Normality and paracompactness in finite and countable Cartesian products

by

Teodor C. Przymusiński (Pittsburgh * and Warszawa)

Abstract. We prove the following two theorems.

THEOREM 1. *For every k and m such that $1 \leqslant k \leqslant m \leqslant \omega$ there exists a separable and first countable space $X = X(k, m)$ such that*

(a) *X^n is paracompact (Lindelöf, subparacompact) if and only if $n < k$,*

(b) *X^n is normal (collectionwise normal) if and only if $n < m$.*

THEOREM 2. *There exists a separable metric space M and a separable and first countable Lindelöf space Y such that $M \times Y$ is not subparacompact.*

§ 1. Introduction. Throughout this paper k, m and n denote natural numbers $0, 1, 2, \ldots$ or the first infinite ordinal (cardinal) ω. Unless otherwise stated, all spaces are completely regular. For undefined notions the reader is referred to [5]. The following results are proved.

THEOREM 1.1. *For every k and m such that $1 \leqslant k \leqslant m \leqslant \omega$ there exists a separable and first countable space $X = X(k, m)$ such that*

(a) *X^n is paracompact (Lindelöf, subparacompact) if and only if $n < k$;*

(b) *X^n is normal (collectionwise normal) if and only if $n < m$.*

COROLLARY 1.2. *For natural numbers k and m satisfying $k \leqslant m$ there exists a first countable and separable space $X = X(k, m)$ such that*

(a) *X^n is paracompact (Lindelöf, subparacompact) if and only if $n \leqslant k$;*

(b) *X^n is normal (collectionwise normal) if and only if $n \leqslant m$.* ■

COROLLARY 1.3. *There exists a first countable and separable Lindelöf space X such that X^2 is (collectionwise) normal but not paracompact.* ■

COROLLARY 1.4. *There exists a first countable separable space X such that X^n is Lindelöf for all $n < \omega$ but X^ω is not normal.* ■

THEOREM 1.5. *There exists a separable metric space M and a separable and first countable Lindelöf space Y such that $M \times Y$ is not subparacompact.*

COROLLARY 1.6. *There exists a first countable and separable Lindelöf space X such that X^2 is not subparacompact.*

* This paper was completed while the author was visiting the University of Pittsburgh as a Mellon Postdoctoral Fellow (1975–76) and a Visiting Assistant Professor (1976–77).

Remark 1.7. a) In the above theorems *subparacompactness* can be everywhere replaced by *θ-refinability* (see Section 3 and Remark 4.7).

In particular, Theorem 1.5 provides an example of a separable metric space M and a Lindelöf space Y whose product $M\times Y$ is not θ-refinable.

b) The space $X = X(k, m)$ from Theorem 1.1 also has the property that X^n is *countably paracompact* (resp. *ω_1-compact*) if and only if $n<m$ (see Section 4). In particular, it follows from Theorem 1.1 that for every $m\leqslant\omega$ there exists a first countable separable space $X = X(m, m)$ such that X^n is Lindelöf for every $n<m$, but X^m is neither normal, nor countably paracompact, nor ω_1-compact, nor subparacompact.

c) Since countable products of paracompact Čech-complete spaces are paracompact, the space $X = X(k, m)$ cannot be Čech-complete if $k>1$. However, $X(k, m)$ is locally compact (and locally countable) if $k = 1$ (see Remark 4.8). Therefore, for every $m\leqslant\omega$ the space $X(1, m)$ may serve as an example of a first countable, separable, locally compact and locally countable space X such that X^n is collectionwise normal, countably paracompact and ω_1-compact for every $n<m$, but X^m is neither normal, nor countably paracompact nor ω_1-compact. ■

The results obtained in this paper and also its title were inspired by a beautiful paper entitled "Paracompactness and the Lindelöf property in finite and countable cartesian products" written in 1971 by E. Michael [12]. Theorem 1.1 answers some of the questions raised in Michael's paper (see Section 2 for details).

Our paper originated after the author was informed about a recent result (Theorem 2.13) obtained by K. Alster and P. Zenor under the assumption of the Continuum Hypothesis. Methods and ideas from their paper [3] are frequently exploited in the proof of Theorem 1.1.

Throughout this paper we make an extensive use of the techniques used by E. van Douwen in his paper [4], where a collectionwise normal non-paracompact topology, stronger than the usual one, is constructed on the real line. We also exploit ideas from K. Kunen's paper [9] (where, under the assumption of the Continuum Hypothesis, a technique for constructing spaces Z such that Z^n is an S-space for every $n<\omega$ is described).

An important role in the proof of Theorem 1.1 plays the notion of *n-cardinality*, which is introduced and investigated by the author in [18].

The author is deeply grateful to K. Alster and E. van Douwen for many helpful discussions.

Our paper is organized as follows. In Section 2 we review some of the known results involving covering (and separation) properties of product spaces, pointing out problems solved by our theorems.

Section 3 is devoted to the proof of Theorem 1.5 and in Section 4 Theorem 1.1 is proved.

Section 5 is devoted to final remarks and a discussion of problems that remain open.

§ 2. **Review of known results.** In this section we discuss some of the known results involving covering (and separation) properties of finite and countable cartesian products. The following diagram describes the relationship between different properties considered in this section.

Lindelöf
↓
paracompact
↓ ↘
collectionwise normal   subparacompact
↙ ↓
collectionwise Hausdorff   normal

Let us recall that a regular space X is *subparacompact* if every open covering of X admits a σ-discrete closed refinement. A T_1-space X is *collectionwise Hausdorff* if every discrete collection of points of X can be separated by disjoint open sets. It is well-known that a space is paracompact if and only if it is collectionwise normal and subparacompact. It is also easy to check that a separable paracompact space is Lindelöf.

Most of the results reviewed below were obtained under additional set-theoretic assumptions such as CH (Continuum Hypothesis), MA (Martin's Axiom), $E(\omega_2)$ or a conjunction of some of them or their negations, e.g. MA+$\neg$CH (Martin's Axiom plus the negation of the Continuum Hypothesis). The reader should consult corresponding references for definitions of these statements.

A. The following question, attributed by E. Michael to M. Maurice, was often asked (cf. Michael [12], Tamano [23], Rudin [19]).

Question 2.1. Assume that X and Y are paracompact spaces. Can $X\times Y$ be normal without being paracompact? What if Y is metrizable?

The second part of Question 2.1 was answered negatively by the following result.

Theorem 2.2 (Tamano (1963) [22], Morita (1963) [13], Rudin and Starbird (1975) [20]). *Let X be a paracompact and M a metric space. The product space $X\times M$ is paracompact if and only if it is normal* (¹).

Corollary 2.3 (cf. [2]). *Let X be a paracompact space and Y a paracompact p-space* (²). *The product space $X\times Y$ is paracompact if and only if it is normal.*

Our Corollary 1.3 gives a positive answer to the first part of Question 2.1. The following results — depending upon additional set-theoretic assumptions — were known before.

(¹) An analogous theorem is valid for open subsets of a product $X\times M$ of a hereditarily paracompact space X and a metric space M (see [17]).

(²) A space Y is a *paracompact p-space* if there exists a perfect mapping of Y onto a metric space. Every metric space and every paracompact Čech-complete space is a paracompact p-space.

