Universal continua *

by

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Abstract. It is shown that there is a metric continuum which can be mapped onto all $\mathcal{T}$-like continua if and only if $\mathcal{T}$ is the collection {arc}, {circle}, or {arc, circle}.

Introduction. A continuum is a nondegenerate compact connected metric space. A map is a continuous function. All polyhedra in this paper are connected, finitely triangulable spaces. A continuum X is $\mathcal{P}$-like [10], where $\mathcal{P}$ is a collection of polyhedra, provided X is homeomorphic to the inverse limit of an inverse sequence whose factor spaces are elements of $\mathcal{P}$ and whose bonding maps are onto. Let $\mathcal{C}$ be a collection of continua. A space X is a model for $\mathcal{C}$ provided X is a continuum, and if $Y\in\mathcal{C}$, then X can be mapped onto Y. A universal continuum for $\mathcal{C}$ is a space X such that X is a model for $\mathcal{C}$ and $X\in\mathcal{C}$.

It is well known from the Hahn–Mazurkiewicz theorem that an arc is a universal continuum for locally connected continua. W. Kuperberg [8] has shown that the cone over the Cantor set is universal for uniformly pathwise connected continua. Mioduszewski [11], Lelek [9], and Fearnley [4] have shown that the pseudo-arc is universal for arc-like continua, in fact for weakly chainable or p-chainable continua. J. T. Rogers [12] has shown that there is a pseudo-solenoid which is universal for circle-like continua, in fact for $\mathcal{P}$-like continua where $\mathcal{P}=$ {arc, circle}, or even further for q-chainable continua.

On the other hand, in 1934 Waraszkiewicz [14] exhibited a collection of plane continua which has no model. Based on this result, Bellamy [2] has shown that there is no model for indecomposable continua. In this paper it is shown that:

(i) there is no model for planar tree-like continua;

(ii) there is no model for arc-connected continua;

(iii) there is no model for planar indecomposable tree-like continua;

(iv) if $\mathcal{P}$ is a collection of polyhedra containing a polyhedron of dimension greater than 1, then every tree-like continuum is $\mathcal{P}$-like;

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(v) if $\mathscr{P}$ is a collection of polyhedra other than {arc}, {circle}, or {arc, circle}, then there is no model for $\mathscr{P}$-like continua.

Hereditarily indecomposable tree-like continua which are not arc-like play a central role in unsolved problems concerning hereditarily equivalent continua [3], homogeneous plane continua [7], and the fixed point property [1]. Mioduszewski and Rogers have shown that the universal property can be used to construct hereditarily indecomposable continua. Our results show that this approach cannot be used to construct hereditarily indecomposable $\mathscr{P}$-like continua except in the cases $\mathscr{P}$ = {arc}, {circle}, or {arc, circle}.

J. W. Rogers [13] has posed the question: given a collection $\mathscr{G}$ of continua, is there an element of $\mathscr{G}$ which can be mapped onto every element of $\mathscr{G}$? Our results provide a negative answer for many collections.

Section 1. In this section we define a plane continuum M, and an uncountable collection $\mathscr{T}$ of subcontinua of M. Every $T\in\mathscr{T}$ will be a tree-like continuum and will consist of a piecewise linear spiral about a simple triod. $\mathscr{T}$ has the property that if H is any continuum and f and g are two maps of H into M such that $f(H), g(H)\in\mathscr{T}$ and

$$\varrho(f,g) = \sup_{x\in H} d(f(x), g(x)) < \tfrac{1}{1500},$$

where d is the usual metric on the plane, then $f(H) = g(H)$. Thus, if H is a continuum which can be mapped onto every element of $\mathscr{T}$, then there exists an uncountable collection $\mathscr{T}'$ of maps from H into M such that if $f, g\in\mathscr{T}'$, then $\varrho(f,g)\geqslant\frac{1}{1500}$. There is a theorem due to Borsuk that if H and M are two compact metric spaces, then $C(H,M)$, the space of all maps from H into M metrized by ϱ, is separable. However, $\mathscr{T}'\subseteq C(H,M)$ and thus $C(H,M)$ is not separable. This contradiction shows that no such H can exist.

