On total paracompactness and total metacompactness

by

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Abstract. We first generalize the countable sum theorem concerning totally paracompact spaces by two order locally finite covers. Secondly, we show that there exists a separable metric space with a base containing no weakly uniform base.

0. Introduction. The present paper is composed of two main results. One is concerned with total paracompactness. Many results on a space X which has two order locally finite covers $\{F_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ such that F_ξ is closed and has a topological property $\mathfrak{P}$ and U_ξ is an open nbd of F_ξ for each $\xi<\alpha$ have been studied by Y. Katuta [8] and R. Telgársky [10], [12]. We show that X is totally paracompact if X has the above two covers $\{F_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ and the above property $\mathfrak{P}$ is total paracompactness relative to X. This is a generalization of the countable sum theorem concerning total paracompactness by D. W. Curtis [3].

The other result is concerned with total metacompactness. It is well-known that the space N^N of all irrational numbers with a usual topology is not totally metacompact [2], [4]. R. W. Heath and W. F. Lindgren raised the question of whether or not every base of a space with a weakly uniform base contains a weakly uniform base ([6] Question 4). We give a negative answer to this question. We shall show that the space N^N of irrational numbers is a desired counterexample for it.

When $\mathfrak{A}$ is a collection of subsets of a space X, let $\mathfrak{A}^{\#} = \bigcup\{A\colon A\in\mathfrak{A}\}$. Unless otherwise stated, no separation axioms are assumed. However, compact, paracompact and totally paracompact spaces are always Hausdorff. N denotes the set of all natural numbers. Natural numbers are denoted by $m, n, i, j, \ldots$ and ordinal numbers are denoted by $\alpha, \beta, \gamma, \ldots, \xi, \eta, \zeta, \ldots$

1. Total paracompactness. A Hausdorff space X is said to be *totally paracompact* [5] if every open base of X contains a locally finite cover of X. A closed set F of X is said to be *totally paracompact relative to* X [10] if every open base of X contains a locally finite (in X) cover of F. Then F is clearly totally paracompact. Recall that a collection $\{A_\lambda\colon \lambda\in\Lambda\}$ of subsets of a space X is said to be *order locally finite* [8] if we can introduce a well-ordering $<$ in the index set Λ such that for each $\lambda\in\Lambda$ the collection $\{A_\mu\colon \mu<\lambda\}$ is locally finite at each point of A_λ. Since every well-ordered set is order-isomorphic to an initial segment of ordinal numbers, we use the notation

$\{A_\xi: \xi<\alpha\}$ instead of $\{A_\lambda: \lambda \in \Lambda\}$. Every countable collection of subsets of X is clearly order locally finite.

Our main theorem is as follows.

THEOREM 1. *If a regular T_1-space X has two order locally finite covers $\{F_\xi: \xi<\alpha\}$ and $\{U_\xi: \xi<\alpha\}$ such that F_ξ is closed and totally paracompact relative to X and U_ξ is an open nbd of F_ξ for each $\xi<\alpha$, then X is totally paracompact.*

We need the following two lemmas to prove Theorem 1.

LEMMA 2 (Y. Katuta [8]). *Let $\{A_\lambda: \lambda \in \Lambda\}$ be an order locally finite collection of subsets of a space X, and let $\{B_\xi: \xi \in \Xi_\lambda\}$ be a collection of subsets of A_λ which is locally finite in X for each $\lambda \in \Lambda$. Then the collection $\{B_\xi: \xi \in \Xi\}$ is order locally finite, where Ξ is the disjoint union of Ξ_λ.*

LEMMA 3 (Y. Katuta [8]). *A regular T_1-space X is paracompact if and only if any open cover of X has an order locally finite open refinement.*

Proof of Theorem 1. Let $\mathfrak{B}$ be an open base of X. We can choose a subcollection $\mathfrak{B}'_\xi$ of $\mathfrak{B}$ such that $\mathfrak{B}'_\xi$ is locally finite in X and $F_\xi \subset \mathfrak{B}'^{\#}_\xi \subset U_\xi$ for each $\xi<\alpha$. Then it follows from Lemma 2 that the collection $\bigcup_{\xi<\alpha} \mathfrak{B}'_\xi$ is an order locally finite open cover of X. So any open base of X contains an order locally finite cover of X. By Lemma 3, X is paracompact. We first construct transfinite sequences $\{\mathfrak{B}_\xi: \xi<\alpha\}$ of subcollections of $\mathfrak{B}$ and $\{V_\xi: \xi<\alpha\}$ of open sets in X satisfying the following conditions: For each $\xi<\alpha$,

(1) $\mathfrak{B}_\xi$ is locally finite in X.