THEOREM 2.4 (MA+¬CH) (Przymusiński (1973) [15]). *There exists a first countable separable paracompact space X such that X^2 is normal but not collectionwise Hausdorff (consequently X^2 is not paracompact).*

THEOREM 2.5 ($E(\omega_2)$+MA+¬CH) (Przymusiński (1976) [16]). *There exists a first countable paracompact space X such that X^2 is normal and collectionwise Hausdorff but X^2 is not collectionwise normal.*

THEOREM 2.6 (CH) (Alster and Zenor (1977) [3]). *There exists a first countable separable paracompact space X such that X^2 is collectionwise normal but not paracompact.*

THEOREM 2.7 ($E(\omega_2)$+CH) (Przymusiński (1976) [16]). *There exists a first countable paracompact space X such that X^2 is collectionwise Hausdorff but not normal.*

THEOREM 2.8 (MA+¬CH) (Alster and Przymusiński (1976) [2]). *There exists a first countable separable paracompact space X such that X^ω is normal but X^2 is not paracompact.*

B. Theorems presented below were proved by Michael [12] in 1971 mostly under the assumption of the Continuum Hypothesis. Michael also raised the question whether this assumption is necessary.

THEOREM 2.9 (Michael [12]). *There exists a first countable space X such that X^n is paracompact for all $n<\omega$ but X^ω is not normal.*

THEOREM 2.10 (CH) (Michael [12]). *There exists a first countable space X such that X^n is Lindelöf for all $n<\omega$ but X^ω is not normal.*

THEOREM 2.11 (CH) (Michael [12]). *For every $k<\omega$ there exists a first countable space $X = X(k)$ such that X^k is Lindelöf but X^{k+1} is not normal.*

THEOREM 2.12 (CH) (Michael [12]). *For every $k<\omega$ there exists a first countable space $X = X(k)$ such that X^k is Lindelöf and X^{k+1} is paracompact but X^{k+1} is not Lindelöf.*

Recently, Alster and Zenor obtained

THEOREM 2.13 (CH) (Alster and Zenor (1977) [3]). *For every $k<\omega$ there exists a first countable separable space $X = X(k)$ such that X^k is Lindelöf and X^{k+1} is (collectionwise) normal, but X^{k+1} is not paracompact (in fact also X^ω is normal).*

Corollaries 1.2 and 1.4 show that the assumption of the Continuum Hypothesis in Theorems 2.10, 2.11 and 2.13 can be omitted ([3]), thus partially answering Michael's question.

C. Alster and Engelking [1] showed in 1972 that there exists a subparacompact space X such that X^2 is not subparacompact. Their space X is neither Lindelöf, nor first countable.

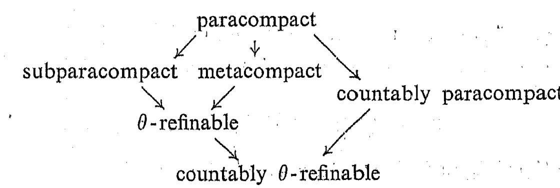
([3]) Spaces X (and also X^ω) appearing in Theorems 2.4, 2.5, 2.6, 2.7, 2.8, 2.11 and 2.13 are additionally *perfect*, i.e. all open subsets are F_σ sets. The space X from our Theorem 1.1 is not perfect.

THEOREM 2.14 (Alster and Engelking [1]). *There exists a paracompact space X such that X^2 is not subparacompact.*

F. Tall [21] raised the question whether there exists such a Lindelöf space. It was also unknown whether there exists a metric space M and a subparacompact space Y such that $M\times Y$ is not subparacompact (this question was communicated to the author by K. Alster) and whether there exists a Lindelöf space X such that X^2 is not θ-refinable. Theorem 1.5 and Corollary 1.6 give positive answers to all of these questions (cf. Remark 1.7).

§ 3. Proof of Theorem 1.5. Let us recall that a Hausdorff space X is (*countably*) *θ-refinable* if for every (countable) open covering $\mathcal{U}$ of X there exists a sequence $\{\mathcal{U}_n\}_{n=1}^\infty$ of open coverings $\mathcal{U}_n$ refining $\mathcal{U}$ and such that for every point $x\in X$ there exists an n so that only finitely many members of $\mathcal{U}_n$ contain x. Countable θ-refinability coincides with countable metacompactness.

It is known [24] that a space is paracompact if and only if it is collectionwise normal and θ-refinable. The following diagram describes the relationship between (countable) θ-refinability and other covering properties (cf. [24]).



The proof of the lemma below is a slight modification of the construction due to van Douwen [4].

LEMMA 3.1. *Let A be such a subset of the real line R that $|F\cap A| = c$ for every closed uncountable subset of R.*

There exists a first countable, separable and locally compact topology $\mathcal{T}$ on A which is stronger than the usual topology on A and is neither countably θ-refinable nor normal.

Proof. Let A_0 be a countable dense subset of A. For every $x\in A$ we choose a sequence $\{a_k(x)\}_{k=1}^\infty$ of points from A_0 such that $|x-a_k(x)|<1/k$. Let us enumerate by $\{P_\alpha\}_{\alpha<c}$ all countable subsets of A such that the closure $\bar{P}_\alpha$ of P_α in R is uncountable. It follows from our assumption that $|\bar{P}_\alpha\cap A| = c$, for every $\alpha<c$, therefore by transfinite induction we can find for every $\alpha<c$ and $m = 0, 1, 2, \ldots$ a point

$$x_{\alpha m}\in \bar{P}_\alpha\cap A\setminus(A_0\cup\{x_{\beta i}\colon \beta<\alpha \text{ or } \beta=\alpha \text{ and } i<m\}).$$

We put

$$S_0 = \{x_{\alpha 0}\colon \alpha<c\},$$

$$S_m = \{x_{\alpha m}\colon \alpha<c \text{ and } P_\alpha\subseteq S_0\},\ \text{for } m\geqslant 1,$$

$$S = A\setminus\bigcup_{m=1}^{\infty} S_m.$$

Clearly $A_0 \cup S_0 \subset S$ and $S_m \cap S_{m'} = \emptyset$ if $m \neq m'$.

For every point $x = x_{\alpha m} \in A \setminus S$ we choose a sequence $\{p_s(x)\}_{s=1}^{\infty}$ of points from P_α such that $|p_s(x) - x| < 1/s$. Let us note that points $p_s(x)$ belong to $S_0 \subset S$.

Topology $\mathcal{T}$ on A is generated by the system $\{\mathcal{B}(x)\}_{x \in A}$ of bases $\mathcal{B}(x)$ of neighbourhoods of points $x \in A$, where

$$\mathcal{B}(x) = \{B_l(x)\}_{l=1}^{\infty},$$

and

$$(1) \qquad B_l(x) = \begin{cases} \{x\}, & \text{if } x \in A_0, \\ \{x\} \cup \{a_k(x)\}_{k=l}^{\infty}, & \text{if } x \in S \setminus A_0, \\ \{x\} \cup \{p_s(x)\}_{s=l}^{\infty} \cup \{a_k(p_s(x))\}_{s=l,k=s}^{\infty\ \ \infty}, & \text{if } x \in A \setminus S. \end{cases}$$

It is easy to check that the topology $\mathcal{T}$ is well-defined and stronger than the usual topology on A. Sets $B_l(x)$ are compact subsets of $(A, \mathcal{T})$ and therefore $\mathcal{T}$ is locally compact. Clearly $\mathcal{T}$ is first countable, locally countable and separable. Since the sets S_m, $m = 1, 2, \ldots$ are closed, discrete and of cardinality continuum, the space $(A, \mathcal{T})$ is not normal by Jones' Lemma [7]. It remains to show that $\mathcal{T}$ is not countably θ-refinable.