In the plane let Y be the simple triod defined in polar coordinates by

$$Y = \{(r,\theta)\colon 0\leqslant r\leqslant 1,\ \theta = 0,\ \tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi\}.$$

For $n = 0, 1, 2, \ldots$ let $q(n)$ be the point in the plane defined by

$$q(n) = \left(1+\frac{1}{n+1}, 0\right).$$

Note that the distance from $q(0)$ to $q(1)$ is $\frac{1}{2}$.

For $n = 1, 2, 3, \ldots$ define in polar coordinates

$$s(n) = \left(\frac{1}{n}, \frac{\pi}{3}\right),$$

$$t(n) = \left(1+\frac{1}{n}, \frac{2\pi}{3}\right),$$

$$u(n) = \left(\frac{1}{n}, \pi\right),$$

$$v(n) = \left(1+\frac{1}{n}, \frac{4\pi}{3}\right),$$

$$w(n) = \left(\frac{1}{n}, \frac{5\pi}{3}\right).$$

If p and q are two points in the plane, let $\overline{pq}$ be the straight line segment joining p to q. Let W be the spiral about Y defined by

$$W = \bigcup_{n=1}^{\infty} \overline{q(n-1)s(n)} \cup \overline{s(n)t(n)} \cup \overline{t(n)u(n)} \cup \overline{u(n)v(n)} \cup \overline{v(n)w(n)} \cup \overline{w(n)q(n)}.$$

Let V be the spiral about Y defined by

$$V = \bigcup_{n=1}^{\infty} \left(\overline{q(n-1)w(n)} \cup \overline{w(n)v(n)} \cup \overline{v(n)u(n)} \cup \overline{u(n)t(n)} \cup \overline{t(n)s(n)} \cup \overline{s(n)q(n)}\right).$$

We define $M = Y\cup W\cup V$ (see Fig. 1).

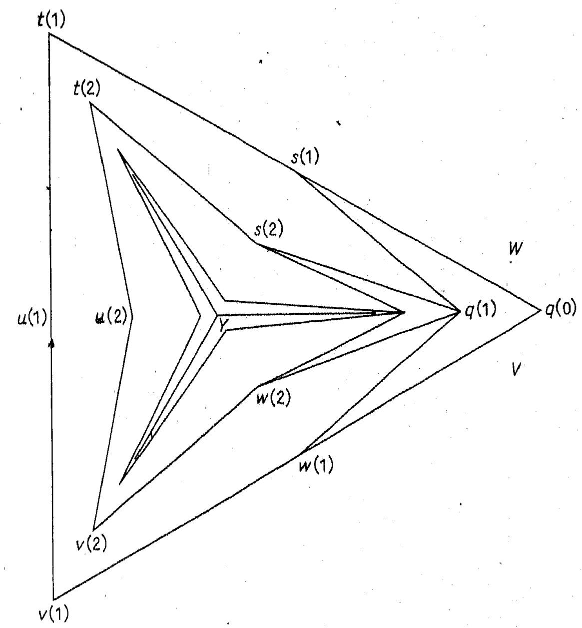

Fig. 1

To every $x \in [1,2)$, associate a sequence, denoted by $\operatorname{seq} x$, where $\operatorname{seq} x = 1$, $E(10\cdot x)$, $E(10^2\cdot x)$, ..., $E(10^n\cdot x)$, ..., and $E(\ )$ means the integer part. If x has two decimal representations, use only the one that does not terminate in an infinite string of nines. Thus

$$\operatorname{seq} 1.246 = 1, 12, 124, 1246, 12460, \ldots$$

Note that if $\operatorname{seq} x = 1, n_1, n_2, \ldots, n_k, \ldots$, then

$$\ldots < 1-n_1+n_2-n_3 < 1-n_1 < 0 < 1 < 1-n_1+n_2 < 1-n_1+n_2-n_3+n_4 < \ldots;$$

in fact, $n_k > 1+n_1+\ldots+n_{k-1}$. Also, define

$$\nu(k, x) = 1-n_1+n_2-\ldots+(-1)^k n_k,$$
$$\eta(k, x) = 1+n_1+n_2+\ldots+n_k.$$