(2) $F_\xi - \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta \subset \mathfrak{B}^{\#}_\xi \subset U_\xi$.

(3) $F_\xi \subset V_\xi \subset \mathrm{Cl}\, V_\xi \subset \bigcup_{\eta\leq\xi} \mathfrak{B}^{\#}_\eta \cap U_\xi$.

(4) $\mathrm{Cl}(\bigcup_{\eta<\xi} V_\eta) \cap \mathfrak{B}^{\#}_\xi = \emptyset$.

We can choose a subcollection $\mathfrak{B}_0$ of $\mathfrak{B}$ and an open set V_0 such that $\mathfrak{B}_0$ is locally finite in X and $F_0 \subset V_0 \subset \mathrm{Cl}\, V_0 \subset \mathfrak{B}^{\#}_0 \subset U_0$. Assume that $\{\mathfrak{B}_\eta: \eta<\xi\}$ and $\{V_\eta: \eta<\xi\}$ have already been constructed. We show that

$$\mathrm{Cl}(\bigcup_{\eta<\xi} V_\eta) \subset \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta .$$

Let $x \in X - \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta$. We take some $\zeta<\alpha$ such that $x \in F_\zeta$. By (3),

$$\bigcup_{\eta<\xi} F_\eta \subset \bigcup_{\eta<\xi} \mathrm{Cl}\, V_\eta \subset \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta .$$

So we have $\zeta \geq \xi$. By order locally finiteness of $\{U_\xi: \xi<\alpha\}$, we can choose an open nbd G'_x of x such that for some $\eta_1, \ldots, \eta_m < \zeta$, $G'_x \cap U_\eta = \emptyset$ if $\eta \neq \eta_1, \ldots, \eta_m$. Let $G_x = G'_x - \bigcup \{\mathrm{Cl}\, V_{\eta_i}: i = 1, \ldots, m, \eta_i<\xi\}$. Then G_x is an open nbd of x such that $G_x \cap \bigcup_{\eta<\xi} V_\eta = \emptyset$. Hence we have $x \in X - \mathrm{Cl}(\bigcup_{\eta<\xi} V_\eta)$. $F_\xi - \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta$ is clearly totally paracompact relative to X. It is easy to show that there exists a subcollection $\mathfrak{B}_\xi$

of $\mathfrak{B}$ such that $\mathfrak{B}_\xi$ is locally finite in X, $F_\xi - \bigcup_{\eta<\xi} \mathfrak{B}^{\#}_\eta \subset \mathfrak{B}^{\#}_\xi \subset U_\xi$ and $\mathfrak{B}^{\#}_\xi \cap \mathrm{Cl}(\bigcup_{\eta<\xi} V_\eta) = \emptyset$. Then $F_\xi \subset \bigcup_{\eta\leq\xi} \mathfrak{B}^{\#}_\eta \cap U_\xi$. Since X is normal, we can take an open set V_ξ in X such that $F_\xi \subset V_\xi \subset \mathrm{Cl}\, V_\xi \subset \bigcup_{\eta\leq\xi} \mathfrak{B}^{\#}_\eta \cap U_\xi$. Thus we have constructed the desired transfinite sequences $\{\mathfrak{B}_\xi: \xi<\alpha\}$ and $\{V_\xi: \xi<\alpha\}$. Put $\tilde{\mathfrak{B}} = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$. By (3), $\tilde{\mathfrak{B}}$ is a subcover of $\mathfrak{B}$. Let $x \in X$. We take some $\xi_0<\alpha$ such that $x \in F_{\xi_0}$. By (4), $V_{\xi_0} \cap \mathfrak{B}^{\#}_\xi = \emptyset$ for any $\xi>\xi_0$. We can choose an open nbd H'_x of x such that for some $\xi_1, \ldots, \xi_n<\xi_0$ $H'_x \cap U_\xi = \emptyset$ if any $\xi \neq \xi_1, \ldots, \xi_n$. By (1), $\mathfrak{B}_{\xi_0} \cup (\bigcup_{i=1}^{n} \mathfrak{B}_{\xi_i})$ is locally finite in X. Hence we can also choose an open nbd H_x of x which intersects at most finite members of $\bigcup_{\xi\leq\xi_0} \mathfrak{B}_\xi$. Then $H_x \cap V_{\xi_0}$ is an open nbd of x which intersects at most finite members of $\tilde{\mathfrak{B}}$. We have shown that $\tilde{\mathfrak{B}}$ is a locally finite subcover of $\mathfrak{B}$. The proof is complete.