Let $\mathcal{U} = \{U_m\}_{m=1}^{\infty}$ be a countable open covering of $(A, \mathcal{T})$, where $U_m = S \cup S_m$. Let $\{\mathcal{U}_n\}_{n=1}^{\infty}$ be a sequence of open covering refining $\mathcal{U}$. For every $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ let

$$(2) \qquad W_{nm} = \bigcup \{U \in \mathcal{U}_n:\ U \cap S_m \neq \emptyset\}.$$

We have $S_m \subset W_{nm}$ and $S_{m'} \cap W_{nm} = \emptyset$, if $m' \neq m$. We will show first that for arbitrary n and m

$$|S_0 \setminus W_{nm}| \leqslant \omega.$$

Indeed, otherwise there would exist a countable dense subset $P \subset S_0 \setminus W_{nm}$ and an $\alpha < c$ such that $P = P_\alpha \subset S_0$. Therefore we would have $x_{\alpha m} \in S_m \cap W_{nm}$ and $P_\alpha \cap W_{nm} = \emptyset$, which by (1) contradicts the openess of W_{nm}.

Let

$$K = S_0 \cap \bigcap_{\substack{n=1\\m=1}}^{\infty} W_{nm} = S_0 \setminus \bigcup_{\substack{n=1\\m=1}}^{\infty} (S_0 \setminus W_{nm}).$$

Since $|S_0| = c$ it follows that $K \neq \emptyset$. Let $x_0 \in K$ and $n = 1, 2, \ldots$ Since $x_0 \in \bigcap_{m=1}^{\infty} W_{nm}$, by (2) there exist sets $U_m \in \mathcal{U}_n$, $m = 1, 2, \ldots$, such that $x_0 \in U_m$, $U_m \cap S_m \neq \emptyset$ and $U_m \cap S_{m'} = \emptyset$, if $m \neq m'$. Therefore, for every $n = 1, 2, \ldots$ there exist infinitely many different members of $\mathcal{U}_n$ containing x_0, which shows that $\mathcal{T}$ is not countably θ-refinable. ■

We proceed to the proof of Theorem 1.5. Let A be a subset of the real line such that $|A \cap F| = c = |F \setminus A|$, for every closed uncountable subset F of R (see [10], Ch. III, § 40, I, Theorem 1).

By Lemma 3.1 there exists a first countable, locally compact, separable topology $\mathcal{T}$ on A which is stronger than the usual topology on A and is not countably θ-refinable. Denote by $\mathcal{B}$ the family of all euclidean-open subsets of R and let $\mathcal{D}$ be a new topology on R generated by the family $\mathcal{T} \cup \mathcal{B}$. One easily checks that the space $Y = (R, \mathcal{D})$ is regular first countable and separable. We shall show that Y is Lindelöf.

Let $\mathcal{U}$ be an open covering of Y. There exists a countable refinement $\mathcal{V}$ of $\mathcal{U}$ covering $R \setminus A$ and consisting of euclidean-open sets. The set $F = R \setminus \bigcup \mathcal{V}$ is a euclidean-closed subset of R contained in A. Therefore F is countable and can be covered by countably many elements of $\mathcal{U}$.

Let M be the set A with the usual topology. The closed subspace

$$\Delta = \{(x, x) \in M \times Y:\ x \in A\}$$

of $M \times Y$ is homeomorphic to the space $(A, \mathcal{T})$, what implies that the space $M \times Y$ is not countably θ-refinable and, hence, $M \times Y$ is not subparacompact. The proof of Theorem 1.5 is completed. ■

Corollary 1.6 is an easy consequence of Theorem 1.5. Let $X = M \oplus Y$ be the topological sum of spaces M and Y. Clearly X^2 is not countably θ-refinable.

Corollary 1.6 follows also from Corollary 1.3 but the latter is more difficult to prove than Theorem 1.5.

§ 4. Proof of Theorem 1.1. In [18] the author introduces and investigates the notion of n-cardinality, which turned out to be very useful in constructions involving product spaces. We will use this notion in the proof of Theorem 1.1.

DEFINITION 4.1 [18]. Let A be a subset of X^n, where X is an arbitrary set and $n < \omega$. The *n-cardinality* $|A|_n$ of A (with respect to X^n) is defined by

$$|A|_n = \max\{|B|:\ B \subset A \text{ and } p_i \neq q_i, \text{ for every } i = 1, \ldots, n \text{ and any two distinct points } p = (p_1, \ldots, p_n) \text{ and } q = (q_1, \ldots, q_n) \text{ from } B\}.$$

It is shown in [18] that $|A|_n$ is well-defined, i.e. that the maximum always exists. We say that a subset A of X_n is *n-countable* (*n-uncountable*) if $|A|_n \leqslant \omega$ ($|A|_n > \omega$).

For a point $p = (p_1, \ldots, p_n) \in X^n$ by $\hat{p}$ we will denote the set $\{p_1, \ldots, p_n\}$ of coordinates of p. The following lemma is proved in [18].

LEMMA 4.2. *For a subset A of X^n we have:*

$$|A|_n = \max\{|B|:\ B \subset A \text{ and } \hat{p} \cap \hat{q} = \emptyset, \text{ for every two distinct points } p \text{ and } q \text{ from } B\}$$
$$= \min\{|Y|:\ Y \subset X \text{ and } A \subset \bigcup_{i=1}^{n} (X^{i-1} \times Y \times X^{n-i})\},$$

provided that $|A|_n$ is infinite. ■

The following theorem has been (implicitly) proved by van Douwen [4] (see [18]; Theorem 1).

THEOREM 4.3. *A closed subset F of the n-dimensional euclidean space R^n is either n-countable or has n-cardinality continuum.* ■

We now proceed to the proof of Theorem 1.1.

Theorem 1.1 becomes trivial if $m = k = 1$, therefore we will assume that $m \geq 2$. By R we denote the real line and by Q the set of rationals. Let $\prec$ be an arbitrary well-ordering of R of type c such that for each $q \in Q$ and $p \in R \setminus Q$ we have $q \prec p$. For $x \in R$ by $R(x)$ we denote the set $\{y \in R: y \prec x\}$.