If there is no chance of confusion, we will simply write $\nu(k)$ for $\nu(k, x)$ and $\eta(k)$ for $\eta(k, x)$. If $y \in [1,2)$ and $\operatorname{seq} y = 1, m_1, m_2, \ldots, m_k, \ldots$, then $m_k > \eta(k-1, x)$ and $0 < \nu(k, x) < \nu(k+2, y)$ if k is even, and $0 > \nu(k, x) > \nu(k+2, y)$ if k is odd.

We associate with x a subcontinuum T_x of M determined by $\operatorname{seq} x$ in the following manner:

$$T_x = Y \cup \overline{q(0)q(1), W} \cup \overline{q(1)q(\eta(1)), V} \cup$$
$$\cup \overline{q(\eta(1))q(\eta(2)), W} \cup \overline{q(\eta(2))q(\eta(3)), V} \cup \ldots,$$

where $\overline{rs, D}$ is the arc from r to s in D. Less precisely but more clearly, T_x is Y plus a spiral about Y determined by one "turn" on W starting at $q(0)$ followed by n_1 turns on V, then by n_2 turns on W, etc. Define

$$\mathscr{T} = \{T_x \colon x \in [1,2)\},$$

then $\mathscr{T}$ is an uncountable collection of subcontinua of M.

THEOREM 1. *Let H be a continuum and $T_x \in \mathscr{T}$. If $f\colon H \to T_x$ is a map, then for any arc $\overline{ab} \subseteq f(H) \setminus Y$, there exists a subcontinuum H_1 of H so that $f(H_1) = \overline{ab}$.*

Proof. Without loss of generality, let a lie between $q(0)$ and b in T_x. Now, $f^{-1}(\overline{ab})$ is closed. Further, some component H_1 of $f^{-1}(\overline{ab})$ intersects both $f^{-1}(a)$ and $f^{-1}(b)$. (Suppose not. Then $f^{-1}(\overline{ab}) = P \cup Q$, where P and Q are closed, $P \cap Q = \emptyset$, and $f^{-1}(a) \subseteq P$ and $f^{-1}(b) \subseteq Q$. But this means we can disconnect H since

$$H = \big(P \cup f^{-1}(\overline{q(0)a})\big) \cup \big(Q \cup f^{-1}(T_x \setminus \overline{q(0)b})\big).$$

This is a contradiction since H is connected.) In fact, H_1 is a subcontinuum of H. Also, $f(H_1)$ is a subcontinuum of $\overline{ab}$ containing a and b, and $\overline{ab}$ is irreducible between a and b. Therefore, $f(H_1) = \overline{ab}$. ■

The next theorem and its proof are essentially translations of the originals given by Waraszkiewicz in [14].

THEOREM 2. *Let H be a continuum, and let $f\colon H \to T_x \setminus Y$ and $g\colon H \to T_y \setminus Y$ be maps so that*

(i) $f(H) = \overline{ab, T_x}$ where a precedes b (considering $T_x \setminus Y$ ordered linearly from $q(0)$ outward) and $g(H) = \overline{ce, T_y}$ where c precedes e;

(ii) $|\overline{q(0)a, T_x}| < \frac{1}{150}$;

(iii) for every $r \in g(f^{-1}(a))$, $|\overline{q(0)r, T_y}| < \frac{1}{150}$ where $|\text{arc}|$ means the length of the arc.

If $\varrho(f, g) < \frac{1}{150}$, then there are two maps $\varphi\colon I \to T_x$ and $\psi\colon I \to T_y$, where $I = [0, 1]$, such that

(a) $\varrho(\varphi, \psi) \leqslant \frac{1}{30}$,

(b) $\varphi(0) = \psi(0) = q(0)$ and $\varphi(1) = b$,

(c) $\varphi(I) = \overline{q(0)b, T_x}$ and $\psi(I) = \overline{q(0)e, T_y}$.

