Homotopy groups of surfaces whose boundary is the union of 1-spheres

by

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Abstract. Let $M$ be a surface (connected, separable, metrizable 2-manifold) with non empty boundary $bd M$ which is the union of a collection of 1-spheres $(S_j)_{j=1}^{n}$. Let $D_j$ be the 2-cell obtained by taking the cone over $S_j$ and let $M$ denote the surface $M \cup \left( \bigcup_{j=1}^{n} D_j \right)$ where $f: (\bigcup_{j=1}^{n} S_j) \to M$ is defined by $f(x) = x$. As usual, identify $D_j$ and $M$ with the appropriate subsets of $\tilde{M}$ and for each $j$ let $g_j \in \pi_1(M, M)$. The principal result of this paper is that the homotopy group $\pi_1(M)$ of $M$ is isomorphic to the group $\pi_1(\tilde{M}, F)$ where $F = \langle g_j \rangle$. This generalizes the results of D. J. Sproville concerning $\pi_1(M)$ where $M$ is a compact surface, $bd M \neq \emptyset$.

1. Introduction. Let $M$ be an $n$-manifold (connected, separable, metrizable), $bd M \neq \emptyset$, $n \geq 2$. $M$ will be called sphere bounded if each component of $bd M$ is an $(n-1)$-sphere. If $M$ is a sphere bounded $n$-manifold, the capping of $M$, denoted by $\bar{M}$, is the $n$-manifold defined by

$$\bar{M} = M \cup \left( \bigcup_{j=1}^{n} D_j \right)$$

where for each $j$, $D_j$ is the cone over the $j$th boundary component $S_j$ of $M$ and

$$f: (\bigcup_{j=1}^{n} S_j) \to M$$

is defined by $f(x) = x$. As usual, we will identify $S_j$, $D_j$ and $M$ with the appropriate subspaces of $\bar{M}$.

Let $M$ be an $n$-manifold, $F$ a subset of $\text{int} M$. As usual, the homotopy group $\pi_1(M)$ of $M$ is the quotient group $G(M)/G_0(M)$ where $G(M)$ is the group of all homeomorphisms of $M$ onto itself and $G_0(M)$ is the normal subgroup of $G(M)$ consisting of those homeomorphisms $g$ which are isotopic to the identity (denoted $\simeq_1 M$). Also $\pi_1(M, F)$ denotes the quotient group $G(M, F)/G_0(M, F)$ where $G(M, F)$ is the subgroup of $G(M)$ consisting of those $g$ which map $F$ onto $F$ and $G_0(M, F)$ is the normal subgroup of $G(M, F)$ consisting of those homeomorphisms $h$ in $G(M, F)$ which are isotopic to the identity by an isotopy which keeps $F$ pointwise fixed (denoted $h \simeq_1 F$ rel $F$).
The main results of this paper are the following:

**Theorem 1.** Let \( M \) be a sphere bounded \( n \)-manifold, \( n \geq 2 \) and \( n \neq 4 \), \( \tilde{M} \) the capping of \( M \). For each \( j \) such that \( S_j \) is a component of \( \partial M \), let \( p_j \in \text{int } D_j \subset \tilde{M} \) and let \( F = \{ p_j \}_{j=1}^{\infty} \). Then there is an epimorphism \( \psi: H(M) \to H(M, F) \).

**Theorem 2.** Let \( M \) be a sphere bounded 2-manifold, \( \tilde{M} \) the capping of \( M \). Then \( H(M) \cong H(M, F) \).

**Theorem 3.** Let \( Q \) be a 2-manifold, \( \partial Q = \emptyset \) and let \( F \) be a non-empty closed discrete subset of \( Q \). Then

(i) \( H(Q, F) \cong H(M) \) for some sphere bounded 2-manifold \( M \).

(ii) If \( F' \) is a non-empty, closed discrete subset of \( Q \) such that \( F \) is homeomorphic to \( F' \). If \( F \) and \( F' \) determine the same subspace of the space of ends \( e(Q) \) of \( Q \), then there is a homeomorphism \( h \) of the pair \( (Q, F) \) onto \( (Q, F') \) and hence \( H(Q, F) \cong H(Q, F') \).

2. **Proof of the Results.** We will follow the procedure of Sprows in [6] with certain crucial modifications.