For $n < \omega$ let $\mathscr{B}_n$ be the collection of all pairs (A, B) of *countable* subsets of R^n such that the subset $F = \bar{A} \cap \bar{B}$ (4) of R^n is n-uncountable (cf. [4]). Let $\{(A_\alpha, B_\alpha)\}_{\alpha<c}$ be such an enumeration of all pairs belonging to $\mathscr{B} = \bigcup_{n<\omega} \mathscr{B}_n$ that each pair from $\mathscr{B}$ is listed continuum many times. For each $\alpha < c$ there exists an $n = n(\alpha)$ such that the pair (A_α, B_α) belongs to $\mathscr{B}_n$ and since A_α and B_α are countable there exists an *irrational* number $x_\alpha \in R$ such that

(1) $$A_\alpha \cup B_\alpha \subset R(x_\alpha)^n .$$

By transfinite induction, for every $\alpha < c$ and $i = 1, 2, \ldots, k+m$ (5) we will choose a point $p(\alpha, i) \in F_\alpha = \bar{A}_\alpha \cap \bar{B}_\alpha$ so that the following conditions are satisfied

(2) $$\hat{p}(\alpha, i) \cap \hat{p}(\alpha', i') = \emptyset \quad \text{if} \quad (\alpha, i) \neq (\alpha', i') ;$$

(3) $$\hat{p}(\alpha, i) \cap R(x_\alpha) = \emptyset .$$

Assume that $\alpha < c$ and that for all $\beta < \alpha$ and $i = 1, 2, \ldots, k+m$ we have already chosen points $p(\beta, i)$. The set

$$Y = R(x_\alpha) \cup \bigcup \{\hat{p}(\beta, i): \beta < \alpha,\ i = 1, 2, \ldots, k+m\}$$

has cardinality less than c and therefore by Theorem 4.3 the set

$$F_\alpha^* = F_\alpha \setminus \bigcup_{j=1}^{n} (R^{j-1} \times Y + R^{n-j}), \quad \text{where} \quad n = n(\alpha) ,$$

has n-cardinality continuum. By Lemma 4.2 we can find points $\hat{p}(\alpha, i)$, $i = 1, 2, \ldots, k+m$ from F_α^* such that $\hat{p}(\alpha, i) \cap \hat{p}(\alpha, i') = \emptyset$, provided $i \neq i'$, which completes the inductive construction.

One easily verifies that the collection $\{p(\alpha, i)\}_{\alpha<c', i=1,2,\ldots,k+m}$ of points satisfies conditions (2) and (3).

The sets $D_i = \bigcup \{\hat{p}(\alpha, i): \alpha < c\}$, where $i = 1, 2, \ldots, k+m$, have the following properties

(4) $$D_i \cap D_j = \emptyset, \quad \text{if} \quad i \neq j ,$$

(5) for every $i = 1, 2, \ldots, k+m$, $n < \omega$ and any pair (A, B) of countable subsets of R^n such that the set $\bar{A} \cap \bar{B}$ is n-uncountable there exists a subset L of c of cardinality continuum such that $A = A_\alpha$, $B = B_\alpha$, $p(\alpha, i) \in \bar{A} \cap \bar{B}$ and $\hat{p}(\alpha, i) \subset D_i$, for every $\alpha \in L$.

(4) In the sequel $\bar{A}$ always denotes the *euclidean closure* of A.

(5) The symbol $k+m$ denotes the *ordinal number* being the sum of two ordinal numbers k and m, e.g. $l+\omega \neq \omega+l$.

The property (4) is obvious and (5) follows from the definition of D_i and our assumption that every pair (A, B) appears continuum many times in the transfinite sequence $\{(A_\alpha, B_\alpha)\}_{\alpha<c}$. Clearly we can additionally assume that

(6) $$\bigcup_{i=1}^{k+m} D_i = R \setminus Q ,$$

and (4) and (5) remain valid.

Let us fix $\alpha < c$, $n = n(\alpha)$ and $i = 1, 2, \ldots, k+m$. Since $p(\alpha, i) \in F_\alpha = \bar{A}_\alpha \cap \bar{B}_\alpha \subset R^n$, there exist two sequences $\{v^s(\alpha, i)\}_{s=1}^{\infty}$ and $\{w^s(\alpha, i)\}_{s=1}^{\infty}$ of points from A_α and B_α, respectively, converging to $p(\alpha, i)$; i.e.

(7) $$v^s(\alpha, i) \in A_\alpha, \quad w^s(\alpha, i) \in B_\alpha , \qquad \lim_{s\to\infty} v^s(\alpha, i) = p(\alpha, i) = \lim_{s\to\infty} w^s(\alpha, i) .$$

Let us put $p(\alpha, i) = (p_1(\alpha, i), \ldots, p_n(\alpha, i))$, $v^s(\alpha, i) = (v_1^s(\alpha, i), \ldots, v_n^s(\alpha, i))$ and $w^s(\alpha, i) = (w_1^s(\alpha, i), \ldots, w_n^s(\alpha, i))$. We have

(8) $$\lim_{s\to\infty} v_j^s(\alpha, i) = p_j(\alpha, i) = \lim_{s\to\infty} w_j^s(\alpha, i) \quad \text{for all } j \leq n ,$$

and by (1) and (3)

(9) $$v_j^s(\alpha, i) \prec p_j(\alpha, i) \quad \text{and} \quad w_j^s(\alpha, i) \prec p_j(\alpha, i) \quad \text{for} \quad s = 1, 2, \ldots$$

For each $x \in R \setminus Q$ let $\{q^s(x)\}_{s=1}^{\infty}$ be a sequence of rationals converging to x. Our next goal is to define for every $x \in R \setminus Q$ a certain sequence $T(x) = \{t^s(x)\}_{s=1}^{\infty}$ of real numbers converging to x satisfying

(10) $$t^s(x) \prec x \quad \text{for every } s = 1, 2, \ldots$$

Let $x \in R \setminus Q$. If there is no $\alpha < c$ and $i = 1, 2, \ldots, k+m$ such that $x \in \hat{p}(\alpha, i)$, then we put $t^s(x) = q^s(x)$. Clearly (10) is satisfied. Otherwise, by (2), there is exactly one $\alpha < c$ and $i = 1, 2, \ldots, k+m$ so that $x \in \hat{p}(\alpha, i)$. In that case we put $n = n(\alpha)$, $J = \{j = 1, \ldots, n: x = p_j(\alpha, i)\}$ and the sequence $T(x)$ is the "union" of sequences $\{q^s(x)\}_{s=1}^{\infty}$, $\{v_j^s(\alpha, i)\}_{s=1}^{\infty}$ and $\{w_j^s(\alpha, i)\}_{s=1}^{\infty}$, for $j \in J$. More precisely, we put $r = |J|$, $J = \{j(1), \ldots, j(r)\}$ and we define

(11) $$t^s(x) = \begin{cases} q^l(x) , & \text{if } s = l(2r+1) , \\ v_{j(u)}^l(\alpha, i) , & \text{if } s = l(2r+1)+u , \\ w_{j(u)}^l(\alpha, i) , & \text{if } s = l(2r+1)+2u , \end{cases}$$

where $u = 1, 2, \ldots, r$.

It follows from (8) and (9) that the sequence $T(x) = \{t^s(x)\}_{s=1}^{\infty}$ converges to x and that (10) is satisfied.

LEMMA 4.4. *Let us fix an $i = 1, 2, \ldots, k+m$ and let $\{\mathscr{T}_j\}_{j=1}^{n}$ be a family of topologies on R such that for every $j = 1, 2, \ldots, n$ and $x \in D_i$ every neighbourhood U of x in $(R, \mathscr{T}_j)$ contains a "tail" of the sequence $T(x)$ (i.e. there exists an s_0 such that $t^s(x) \in U$, for $s \geq s_0$).*

For any two subsets A and B of the space $Y = \prod_{j=1}^{n} (R, \mathcal{T}_j)$ with $\bar{A} \cap \bar{B}$ (see footnote (4)) n-uncountable, the set $\mathrm{Cl}_Y A \cap \mathrm{Cl}_Y B$ is also n-uncountable. In particular, if A is a subset of Y such that $\bar{A}$ is n-uncountable, then $\mathrm{Cl}_Y A$ is also n-uncountable.