Proof. If $X \subseteq T_x \setminus Y$ or $T_y \setminus Y$, then $\overline{X, x}$ or $\overline{X, y}$ will denote the smallest arc containing X in $T_x \setminus Y$ or $T_y \setminus Y$, respectively. Let $h \in H$, then there exist points $u, v \in T_x \setminus Y$ and $w, z \in T_y \setminus Y$ such that $f(h) \in \operatorname{Int}\overline{uv, T_x}$ and $g(h) \in \operatorname{Int}\overline{wz, T_y}$ and $|\overline{uv, T_x}| < \frac{1}{150}$ and $|\overline{wz, T_y}| < \frac{1}{150}$. Now, $L = \operatorname{Int}\overline{uv, T_x}$ and $N = \operatorname{Int}\overline{wz, T_y}$ are open. Let $U(h) = f^{-1}(L) \cap g^{-1}(N)$, then $U(h)$ is an open neighborhood of h;

$$|\overline{f(U(h)), x}| < \tfrac{1}{150}; \quad \text{and} \quad |\overline{g(U(h)), y}| < \tfrac{1}{150}.$$

The collection of all such $U(h)$ as h ranges over H is an open cover of H. But H is compact, so some finite collection $\mathscr{U}$ of such sets covers H. Since H is connected we can choose a sequence $U_1, \ldots, U_n$ of elements of $\mathscr{U}$ satisfying the conditions:

1° $f^{-1}(a) \cap U_1 \neq \emptyset \neq f^{-1}(b) \cap U_n$,

2° $U_k \cap U_{k+1} \neq \emptyset$ for $k = 1, 2, \ldots, n-1$,

3° every element of $\mathscr{U}$ occurs at least once in the sequence.

We will denote the endpoints of $\overline{f(U_k), x}$ by $a(k)$ and $b(k)$ where $a(k)$ precedes $b(k)$, and the endpoints of $\overline{g(U_k), y}$ by $c(k)$ and $e(k)$ where $c(k)$ precedes $e(k)$. Thus $\overline{f(U_k), x} = \overline{a(k)b(k), T_x}$ and $\overline{g(U_k), y} = \overline{c(k)e(k), T_y}$. Further, we have $|\overline{q(0)a(k), T_x}| \leqslant |\overline{q(0)b(k), T_x}|$ for all k. Now,

$$\overline{ab, T_x} = \bigcup_{k=1}^{n} \overline{a(k)b(k), T_x} \quad \text{and} \quad \overline{ce, T_y} = \bigcup_{k=1}^{n} \overline{c(k)e(k), T_y}.$$

Clearly, $a = a(1)$ and $b = b(n)$ and $c = c(l)$ for some $l \leqslant n$. Now, 2° implies that

$$\overline{a(k)b(k), T_x} \cap \overline{a(k+1)b(k+1), T_x} \neq \emptyset$$

and

$$\overline{c(k)e(k), T_y} \cap \overline{c(k+1)e(k+1), T_y} \neq \emptyset$$

for $k = 1, 2, \ldots, n-1$. Let $b(0) = a(1)$, $e(0) = c(1)$, $a(0) = a(1)$, and $c(0) = c(1)$. Then for every $k = 1, 2, \ldots, n$

$$(1)\qquad \begin{aligned} |\overline{b(k-1)b(k), T_x}| &\leqslant \max\{|\overline{a(k-1)b(k-1), T_x}|, \quad |\overline{a(k)b(k), T_x}|\} < \tfrac{1}{150},\\ |\overline{e(k-1)e(k), T_y}| &\leqslant \max\{|\overline{c(k-1)e(k-1), T_y}|, \quad |\overline{c(k)e(k), T_y}|\} < \tfrac{1}{150}. \end{aligned}$$