**Proof of Theorem 1.** We first note that we may assume that \( p_j \) is the vertex of the cone \( D_j \) over \( S_j \). For each \( j \), let \( E_j \) be a fixed homeomorphism of \( D_j \) onto the closed unit ball in some space \( M \) which carries \( p_j \) to the origin. If \( h \in G(M) \), then we extend \( h \) to \( h_\infty \in G(\tilde{M}, F) \) by coning; i.e., \( h(x) = h(x) \) for all \( x \in S_j \) and \( h(x) = c_j^{-1}(h(x)) + (1-e_j)(x) \) for \( x \in S_j \) and \( e_j \in [0,1] \) and \( h(x) = c_j^{-1}(h(x)) + (1-e_j)(x) \). Since \( M \) together with the collection \( \{D_j\} \) is a closed, nbd-finite family \( F \), \( h_\infty \in G(\tilde{M}, F) \).

For \( f \in G(M) \) and \( g \in G(\tilde{M}, F) \), let \( f \) and \( g \) denote the equivalence classes in \( H(M) \) and \( H(\tilde{M}, F) \) respectively. We define \( \psi: H(M) \to H(\tilde{M}, F) \) by \( \psi(f) = f_\infty \).

1. \( \psi \) is well defined. Suppose that \( f, f' \in G(M) \) such that \( f = f' \). Then \( f^*f^{-1} \in G(M) \), and thus \( f^*f^{-1} \) is isotopic to \( 1 \) by some isotopy \( H_t \). Since \( H_t(b_d D_i) = b_d D_i \) for all \( i \) and all \( t \in [0,1] \), we let \( H_t = H_t(b_d D_i) \), and since \( F \) is a closed, nbd finite family, it follows that \( H_t \) yields an isotopy \( f^*f^{-1} \) is isotopic to \( 1 \) by \( F \).

2. \( \psi \) is a homomorphism. This follows since \( f^*f = f^*f' \) for any \( f, f' \in G(M) \).

3. \( \psi \) is an epimorphism. In order to show this, we first establish the following lemma:

**Lemma 2.1.** Suppose that \( g \in G(\tilde{M}, F) \). Then there exists \( f \in G(M) \) such that \( g = f_\infty \).

**Proof.** By taking an appropriate collar of \( b_d M \), for each \( i \) we get a closed annulus \( A_i \subset M \) such that \( E_i = A_i \cup D_i \) is a disjoint collection of closed \( n \)-cells with bicellular boundary and the collection \( \mathcal{E} \) consisting of \( M = \bigcup (\text{int } E_i) \) together with all the \( E_i \) is a closed, nbd-finite family. Now let \( g \in G(\tilde{M}, F) \). For each \( i \), let \( g(i) \) denote the index such that \( g(E_i) = p_i \). Then for each \( i \),

\[
\theta(p_i) = p_{i0} \in \text{int } D_{i0} \cap \text{int } g(D_i) = \text{int } D_{i0} \cap \theta(\text{int } D_i).
\]

For each \( i \), there exists a closed \( n \)-cell \( F_{i0} \) with bicellular boundary such that \( p_{i0} \in \text{int } F_{i0} \subset F_{i0} \cap \text{int } D_{i0} \cap \text{int } g(D_i) \). Since \( n \geq 2 \), \( n \neq 4 \), it follows from [4] that \( \theta(E_i) = \text{int } F_{i0} \) and \( \theta(E_i) = \text{int } D_{i0} \) are annuli and we can construct a homeomorphism \( \eta_i \) of \( \theta(E_i) \) onto itself such that \( \eta_i \) restricted to \( \partial b_d \theta(E_i) \cup \{ p_{i0} \} \) is the identity and \( \eta_i(\text{int } D_i) = F_{i0} \). It follows from the use of Alexander's technique [1] that \( \eta_i \) is isotopic to \( 1_{\text{bd} \theta(E_i)} \) relative to \( \partial b_d(\theta(E_i) \cup \{ p_{i0} \}) \).

Since \( \eta_i \) is a closed nbd-finite collection and \( g \in G(\tilde{M}, F) \), it follows that the \( \eta_i \) can be extended to a homeomorphism \( \eta \in G(\tilde{M}, F) \) by setting

\[
\phi(x) = \begin{cases} x & \text{for } x \in \tilde{M} - \bigcup \text{int } g(E_i) \\ \phi(x) & \text{for } x \in \theta(E_i) \end{cases}
\]

Also, each \( \eta_i \) has \( \text{bd} b_d \eta_i \cap \partial b_d \theta(E_i) \cup \{ p_{i0} \} \) there is an isotopy \( \eta_i \cong 1_{\text{bd} \theta(E_i)} \).

In a similar manner, we can construct a homeomorphism \( \eta \in G(\tilde{M}, F) \) such that \( \eta = \text{int } F_{i0} \) and \( \eta = \text{int } D_{i0} \) is a homomorphism of \( F_{i0} \) onto itself which is the identity on \( \text{bd} F_{i0} \cup \{ p_{i0} \} \). Now consider \( \eta = \text{int } F_{i0} \). Then clearly \( \eta \cong 1_{\text{bd} \theta(E_i)} \).