Proof. Let us find countable sets A^* and B^* such that $A^* \subset A \subset \bar{A}^*$ and $B^* \subset B \subset \bar{B}^*$. Since the set $F = \bar{A}^* \cap \bar{B}^* = \bar{A} \cap \bar{B}$ is n-uncountable there exists by (5) an uncountable subset L of c such that $A^* = A_\alpha$, $B^* = B_\alpha$ and $\hat{p}(\alpha, i) \subset D_i$ for every $\alpha \in L$. It follows easily from (11) and our assumptions that for each $\alpha \in L$ the point $p(\alpha, i) \in \mathrm{Cl}_Y A^* \cap \mathrm{Cl}_Y B^* \subset \mathrm{Cl}_Y A \cap \mathrm{Cl}_Y B$. Since by (2) $\hat{p}(\alpha, i) \cap \hat{p}(\beta, i) = \emptyset$, if $\alpha \neq \beta$, it follows from Lemma 4.2 that the set $\mathrm{Cl}_Y A \cap \mathrm{Cl}_Y B$ is n-uncountable. ∎

LEMMA 4.5. *Let us fix an $i = 1, 2, \ldots, k+m$ and let $\{\mathcal{T}_j\}_{j=1}^{n}$ be a family of regular and first countable topologies on R which are stronger than the usual topology on R and such that for every $j = 1, 2, \ldots, n$ and $x \in D_i$ every neighbourhood U of x in $(R, \mathcal{T}_j)$ contains a "tail" of the sequence $T(x)$ (i.e. there exists an s_0 such that $t^s(x) \in U$, for $s \geq s_0$).*

Then the space $Y = \prod_{j=1}^{n} (R, \mathcal{T}_j)$ is collectionwise normal, countably paracompact and ω_1-compact (6).

Proof. The proof is by induction on n. Assume that $l = 1, 2, \ldots$ and that our lemma has been proved for $n < l$. We will prove it for $n = l$.

(a) *Y is ω_1-compact.* Let A be a closed discrete subset of Y. Choose a countable subset $\bar{A}$ of A such that $A \subset \bar{A}^*$. Clearly A^* is closed in Y. It easily follows from our inductive assumption that if A is n-countable, then A is countable, therefore we will assume that A is n-uncountable.

Since the set $\bar{A}^*$ is n-uncountable, by Lemma 4.4 the set $\mathrm{Cl}_Y A^* = A^*$ is n-uncountable, which contradicts the countability of A^*.

(b) *Y is collectionwise normal.* Since Y is ω_1-compact it suffices to prove that Y is normal (6). Let A and B be two disjoint closed subsets of Y. To show that A and B can be separated by open sets it suffices to find a countable open covering $\mathcal{U}$ of Y such that for every $U \in \mathcal{U}$:

(12) either $\mathrm{Cl}_Y U \cap A = \emptyset$ or $\mathrm{Cl}_Y U \cap B = \emptyset$ (see [5], Lemma 1.5.14). Since $A \cap B = \emptyset$ it follows from Lemma 4.4 that the set $F = \bar{A} \cap \bar{B}$ is n-countable. Every point $p \in Y \setminus F$ has a euclidean-open neighbourhood U_p such that either $U_p \cap \bar{A} = \emptyset$ or $U_p \cap \bar{B} = \emptyset$. Since the topology of Y is stronger than the euclidean topology on R^n the sets U_p are also open in Y. Let $\mathcal{U}_1$ be a countable subfamily of $\{U_p\}_{p \in Y \setminus F}$ covering $Y \setminus F$.

Let E be a countable subset of R such that $F \subset \bigcup_{j=1}^{n} (R^{j-1} \times E \times R^{n-j})$. It suffices to find a countable open in Y covering $\mathcal{U}_2$ of F such that for every $U \in \mathcal{U}_2$ condition (12) is fulfilled. To this end it suffices to find for each $x \in E$ and $j = 1, \ldots, n$ a countable open in Y covering $\mathcal{U}_{xj}$ of $R^{j-1} \times \{x\} \times R^{n-j}$ such that for every $U \in \mathcal{U}_{xj}$ the condition (12) if fulfilled. Without loss of generality we can assume that $j = n$.

(6) A Hausdorff space is *ω_1-compact* if its every closed discrete subset is countable. One easily sees that a normal ω_1-compact space is collectionwise normal.

Let $Z = \prod_{j=1}^{n-1} (R, \mathcal{T}_j)$, $T = (R, \mathcal{T}_n)$ and let $\{G_s\}_{s=1}^{\infty}$ be a countable base of x in $\mathcal{T}$. For every $s = 1, 2, \ldots$ let

$$V_s(A) = \bigcup \{V\colon V \text{ is open in } Z \text{ and } (V \times \mathrm{Cl}_T G_s) \cap A = \emptyset\},$$

$$V_s(B) = \bigcup \{V\colon V \text{ is open in } Z \text{ and } (V \times \mathrm{Cl}_T G_s) \cap B = \emptyset\}.$$

Clearly the family $\{V_s(A)\}_{s=1}^{\infty} \cup \{V_s(B)\}_{s=1}^{\infty}$ is a countable open covering of Z. Since Z is, by our inductive assumption, normal and countably paracompact, there exist open in Z sets $W_s(A)$ and $W_s(B)$ such that

$$\mathrm{Cl}_Z W_s(A) \subset V_s(A), \quad \mathrm{Cl}_Z W_s(B) \subset V_s(B) \quad \text{and} \quad \bigcup_{s=1}^{\infty} W_s(A) \cup \bigcup_{s=1}^{\infty} W_s(B) = Z.$$

It suffices to put

$$\mathcal{U}_{xn} = \{W_s(A) \times G_s\}_{s=1}^{\infty} \cup \{W_s(B) \times G_s\}_{s=1}^{\infty}.$$

(c) *Y is countably paracompact.* Let $\{F_s\}_{s=1}^{\infty}$ be a decreasing sequence of closed subsets of Y with empty intersection. We have to show that there exists a sequence $\{U_s\}_{s=1}^{\infty}$ of open subsets of Y such that $F_s \subset U_s$ and $\bigcap_{s=1}^{\infty} U_s = \emptyset$ (see [5], Corollary 5.2.2).

We shall first prove that the euclidean-closed set $F = \bigcap_{s=1}^{\infty} \bar{F}_s$ is n-countable. Let $A = \{a_s\}_{s=1}^{\infty}$ be a dense subset of F. Without loss of generality we can assume that $a_s \notin F_s$. For every $s = 1, 2, \ldots$ choose a sequence S_s of points of F_s converging to a_s in R^n. Clearly S_s, considered as a set, is a closed subset of F_s in Y and therefore, since $\bigcap_{s=1}^{\infty} F_s = \emptyset$, the countable set $S = \bigcup_{s=1}^{\infty} S_s$ is closed in Y. Lemma 4.4 implies that the set $\bar{S}$ is n-countable and therefore $F = \bar{A} \subset \bar{S}$ is n-countable.