The following two corollaries are immediate consequences of Theorem 1.

COROLLARY 4 (D. W. Curtis [3]). *Every paracompact space X which is the countable union of closed subspaces totally paracompact relative to X is totally paracompact.*

COROLLARY 5 (R. Telgársky [12]). *If a regular T_1-space X has two order locally finite covers $\{C_\xi: \xi<\alpha\}$ and $\{U_\xi: \xi<\alpha\}$ such that C_ξ is compact and U_ξ is an open nbd of C_ξ for each $\xi<\alpha$, then X is totally paracompact.*

R. Telgársky showed in [12] that the space X described in Corollary 5 has a closure-preserving cover by compact sets. So the following theorem is another generalization of Corollary 5.

THEOREM 6 (Y. Yajima [14]). *If a paracompact space X has a σ-closure-preserving cover by compact sets, then X is totally paracompact.*

A regular T_1-space X is said to be a *Hurewicz space* [7] if for every sequence $\mathfrak{U}_1, \mathfrak{U}_2, \ldots$ of open covers of X there exists a cover $\mathfrak{B}$ of X such that $\mathfrak{B} = \bigcup_{i=1}^{\infty} \mathfrak{B}_i$, where $\mathfrak{B}_i$ is a finite subcollection of $\mathfrak{U}_i$ for each $i \in N$.

Let $\boldsymbol{H}$ be the class of all Hurewicz spaces. $\boldsymbol{H}$*-like space* is defined as in [13] (p. 195).

THEOREM 7 [1]. *The following are equivalent for a space X.*

(a) *X is a Hurewicz space.*

(b) *X is $\boldsymbol{H}$-like.*

Proof. (a)→(b): It is obvious.

(b)→(a): Let s be a winning strategy of player I in $G(\boldsymbol{H}, X)$ and let $\{\mathfrak{U}_n: n\geq 1\}$ be a sequence of open covers of X. We set $E_0 = X$ and $E_1 = s(E_0)$. Since $E_1 \in \boldsymbol{H}$,

(1) Theorem 7 and its proof have been pointed out to the authors by the referee. This theorem has a better form than the original one. The authors would like to thank the referee for several helpful comments.

there exists a sequence $\{\mathfrak{U}_{n1}: n\geqslant 1\}$ such that $\mathfrak{U}_{n1}$ is a finite subcollection of $\mathfrak{U}_n$ for each $n\geqslant 1$, and $E_1\subset \bigcup_{n=1}^{\infty} \mathfrak{U}_{n1}^{\#}$. We set $E_2 = X-\bigcup_{n=1}^{\infty} \mathfrak{U}_{n1}^{\#}$ and $E_3 = s(E_0, E_1, E_2)$. Since $E_3 \in \boldsymbol{H}$, there exists a sequence $\{\mathfrak{U}_{n2}: n\geqslant 2\}$ such that $\mathfrak{U}_{n2}$ is a finite subcollection of $\mathfrak{U}_n$ for each $n\geqslant 2$, and $E_3\subset \bigcup_{n=2}^{\infty} \mathfrak{U}_{n2}^{\#}$. We set

$$E_4 = E_2-\bigcup_{n=2}^{\infty} \mathfrak{U}_{n2}^{\#} \quad \text{and} \quad E_5 = s(E_0, E_1, E_2, E_3, E_4).$$