Since \( \eta_i \) is a closed nbd-finite collection and \( g \in G(\tilde{M}, F) \), it follows that \( \eta = \text{int } F_{i0} \).

Thus \( \eta \) is a closed nbd-finite collection and \( g \in G(\tilde{M}, F) \), it follows that the \( \eta_i \) can be extended to a homeomorphism \( \eta \in G(\tilde{M}, F) \) by setting

\[
\phi(x) = \begin{cases} x & \text{for } x \in \tilde{M} - \bigcup \text{int } g(E_i) \\ \phi(x) & \text{for } x \in \theta(E_i) \end{cases}
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**Theorem 2.** In view of Theorem 1.1, it suffices to establish that \( \psi \) is a monomorphism when \( n = 2 \). We need only note that the construction used by Sprows in [6] can be used in the present case. Thus suppose \( f \in H(M) \) and \( f \in \text{ker } \psi \), i.e., \( f \) is kernel. As in [6], let \( \hat{\rho} \in \text{ker } \psi \) be defined by setting \( \hat{\rho}(x) = x \) for \( x \in M \) and let \( \rho_\infty(b_d D_i) \) be the retraction of \( D_i \) onto \( b_d D_i \), if \( f \in \text{ker } \psi \), \( \rho_\infty(b_d D_i) \) by \( \rho_\infty(b_d D_i) \), \( H_i \), \( H_j \), \( G_i \), \( G_j \) are homotopy on \( M \) with \( G_0 = f \) and \( G_1 = 1_M \). Arguing as in [6], it follows that \( \rho \in 1_M \).

**Proof of Theorem 3.** Let \( F \) be a non-empty closed discrete subset of \( Q \). Then \( F = (\theta(x) \cap \tilde{M}) \cap \text{bd} M \) is at most countably infinite.

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(i) each component of \( U \) is a sphere-bounded surface;
(ii) \( \text{Fr} \ U \) in \( M \) is compact; and
(iii) \( U \cap \partial M = \emptyset \), define
\[ [U] = \{ [p] \mid p = P_1 \circ \ldots \circ P_n \Rightarrow \ldots \Rightarrow P_{n} \Rightarrow \ldots \} \]
is a boundary component, then
\[ P_n \in U \text{ for } N \text{ sufficiently large}. \]

Then it is easily established that the collection \( B \) of all such \([U]\) is a basis for a topology \( T \).

As in [5], a point \( [p] \in e(M) \), where \( p = P_1 \Rightarrow \ldots \Rightarrow P_n \Rightarrow \ldots \), is called planar if \( P_n \) is planar for all \( n \) sufficiently large and \([p]\) is called orientable if \( P_n \) is orientable for all \( n \) sufficiently large. We set
\[ e'(M) = e(M) - \{ [p] \mid [p] \text{ is planar} \} \]
and
\[ e''(M) = e(M) - \{ [p] \mid [p] \text{ is orientable} \}. \]

It is easily seen that \( e'(M) \) and \( e''(M) \) are closed subsets of \( e(M) \). The following proposition establishes an important relationship between \( e(M) \) and \( e(M) \).

**Proposition A.3.** Let \( M \) be a sphere bounded 2-manifold. Then there is a homeomorphism \( h \) of the triple \( (e(M), e'(M), e''(M)) \) onto the triple \( (e(M), e(M), e(M)) \) where \( M \) is the capping of \( M \).

**Proof.** First note that in Definition 2 of [5], one can make the further requirement that \( U \) be open and the same topology will be generated by the basis obtained.

Let \( X = \overline{\sum D_j} \) and let \( X = X + \overline{\sum D_j} \) be the quotient map. Let \( U \) be a domain (open connected subset) of \( M \) which is a sphere bounded 2-manifold. Then \( U \cap S_i = \emptyset \) for any component \( S_i \) of \( \partial M \). Then \( U = g(D(U)) \) is a domain of \( M \) where \( D(U) = (\sum D_j \cup S_i \cap \emptyset) + U \). If \( g = g(D(U)) \) is a boundary component of \( M \), then clearly \( g = g(D(U)) \) is a boundary component of \( M \). We define \( h : e(M) \rightarrow e(M) \) by \( h([p]) = [p] \). Then \( h \) is well defined. Straightforward arguments can now be given to show that \( h \) carries the triple \( (e(M), e'(M), e''(M)) \) homeomorphically onto the triple \( (e(M), e(M), e(M)) \).