Note that

$$(2)\qquad d(b(k), e(k)) \leqslant d(b(k), f(u)) + d(f(u), g(u)) + d(g(u), e(k)) < \tfrac{3}{150} = \tfrac{1}{50}$$

where $u \in U_k$. Also,

$$(3)\qquad |\overline{q(0)b(1), T_x}| = |\overline{q(0)b(0), T_x}| + |\overline{b(0)b(1), T_x}| < \tfrac{2}{150}$$

since $b(0) = a(1) = a$ and by (ii) and (1). Moreover,

$$(4)\qquad |\overline{q(0)e(1), T_y}| = |\overline{q(0)e(0), T_y}| + |\overline{e(0)e(1), T_y}| < \tfrac{2}{150}$$

since $e(0) = c(1)$ and by (1) since there is a $u \in U_1$ so that $f(u) = a$ by 1° and thus by (iii) $|\overline{q(0)g(u), T_y}| < \frac{1}{150}$, but

$$g(u) \in g(U_1), y = \overline{c(1)e(1), T_y}$$

and thus $|\overline{q(0)c(1), T_y}| \leqslant |\overline{q(0)g(u), T_y}|$. Now,

$$\overline{q(0)b, T_x} = \overline{q(0)b(1), T_x} \cup \Big(\bigcup_{k=2}^{n} \overline{b(k-1)b(k), T_x}\Big)$$

and

$$\overline{q(0)e, T_y} = \overline{q(0)e(1), T_y} \cup \Big(\bigcup_{k=2}^{n} \overline{e(k-1)e(k), T_y}\Big).$$

Divide I into n equal intervals $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$. Define $\varphi\colon I \to T_x$ for $z \in \left[0, \frac{1}{n}\right]$ by $\varphi(z) \in \overline{q(0)b(1), T_x}$ such that

$$\left(\frac{1}{n} - z\right)|\overline{q(0)\varphi(z), T_x}| = (z-0)|\overline{\varphi(z)b(1), T_x}|;$$

and for $z \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $k = 2, 3, \ldots, n$ by $\varphi(z) \in \overline{b(k-1)b(k), T_x}$ such that

$$\left(\frac{k}{n} - z\right)|\overline{b(k-1)\varphi(z), T_x}| = \left(z - \frac{k-1}{n}\right)|\overline{\varphi(z)b(k), T_x}|.$$

Define $\psi\colon I \to T_y$ similarly by replacing b with e and T_x with T_y. Clearly, φ and ψ are continuous and

$$\varphi\left(\left[0, \frac{1}{n}\right]\right) = \overline{q(0)b(1), T_x}, \qquad \psi\left(\left[0, \frac{1}{n}\right]\right) = \overline{q(0)e(1), T_y}$$

and for $k = 2, 3, \ldots, n$

$$\varphi\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \overline{b(k-1)b(k), T_x}, \qquad \psi\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \overline{e(k-1)e(k), T_y}.$$

So $\varphi(0) = \psi(0) = q(0)$, $\varphi(1) = b(n) = b$, and $\varphi(I) = \overline{q(0)b, T_x}$, $\psi(I) = \overline{q(0)e, T_y}$.

Finally, if $z \in \left[0, \frac{1}{n}\right]$ then

$$\begin{aligned} d(\varphi(z), \psi(z)) &\leqslant d(q(0), \varphi(z)) + d(q(0), \psi(z)) \leqslant |\overline{q(0)b(1), T_x}| + |\overline{q(0)e(1), T_y}|\\ &< \tfrac{2}{150} + \tfrac{2}{150} = \tfrac{4}{150} < \tfrac{1}{30} \end{aligned}$$

by (3) and (4) since $\varphi(z) \in \overline{q(0)b(1), T_x}$, $\psi(z) \in \overline{q(0)e(1), T_y}$. If $z \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $k = 2, \ldots, n$, then