Continuing in that matter, we get a play $(E_0, E_1, E_2, \ldots)$ of $G(\boldsymbol{H}, X)$. Since s is a winning strategy, we have $\bigcap_{n=1}^{\infty} E_{2n} = \emptyset$. Therefore $\{\bigcup_{n=k}^{\infty} \mathfrak{U}_{nk}^{\#}: k\geqslant 1\}$ is an open cover of X. So, we set $\mathfrak{V}_n = \bigcup_{m=1}^{n} \mathfrak{U}_{nm}$ for each $n\geqslant 1$. Then $\mathfrak{V}_n$ is a finite subcollection of $\mathfrak{U}_n$ for each $n\geqslant 1$ and $\bigcup_{n=1}^{\infty} \mathfrak{V}_n$ is an open cover of X. The proof is complete.

Remark 8. Theorem 7 is a generalization of 5.5 in [13]. Moreover, since each σ-compact space is a Hurewicz space, it is easy to see from Theorem 7 and 10.2 of [13] that each Lindelöf regular T_1-space with σ-closure-preserving cover by compact sets is a Hurewicz space.

2. Total metacompactness. A space X is said to be *totally metacompact* [12] if every open base of X contains a point-finite cover of X.

THEOREM 9. *If a metacompact T_1-space X has a σ-closure-preserving closed cover by compact sets, then X is totally metacompact.*

The proof is quite similar to that of our Theorem 6. So the detail of it is left to the reader.

A base $\mathfrak{B}$ of a space X is said to be a *uniform base* [1], if for each $x\in X$, any infinite subcollection of $\mathfrak{B}$, each member of which contains x, is a local base at x. A base $\mathfrak{B}$ of a space X is said to be a *weakly uniform base* [6] if for each $x\in X$, the intersection of any infinite subcollection of $\mathfrak{B}$, each member of which contains x, is $\{x\}$. A collection $\mathfrak{A}$ of subsets of X is said to be *weakly uniform* [6] if no two points of X belong to infinitely many members of $\mathfrak{A}$. Clearly, a uniform base of a T_1-space is a weakly uniform base and a weakly uniform base is a base being weakly uniform.

R. W. Heath and W. F. Lindgren [6] raised the question of whether or not every base of a space with a weakly uniform base contains such a base. First, we consider the following generalized form instead of the above.

Let $\mathfrak{P}$ be a property of bases of a space X.

"Suppose that a space X has a base having property $\mathfrak{P}$. Does every base of X contain a base having property $\mathfrak{P}$?"

In this section, we study the two cases that $\mathfrak{P}$ is the property being a uniform base and being a weakly uniform base. They seem to be closely related to total metacompactness.

THEOREM 10. *The following are equivalent for a space X with a uniform base.*

(a) *X is totally metacompact.*

(b) *Every base of X contains a uniform base.*

Proof. (a)→(b): Let $\mathfrak{B}$ be any base of X. It is well-known that a space with a uniform base is developable. So we can easily construct a development $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ such that $\mathfrak{B}\supset\mathfrak{B}_1\supset\mathfrak{B}_2\supset\ldots$ and each $\mathfrak{B}_i$ is a base of X. By (a), $\mathfrak{B}_i$ contains a point-finite subcover $\mathfrak{A}_i$ of X for each $i\in N$. Put $\tilde{\mathfrak{B}} = \bigcup_{i=1}^{\infty} \mathfrak{A}_i$. Every infinite subcollection of $\tilde{\mathfrak{B}}$, each member of which contains a point x of X, can not be contained in $\bigcup_{i=1}^{n} \mathfrak{A}_i$ for each $n\in N$. Since $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ is a development of X, $\tilde{\mathfrak{B}}$ is a uniform base of X.