**Definition A.4.** Let \( M \) be a sphere bounded surface. A point \([p] \in e(M)\) will be called a rim point of \( M \) if \( p = P_1 \Rightarrow \ldots \Rightarrow P_n \Rightarrow \ldots \) has the property that \( P_n \cap \partial M \) is an infinitely large number of components for all \( n \geq 1 \). Clearly, this definition is independent of the choice of the boundary component representing \([p]\). The rim of \( e(M) \), denoted by \( \text{rim}(M) \), is defined by
\[ \text{rim}(M) = \{ [p] \in e(M) \mid [p] \text{ is a rim point of } M \}. \]

**Proposition A.5.** Let \( M \) be a sphere bounded 2-manifold. Then \( \text{rim}(M) \) is a closed subset of \( e(M) \).

**Proof.** This follows easily from the definition.
DEFINITION A.6. Let $M$ be a sphere bounded 2-manifold. The end tuple of $M$ is the ordered 4-tuple of spaces $(e(M), e'(M), e''(M), \text{rim}(M))$.

THEOREM A.7 (Kerekjarto's Theorem). Suppose that $M, M'$ are sphere bounded 2-manifolds such that $M$ and $M'$ are of the same genus and orientability class; i.e., $M, M'$ satisfy this condition. Then $M$ and $M'$ are homeomorphic or $\text{bd}(M)$ and $\text{bd}(M')$ are homeomorphic and there is a homeomorphism $k$ of the end tuple of $M$ onto the end tuple of $M'$.

Proof. Necessity is clear so we only consider the problem of establishing sufficiency.

If $\text{bd}(M)$ and $\text{bd}(M')$ have only a finite number of components, then $\text{rim}(M)$ = $\text{rim}(M')$ = $\emptyset$ and it follows from Proposition A.3 and the main result of [5] that $M$ and $M'$ are homeomorphic. Since $\text{bd}(M)$ and $\text{bd}(M')$ have the same finite number of boundary components, a homeomorphism of $M$ onto $M'$ is easily obtained.

Now suppose that $\text{bd}(M)$ and $\text{bd}(M')$ have an infinite number of components. We outline how the proof of Theorem 1 of [5] can be modified to obtain the result. As in [5], assume that $M$ and $M'$ are both infinitely non-orientable, the proof in other cases being similar.

Let $k$ be a homeomorphism of the end tuple of $M$ onto the end tuple of $M'$. The existence of a homeomorphism $f$ of $M$ onto $M'$ will follow provided we can represent $M$ and $M'$ as the union of compact 2-manifolds $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$, respectively such that for all $n \geq 1$

(i) $A_n \cap \text{bd}(M)$ is either empty or the union of a finite number of components of $\text{bd}(M)$; similarly for $A'_n \cap \text{bd}(M')$.

(ii) $A_n \subset \text{bd}(A_{n+1})$ is interior to $A_{n+1}$, relative to $M$; similarly $A'_n \subset \text{bd}(A'_{n+1})$.

(iii) There is a homeomorphism $f_n$ of the pair $(A_n, A_n \cap \text{bd}(M))$ onto the pair $(A'_n, A'_n \cap \text{bd}(M'))$.

(iv) $f_{n+1} | A_n = f_n$.

(v) Each component $U$ of $M-A_n$ has non-compact closure and simple closed curve $\partial(U)$ comprises $\text{Fr}(U)$. Furthermore, either $U \cap \text{bd}(M) = \emptyset$ or $U$ contains an infinite number of components of $\text{bd}(M)$. Similarly for $M'-A'_n$.

(vi) If $f_n(\partial(U)) = \partial(U')$, then $k([U]) = [U']$.

To do this, we note that we can first construct sequences $\{B_n\}$ and $\{B'_n\}$ of compact 2-manifolds contained in $M$ and $M'$ which satisfy for all $n \geq 1$ the conditions

(vii) $B_n \cap \text{bd}(M)$ is either empty or consists of a finite number of components of $\text{bd}(M)$, $B_n \subset \text{bd}(B_{n+1})$, and $M = \bigcup_{n=1}^{\infty} B_n$. Likewise for $\{B'_n\}$.

(viii) Every component $U$ of $M-B_n$ has non-compact closure, is either of genus zero or infinite genus, is either orientable or infinitely non-orientable, and either $U \cap \text{bd}(M) = \emptyset$ or $U \cap \text{bd}(M)$ consists of an infinite number of components of $\text{bd}(M)$. Furthermore, FrU in $M$ is precisely one simple closed curve. Similarly for each component of $M'-B'_n$.

References


