsilon>0$ be given and put $M=m2^m$ and $\varepsilon^*=\varepsilon/(4^n2^M)$. Let $\{I(j,k)\colon j=1,\dots,n;\ k=1,2,\dots\}$ be a collection of dyadic intervals of $[0,1]$ covering R as in Lemma 5.1 with $\varepsilon=\varepsilon^*$. Without loss of generality we choose all $I(j,k)$ so that $p(j,k)>m$. Putting

$$B(j,k)=\{\alpha\in Z\colon f_\alpha(C(t_j))\cap I(j,k)\neq\emptyset\}$$

we see that $W(R,m,L)\subset\bigcup_{k=1}^{\infty}\bigcap_{j=1}^{n}B(j,k)$ and that

$$P(W(R,m,L))\leqslant\sum_{k=1}^{\infty}P(\bigcap_{j=1}^{n}B(j,k))$$
$$\leqslant 2^M\sum_{k=1}^{\infty}\prod_{j=1}^{n}P(B(j,k))\quad\text{by Lemma 5.2}$$
$$\leqslant 2^M\sum_{k=1}^{\infty}\prod_{j=1}^{n}1/2^m\big(4/(4+p(j,k)-m)\big)\quad\text{by Lemma 2.2}$$
$$<2^M4^n\sum_{k=1}^{\infty}\prod_{j=1}^{n}1/\big(2^m(p(j,k)-m)\big)$$
$$<2^M4^n\sum_{k=1}^{\infty}\prod_{j=1}^{n}1/p(j,k)<\varepsilon.$$

Since $\varepsilon>0$ was arbitrary it follows that $P(W(R,m,L))=0$. We then put $W=\bigcup_{m=n}^{\infty}\bigcup_{\{L\}}W(R,m,L)$ and observe that W has P-measure zero. We claim that every f_α for which $\alpha\notin W$ is independent. This follows because for any set $\{x_1,\dots,x_n\}$ of distinct elements of C with $(f_\alpha(x_1),\dots,f_\alpha(x_n))\in R$ there exists an $m\geqslant n$ and an L such that $\alpha\in W(R,m,L)$. Q.E.D.

Remarks. 1) To get a theorem similar to Mycielski's (see introduction) we can extend Theorem 5.3 to the case where the number of relations is countable. This follows by the countable additivity of the measure P in 2^T.

2) Combining Theorems 3.3 and 5.3 it follows that almost every tree function is independent with perfect range in $[0, 1]$.

3) Theorem 4.3 ensures that Theorem 5.3 remains valid if the Hausdorff $(-1/\log t)^n$-measure of R is zero.

4) Theorem 4.4 ensures that Theorem 5.3 remains valid if R is of measure zero with respect to the product measure (h_0-measure)^n.

5) The mapping $\alpha \to f_\alpha$ from 2^T to the space $[0, 1]^C$ is continuous. For any Borel set $B \subseteq [0, 1]^C$ we define $\mu(B) = P(\{\alpha \in 2^T: f_\alpha \in B\})$. Under this Borel measure, and for any h-null $R \subseteq [0, 1]^n$, almost every $f \in [0, 1]^C$ is independent over R.

References

[1] L. J. Gallagher, *Independence in topological and measure structures*, Notices Amer. Math. Soc. 19 (1972), A-757.
[2] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6 (1958), pp. 731–736.
[3] M. E. Munroe, *Introduction to Measure and Integration*, Addison–Wesley 1953.
[4] J. Mycielski, *Independent sets in topological algebras*, Fund. Math. 55 (1964), pp. 139–147.
[5] — *Algebraic independence and measure*, Fund. Math. 61 (1967), pp. 165–169.
[6] — *Almost every function is independent*, Fund. Math. 81 (1973) pp. 43–48.
[7] C. A. Rogers, *Hausdorff Measures*, Cambridge 1970.

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Sequence of iterates of generalized contractions

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Abstract. The main purpose of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if T is a generalized contraction mapping of closed, bounded and convex subset of a uniformly convex Banach space into itself with nonempty fixed points set, then the mapping T_λ defined by $T_\lambda = \lambda I + (1-\lambda)T$, for any λ such that $0<\lambda<1$ is asymptotically regular. As a corollary of this we get the result of Schaeffer (Jbr. Deutch. Math. Verein. (1957), pp. 131–140). In Section 2, we prove for Hilbert spaces the mapping T_λ as defined above is a reasonable wanderer. As a corollary of this we obtain the result of Browder and Petryshyn (J. Math. Anal. and Appl. 20 (1967), pp. 197–228). Finally in Sections 3 and 4, we have obtained some results for the weak and strong convergence of sequence of iterates for mappings of this type.

Introduction. The main aim of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if T is a generalized contraction mapping of a closed, bounded and convex subset of a uniformly convex Banach space into itself with non-empty fixed point set, then the mapping T_λ defined by $T_\lambda = \lambda I + (1-\lambda)T$, for any λ such that $0<\lambda<1$ is asymptotically regular. In section, it is shown that if T is a generalized contraction self mapping of a closed, convex subset of Hilbert space with non-empty fixed point set, then the mapping T_λ defined as above is a reasonable wanderer with the same fixed point as T. Finally in Sections 3 and 4 we have obtained some results for the weak and strong convergence of sequence of iterates of such kind of mappings.

Definition 1.1. Let C be a closed, bounded and convex subset of a Banach space X. A mapping T: $C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leqslant \|x-y\| \quad \text{for all } x, y \text{ in } C.$$

Definition 1.2. A mapping T: $C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - Ty\| \leqslant a\|x-y\| + b\|x-Tx\| + c\|y-Ty\|$$

for all x, y in C, $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a+b+c \leqslant 1$.

The following example shows that there are quasi-nonexpansive mappings which are not nonexpansive.