Sets $\bar{F}_s \setminus F$ form a decreasing sequence of closed subsets of the open subspace $R^n \setminus F$ of R^n and have empty intersection. Therefore, there exist euclidean-open sets V_s such that $F_s \setminus F \subset \bar{F}_s \setminus F \subset V_s$ and $\bigcap_{s=1}^{\infty} V_s = \emptyset$. To show that Y is countably paracompact it suffices to show that there exist open subsets W_s of Y such that $F_s \cap F \subset W_s$ and $\bigcap_{s=1}^{\infty} W_s = \emptyset$. The proof of the latter fact is analogous to the second part of the proof of normality of Y, and is left to the reader. ∎

LEMMA 4.6. *Let us fix an $i = 1, 2, \ldots, k+m$ and let $\{\mathcal{T}_j\}_{j=1}^{n}$ be a family of regular topologies on R such that for any $j = 1, 2, \ldots, n$ every point $x \in D_i$ has a base of neighbourhoods in $\mathcal{T}_j$ consisting of euclidean-open sets.*

Then the space $Y = \prod_{j=1}^{n} (R, \mathcal{T}_j)$ is Lindelöf.

Proof. The proof is by induction on n. Assume that $l = 1, 2, \ldots$ and that the above lemma has been proved for $n < l$. We will prove it for $n = l$.

Let $\mathscr{U}$ be an open covering of Y. By our assumptions there exists a countable open refinement $\mathscr{V}$ of $\mathscr{U}$ covering D_i^n and consisting of euclidean-open sets. It follows from the inductive assumption that n-countable closed subspaces of Y are Lindelöf. Therefore it suffices to show that the euclidean-closed set $F = Y \setminus \bigcup \mathscr{V}$ is n-countable.

Let A be a countable subset of F such that $F = \bar{A}$. If F were n-uncountable, then by (5) there would exist an $\alpha < c$ such that $A = A_\alpha = B_\alpha$, $p(\alpha, i) \in \bar{A} = F$ and $\hat{p}(\alpha, i) \subset D_i$. Therefore we would have $p(\alpha, i) \in D_i^n \cap F$, which is a contradiction. ■

For every $x \in R \setminus Q$ and $s = 1, 2, \ldots$ let us fix open intervals $I(x, s)$ of diameter $< 1/s$ such that

(13) $$t^s(x) \in I(x, s) \quad \text{and} \quad x \notin I(x, s),$$

(14) $$I(x, s) \cap I(x, s') = \emptyset, \quad \text{if} \quad t^s(x) \neq t^{s'}(x).$$

For every $\nu = 1, 2, \ldots, k$ and $\mu = k+1, \ldots, k+m$ we shall define (cf. [3]) a regular separable and first countable topology $\mathscr{T}_{\nu,\mu}$ on R which is stronger than the usual topology on R and satisfies

(15) every point $x \in \bigcup_{i=1}^{k} D_i \setminus D_\nu$ admits a base of neighbourhoods consisting of euclidean-open sets;

(16) every neighbourhood U of $x \in \bigcup_{i=k+1}^{k+m} D_i \setminus D_\mu$ in $\mathscr{T}_{\nu,\mu}$ contains a "tail" of the sequence $T(x)$; i.e. there exists an l such that $t^s(x) \in U$ for $s \geqslant l$;

(17) every point $x \in Q \cup D_\nu \cup D_\mu$ has a neighbourhood contained in $Q \cup \{x\}$.

Topology $\mathscr{T}_{\nu,\mu}$ will be generated by a collection $\mathscr{B} = \{\mathscr{B}(x)\}_{x \in R}$ of bases $\mathscr{B}(x)$ of neighbourhoods of points $x \in R$, where for each $x \in X$

$$\mathscr{B}(x) = \{B_l(x)\}_{l=1}^{\infty}.$$

We shall define sets $B_l(x)$ by induction on the well-ordering of R. Assume that $x \in R$ and that for all $y \prec x$ and $l = 1, 2, \ldots$ the sets $B_l(y)$ have been defined so that the following conditions are satisfied:

$(18)_y$ $B_{l+1}(y) \subset B_l(y)$ and $B_l(y)$ are euclidean-closed;

$(19)_y$ for every euclidean-open U containing y there exists an l such that $B_l(y) \subset U$.

We shall construct sets $B_l(x)$, for $l = 1, 2, \ldots$, satisfying $(18)_x$ and $(19)_x$.

(20a) If $x \in Q$, then $B_l(x) = \{x\}$, for $l = 1, 2, \ldots$

(20b) If $x \in D_\nu \cup D_\mu$, then $B_l(x) = \{x\} \cup \{q^s(x)\}_{s=l}^{\infty}$.

(20c) If $x \in \bigcup_{i=1}^{k} D_i \setminus D_\nu$, then $B_l(x)$ is an arbitrary closed interval of length $< 1/l$ with rational end-points, containing x and contained in $B_{l-1}(x)$, if $l > 1$.

(20d) If $x \in \bigcup_{i=k+1}^{k+m} D_i \setminus D_\mu$, then for each $s = 1, 2, \ldots$ we first choose a neighbourhood $B_{l(s)}(t^s(x))$ of the point $t^s(x) \prec x$ contained in $I(x, s)$ and we put

$$B_l(x) = \{x\} \cup \bigcup_{s=l}^{\infty} B_{l(s)}(t^s(x)).$$

It follows easily from the definition of $I(x, s)$, $(18)_y$ and (20) that conditions $(18)_x$ and $(19)_x$ are satisfied, which completes the inductive construction.

One easily checks that bases $\mathscr{B}(x)$ of neighbourhoods are well-defined and generate a first countable topology $\mathscr{T}_{\nu\mu}$ which is stronger than the usual topology on R, and satisfies conditions (15), (16) and (17). Moreover, the space $X_{\nu\mu} = (R, \mathscr{T}_{\nu\mu})$ is regular, because the sets $B_l(x)$ are euclidean-closed and therefore open-and-closed in $X_{\nu\mu}$. The set Q of rationals is dense in $X_{\nu\mu}$.

Let X be the topological sum of spaces $X_{\nu\mu}$; i.e.

$$X = \bigoplus_{\nu=1}^{k} \bigoplus_{\mu=k+1}^{k+m} X_{\nu\mu}.$$

Clearly, the space X is regular, first countable and separable.

(A) *X^n is collectionwise normal, countably paracompact and ω_1-compact, for $n < m$.*

Since the space X^n is the topological sum of spaces $Y = \prod_{j=1}^{n} X_{\nu(j),\mu(j)}$, where $1 \leqslant \nu(j) \leqslant k$ and $k+1 \leqslant \mu(j) \leqslant k+m$, for $j = 1, 2, \ldots, n$, it suffices to show that each such a space Y is collectionwise normal, countably paracompact and ω_1-compact.

Since $n < m$ there exists an i such that $k+1 \leqslant i \leqslant k+m$ and $i \neq \mu(j)$, for every $j = 1, 2, \ldots, n$. By (16) for every $j = 1, 2, \ldots, n$ every neighbourhood U of $x \in D_i$ in $\mathscr{T}_{\nu(j),\mu(j)}$ contains a "tail" of the sequence $T(x)$. By Lemma 4.5 the space $Y = \prod_{j=1}^{n} (R, \mathscr{T}_{\nu(j),\mu(j)})$ has the required properties.