(b)→(a): Let $\mathfrak{B}$ be any base of X. By (b), $\mathfrak{B}$ contains a uniform base $\mathfrak{B}'$ of X. It is sufficient to show that $\mathfrak{B}'$ contains a point-finite subcover of X. Put

$$\mathfrak{A} = \{B\in\mathfrak{B}': B \text{ is not properly contained in any set } B',\ B'\in\mathfrak{B}'\}.$$

It is known in [1] that $\mathfrak{A}$ is a point-finite subcover of $\mathfrak{B}'$. The proof is complete.

Let N^N be the space of all irrational numbers with the usual topology. It is known that N^N is not totally metacompact ([2], [4]). By Theorem 10, we can see that N^N has a base containing no uniform base. Moreover, we obtain the following result which is a negative answer for the question of R. W. Heath and W. F. Lindgren ([6], Question 4).

EXAMPLE 11. The space N^N of all irrational numbers has a base containing no weakly uniform base.

It is well-known that N^N is considered as the product space of $\{N_i\}_{i=1}^{\infty}$, where each N_i is a copy of N. Let $B(k) = \{1, \ldots, k\}\times \prod_{i=2}^{\infty} N_i$ and let $B(x_1, \ldots, x_{n-1}; k) = \{x_1\}\times\ldots\times\{x_{n-1}\}\times\{1, \ldots, k\}\times \prod_{i=n+1}^{\infty} N_i$ for each $x_i\in N_i$ and $k\in N$. Moreover, let $\mathfrak{B}_1 = \{B(k): k\in N\}$ and let $\mathfrak{B}_n = \{B(x_1, \ldots, x_{n-1}; k): x_i\in N_i,\ i = 1, \ldots, n-1$ and $k\in N\}$ for $n\geqslant 2$. Then $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ is an open base of N^N (This base $\mathfrak{B}$ was considered in [4] in order to prove non-total metacompactness of N^N). We show that the base $\mathfrak{B}$ contains no weakly uniform base of N^N. Let $\mathfrak{A}$ be any weakly uniform subcollection of $\mathfrak{B}$. Let $\mathfrak{A}_n = \mathfrak{A}\cap\mathfrak{B}_n$. For each $B\in\mathfrak{A}_1$, $(1, 1, 1, \ldots)\in B$ and $(1, 2, 1, 1, \ldots)\in B$. By the assumption of $\mathfrak{A}$, $\mathfrak{A}_1$ is a finite collection. Then we can take some $y_1\in N$ such that $\{y_1\}\times\prod_{i=2}^{\infty} N_i\cap\mathfrak{A}_1^{\#} = \emptyset$. By the same argument,

$$\{B(x_1; k)\in\mathfrak{A}_2: x_1 = y_1, k\in N\}$$

is a finite subcollection of $\mathfrak{A}_2$. Then we can take some $y_2\in N$ such that

$$\{y_1\}\times\{y_2\}\times\prod_{i=3}^{\infty} N_i\cap(\bigcup\{B(x_1; k)\in\mathfrak{A}_2: x_1 = y_1, k\in N\}) = \emptyset.$$

Since $\{y_1\}\times\{y_2\}\times\prod_{i=3}^{\infty}N_i\cap B(x_1;k)=\emptyset$ if $x_1\neq y_1$, we have

$$\{y_1\}\times\{y_2\}\times\prod_{i=3}^{\infty}N_i\cap\mathfrak{A}_2^{\#}=\emptyset.$$

By induction, we can choose a sequence $\{y_1, y_2, \ldots\}\subset N$ such that

$$\{y_1\}\times\ldots\times\{y_n\}\times\prod_{i=n+1}^{\infty}N_i\cap\mathfrak{A}_n^{\#}=\emptyset \quad \text{for each } n\in N.$$

Then $(y_1, y_2, y_3, \ldots)\notin\bigcup_{n=1}^{\infty}\mathfrak{A}_n^{\#}=\mathfrak{A}^{\#}$. Hence we have shown that $\mathfrak{A}$ is not a cover of N^N. Thus the base $\mathfrak{B}$ of N^N contains no weakly uniform cover.

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Accepté par la Rédaction le 21. 3. 1977