(B) *X^n is Lindelöf, for all $n < k$.*

Again it suffices to show that any space Y as described above is Lindelöf. Since $n < k$ there exists an i such that $1 \leqslant i \leqslant k$ and $i \neq \nu(j)$, for all $j = 1, 2, \ldots, n$. By (15) for every $j = 1, 2, \ldots, n$ every point $x \in D_i$ has a base of neighbourhoods in $\mathscr{T}_{\nu(j),\mu(j)}$ consisting of euclidean-open sets. By Lemma 4.6 the space $Y = \prod_{j=1}^{n} (R, \mathscr{T}_{\nu(j),\mu(j)})$ is Lindelöf.

(C) *X^m is neither normal, nor countably paracompact nor ω_1-compact.*

The space $Z = \prod_{j=1}^{k} X_{j,k+j} \times \prod_{j=k+1}^{m} X_{1,k+j}$ is a closed subspace of X^m therefore it suffices to show that Z is neither normal, nor countably paracompact, nor ω_1-compact.

It is known (see [6] and [7]) that a separable space containing a discrete closed subspace of cardinality continuum is neither normal nor countably paracompact. Therefore it suffices to show that Z contains such a subspace. The set

$$F = \{(x, x, \dots, x) \in R^m\colon\ x \in R\setminus Q\}$$

is a closed subset of Z. We shall show that F is discrete. Let us choose a point $x \in R\setminus Q$ and an i such that $x \in D_i$. There exists a j such that $1 \leqslant j \leqslant m$ and either $i = j \leqslant k$ or $i = k+j$. Assume for example that the first case takes place. By (17) every point $x \in D_j = D_i$ has a neighbourhood U in $\mathscr{T}_{j,k+j}$ contained in $Q \cup \{x\}$. Therefore the set

$$V = R^{j-1} \times U \times R^{m-j}$$

is open in Z and $F \cap V = \{(x, x, \dots, x)\}$, which shows that the point $(x, x, \dots, x)$ is isolated in F. We argue similarly if the second case takes place.

It remains to show that X^k is not subparacompact. If $k = \omega$ then $k = m = \omega$ and therefore X^k is not paracompact by (C). Subparacompactness of X^k in this case will be discussed in Remark 4.7. We will first prove

(D) *X^k is not subparacompact if $k < \omega$.*

The space $Z = \prod_{j=1}^{k} X_{j,k+1}$ is a closed separable subspace of X^k, therefore it suffices to show that Z is not subparacompact. Since a collectionwise normal subparacompact space is paracompact and a separable paracompact space is Lindelöf, (D) will be proved if we show that Z is collectionwise normal but not Lindelöf.

Since $m \geqslant 2$, collectionwise normality of Z follows from the fact that for every $j = 1, \dots, k$ every neighbourhood U of $x \in D_{k+2}$ in $\mathscr{T}_{j,k+1}$ contains a "tail" of the sequence $T(x)$ (Lemma 4.5).

We shall show that Z is not Lindelöf. The set $\Delta = \{(x, x, \dots, x) \in R^k\colon\ x \in R\}$ is a closed subspace of Z, therefore it suffices to prove that for every $x \in R$ there exists an open subset $B(x)$ of Z such that

(21) $$B(x) \cap \Delta \subset \Delta(x) = \{(y, y, \dots, y) \in R^k\colon\ y \preccurlyeq x\},$$

because then the family $\{\Delta(x)\}_{x\in R}$ forms an open covering of Δ with no countable subcovering.

Let us denote by $B_l^{(j)}(x)$ the l-th basic neighbourhood of the point $x \in R$ in the topology $\mathscr{T}_{j,k+l}$, where $j = 1, 2, \dots, k$. We will show by induction on the well-ordering of R that for every $x \in R$ we have

$(22)_x$ $$\big(B_l^{(1)}(x) \times \dots \times B_l^{(k)}(x)\big) \cap \Delta \subset \Delta(x),$$

which clearly implies (21).

Assume that $x \in R$ and that $(22)_y$ has been proved for all $y \prec x$. We shall prove $(22)_x$.

(a) If $x \in Q$, then by (20a) $B_l^{(j)}(x) = \{x\}$, for all j and $(22)_x$ is clearly satisfied.

(b) If $x \in \bigcup_{j=1}^{k} D_j$, then there exists a j such that $x \in D_j$ and then $B_l^{(j)}(x) = \{x\} \cup \{q^s(x)\}_{s=l}^{\infty}$ by (20b) and $(22)_x$ follows.

(c) If $x \in \bigcup_{j=k+1}^{k+m} D_j$, then by (20b, d) $B_l^{(j)}(x) = \{x\} \cup \bigcup_{s=l}^{\infty} B_{l(s,j)}^{(j)}(t^s(x))$, where $t^s(x) \prec x$ and $B_{l(s,j)}^{(j)}(t^s(x)) \subset I(x, s)$.

It follows easily from (13), (14) and (18) that

$$B_l^{(1)}(x) \times \dots \times B_l^{(k)}(x) \cap \Delta \subset \{(x, x, \dots, x)\} \cup \bigcup_{s=l}^{\infty}\Big(\prod_{j=l}^{k} B_l^{(j)}(t^s(x)) \cap \Delta\Big).$$

By the inductive assumption, however, we have

$$\bigcup_{s=l}^{\infty}\Big(\prod_{j=1}^{k} B_l^{(j)}(t^s(x)) \cap \Delta\Big) \subset \bigcup_{s=l}^{\infty} \Delta(t^s(x)) \subset \Delta(x)$$

which implies $(22)_x$ and completes the proof of (D).

Remark 4.7. Since a space is paracompact if and only if it is collectionwise normal and θ-refinable (see Section 3), therefore the same proof as in (D) shows that X^k is not θ-refinable if $k < \omega$.

The author does not know whether the space X^ω is subparacompact if $k = \omega$, i.e. if $X = X(\omega, \omega)$, however the construction of X in this case can be easily modified so that all other properties of X are preserved but X^ω is not even countably θ-refinable.

To this end we will change neighbourhoods $\{B_l(x)\}_{l=1}^{\infty}$ of points $x \in \bigcup_{i=k+1}^{k+m} D_i$ in all topologies $\mathscr{T}_{\nu\mu}$. By Lemma 3.1 there exists a first countable, separable and locally compact topology $\mathscr{T}$ on $A = \bigcup_{i=k+1}^{k+m} D_i$ which is stronger than the usual topology on A and is neither normal nor countably θ-refinable. Let us replace neighbourhoods of points $x \in A$ in $\mathscr{T}_{\nu\mu}$ by neighbourhoods of these points in $\mathscr{T}$, for every $\nu = 1, \dots, k$ and $\mu = k+1, \dots, k+m$.

One easily sees that X is again separable, regular and first countable and by Lemma 4.6 X^n is Lindelöf for every $n < \omega$. However, X^k contains the space $Y = \prod_{\nu=1}^{k} X_{\nu 1}$ as a closed subspace and the closed subspace

$$K = \{(x, x, \dots, x) \in Y\colon\ x \in A\}$$

of Y is homeomorphic to $(A, \mathscr{T})$, which implies that X^ω is neither normal nor countably θ-refinable. ■

Remark 4.8. If $k = 1$ then every topology $\mathscr{T}_{\nu\mu}$ is locally compact and locally countable and therefore also the space X is locally compact and locally countable. Indeed, since $\nu = 1$ condition (15) becomes empty and while constructing $\mathscr{T}_{\nu\mu}$ we skip the step (20c). One easily proves by induction on $\prec$ that for every $x \in R$ and $l = 1, 2, \ldots$ the set $B_l(x)$ is a countable compact subspace of $X_{\nu\mu}$. ■

§ 5. Final remarks. The most general question concerning the relationship between normality and paracompactness (resp. the Lindelöf property) in product spaces is probably the following:

5.1. Let $\varkappa$, λ, μ denote cardinal numbers, where $1 \leqslant \lambda \leqslant \mu$. Does there exist a space $X = X(\lambda, \mu)$ such that

(a) $X^\varkappa$ is paracompact (resp. Lindelöf) if and only if $\varkappa < \lambda$;

(b) $X^\varkappa$ is normal if and only if $\varkappa < \mu$?

It is known that if X is paracompact and X^{ω_1} is normal, then X, and all its powers, are compact (see [5], Problem 2.7.16). Therefore we may assume that $1 \leqslant \lambda \leqslant \mu \leqslant \omega_1$.

Theorem 1.1 gives a positive answer to Question 5.1 in case of $\mu \leqslant \omega$. Below we discuss the remaining three cases.

Case 1. ($\lambda = \mu = \omega_1$) The positive answer to Question 5.1 in this case follows from the fact that if N is the space of natural numbers then N^ω is Lindelöf and N^{ω_1} is not normal (cf. [5], Problem 2.7.16).

Case 2. ($\lambda = \omega$, $\mu = \omega_1$) In this case Question 5.1 has a negative answer being a consequence of the following result.

THEOREM 5.2 (Nagami (1968) [14], Zenor [1971] [25]). *Let X^ω be normal. If X^n is paracompact (resp. Lindelöf) for all $n < \omega$, then X^ω is paracompact (resp. Lindelöf).*

Case 3. ($\lambda = k < \omega$, $\mu = \omega_1$) Continuum Hypothesis implies a positive answer to Question 5.1 in this case as follows from the result of Alster and Zenor (Theorem 2.13). However, we do not know whether the assumption of the Continuum Hypothesis is essential (we certainly conjecture that it is not).

It follows from our discussion that to obtain a complete answer to Question 5.1 it suffices to solve the following problem:

PROBLEM 5.3. Does there exist (without any set theoretic assumptions beyond ZFC) for every $k < \omega$ a space $X = X(k)$ such that X^k is paracompact (resp. Lindelöf) and X^ω is normal, but X^{k+1} is not paracompact (resp. Lindelöf)?

*

It is not known whether the assumption of the Continuum Hypothesis in Theorem 2.12 is essential. This leads to the following more general problem:

PROBLEM 5.4. Let $1 \leqslant k \leqslant m \leqslant \omega$. Does there exist (without any set-theoretic assumptions beyond ZFC) a space $Z = Z(k, m)$ such that:

(a) Z^n is Lindelöf if and only if $n < k$,

(b) Z^n is paracompact if and only if $n < m$?

*

In connection with Corollary 1.3 let us note the following result.

THEOREM 5.5. *The existence of a separable paracompact space X such that X^2 is normal but not collectionwise normal is consistent with and independent of the ZFC axioms of set theory.*

Proof. Theorem 2.4 shows that Martin's Axiom plus the negation of the Continuum Hypothesis imply the existence of such a space.

On the other hand, the assumption of $2^{\omega_0} < 2^{\omega_1}$ implies that every separable normal space is collectionwise normal (Jones' Lemma [7]).

*

The metric space M from Theorem 1.5 is not complete. In fact, the answer to the following problem is not known.

PROBLEM 5.6. Do there exist (without any set-theoretic assumptions beyond ZFC) a complete metric space M and a Lindelöf space Y such that $M \times Y$ is not Lindelöf or — equivalently — normal?

Such spaces exist if the Continuum Hypothesis is assumed ([11], footnote 4; see [12], Example 3.2(c), (d) for a proof). The proposition below shows that such spaces cannot be constructed using the "standard" technique which does not require CH.

PROPOSITION 5.7. *Let A be a subset of the real line such that $A \cap F \neq \emptyset$ for every uncountable closed subset of R and let $\mathcal{T}$ be a regular topology on R which coincides with the usual topology at points of A.*

For every complete separable metric space M the product space $M \times Y$, where $Y = (R, \mathcal{T})$, is Lindelöf.

Outline of the proof. Let $\mathcal{U}$ be an open covering of $M \times Y$. Choose a countable refinement $\mathcal{V}$ of $\mathcal{U}$ covering $M \times A$ and consisting of sets open in $M \times R$. The set $F = M \times R \setminus \bigcup \mathcal{V}$ must be 2-countable because otherwise it would contain a graph of a homeomorphic embedding $\psi\colon C \to R$ of the Cantor set $C \subset M$ into R ([18], Theorem 1 and Remark 1). Consequently, we would have $\psi(C) \cap A = \emptyset$, which is impossible. ■

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Almost every tree function is independent

by

Leonard Gallagher * (Washington, D. C.)

Abstract. Points of the Cantor set C may be represented as branches of an infinite dyadic tree. Nodes of the tree may be randomly labeled with 0's and 1's. A tree function is a mapping from C to $[0,1]$ determined by assigning to each branch the real number having binary representation as the labeling of the branch. A tree function f is independent over a relation $R \subseteq [0,1]^n$ if for every sequence $x_1, \ldots, x_n$ of distinct elements of C we have $(f(x_1), \ldots, f(x_n)) \notin R$. We define a Borel probability measure on the set of tree functions and show that if R is null with respect to a special Hausdorff measure on $[0,1]^n$ then almost every tree function is independent over R.

1. Introduction. A generalized notion of independence was introduced by Marczewski in [2] and extended by Mycielski to relational structures in [4]. Following [5] and [6] we consider *relational structures* of the form $\langle M, R_k \rangle_{k<\omega}$ where M is a non-empty, complete metric space, $R_k \subseteq M^{r(k)}$ and $1 \leqslant r(k) < \omega$ for all $k<\omega$. For any set X a function $f\colon X \to M$ is *independent over* the R_k's if for every k and every sequence $x_1, \ldots, x_{r(k)}$ of *distinct* elements of X we have $(f(x_1), \ldots, f(x_{r(k)})) \notin R_k$.

The Cantor set is denoted by the symbol C and is understood to be the discontinuum $\{0,1\}^\omega$ under the usual totally disconnected metrization. M^C is the space of all continuous functions $f\colon C \to M$ with the usual uniform convergence topology.

The main result of this paper is a theorem analogous to the main theorems of [5] and [6]. In [6] Mycielski proves that if each R_k is meagre in $M^{r(k)}$ then the set of functions $f \in M^C$ independent over all R_k's is comeager in the space M^C. In [5] he lets M be Euclidean n-space and shows that if the R_k's are of Lebesgue $r(k)$-dimensional measure zero then there exist independent functions $f \in M^C$. We let $M = [0,1]$ and prove that if each R_k is h_0-null (see below) in $[0,1]^{r(k)}$ then almost every tree function (see below) is also independent over the R_k's (see Remark 1, Section 5).

In Section 2 we define randomly labeled trees and tree functions. We also construct a probability measure over the set of all tree functions and estimate the measure of certain useful subsets. In Section 3 we prove that two interesting properties are true for almost all tree functions. In Section 4 we define h-null sets and compare

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