$$\begin{aligned} d(\varphi(z), \psi(z)) &\leqslant d(\varphi(z), b(k)) + d(b(k), e(k)) + d(e(k), \psi(z))\\ &< |\overline{b(k-1)b(k), T_x}| + \tfrac{1}{50} + |\overline{e(k-1)e(k), T_y}|\\ &< \tfrac{1}{150} + \tfrac{1}{50} + \tfrac{1}{150} = \tfrac{1}{30} \end{aligned}$$

by (2) and (1). Thus $\varrho(\varphi, \psi) \leqslant \frac{1}{30}$. ■

Define $p\colon (\boldsymbol{R}, 0) \to (Y, (1, 0))$ by

$$p(z) = \begin{cases} (6(n+\frac{1}{6}-z), 0) & \text{if } z \in [n, n+\frac{1}{6}],\\ (6(z-n-\frac{1}{6}), \frac{2}{3}\pi) & \text{if } z \in [n+\frac{1}{6}, n+\frac{1}{3}],\\ (6(n+\frac{1}{2}-z), \frac{2}{3}\pi) & \text{if } z \in [n+\frac{1}{3}, n+\frac{1}{2}],\\ (6(z-n-\frac{1}{2}), \frac{4}{3}\pi) & \text{if } z \in [n+\frac{1}{2}, n+\frac{2}{3}],\\ (6(n+\frac{5}{6}-z), \frac{4}{3}\pi) & \text{if } z \in [n+\frac{2}{3}, n+\frac{5}{6}],\\ (6(z-n-\frac{5}{6}, 0) & \text{if } z \in [n+\frac{5}{6}, n+1] \end{cases}$$

in polar coordinates for n an integer. The continuity of p is clear. Define

$$r_x\colon (T_x \setminus Y, q(0)) \to (Y, (1, 0))$$

by

$$r_x(q(n)) = (1, 0), \qquad r_x(s(n)) = r_x(u(n)) = r_x(w(n)) = 0,$$
$$r_x(t(n)) = (1, \tfrac{2}{3}\pi), \qquad r_x(v(n)) = (1, \tfrac{4}{3}\pi),$$

and extend r_x linearly to the rest of $T_x \setminus Y$. So r_x is continuous and is a projection of $T_x \setminus Y$ onto Y. Define $r_x'\colon (T_x \setminus Y, q(0)) \to (\boldsymbol{R}, 0)$ by

$$r_x'(q(0)) = 0, \qquad r_x'(q(1)) = 1, \qquad r_x'(q(\eta(k)+l)) = v(k) + (-1)^{k+1} l$$

where l is an integer and $1\leqslant l\leqslant n_{k+1}$;

$$r'_x(s(1)) = \tfrac16,\qquad r'_x(s(2)) = \tfrac16,$$

$$r'_x(s(\eta(k)+l)) = \begin{cases} v(k)+(-1)^{k+1}l+\frac16 & \text{if } k \text{ is even},\\ v(k)+(-1)^{k+1}l-\frac56 & \text{if } k \text{ is odd};\end{cases}$$

$$r'_x(t(1)) = \tfrac13,\qquad r'_x(t(2)) = \tfrac13,$$

$$r'_x(t(\eta(k)+l)) = \begin{cases} v(k)+(-1)^{k+1}l+\frac13 & \text{if } k \text{ is even},\\ v(k)+(-1)^{k+1}l-\frac23 & \text{if } k \text{ is odd};\end{cases}$$

$$r'_x(u(1)) = \tfrac12,\qquad r'_x(u(2)) = \tfrac12,$$

$$r'_x(u(\eta(k)+l)) = \begin{cases} v(k)+(-1)^{k+1}l+\frac12 & \text{if } k \text{ is even},\\ v(k)+(-1)^{k+1}l-\frac12 & \text{if } k \text{ is odd};\end{cases}$$

$$r'_x(v(1)) = \tfrac23,\qquad r'_x(v(2)) = \tfrac23,$$

$$r'_x(v(\eta(k)+l)) = \begin{cases} v(k)+(-1)^{k+1}l+\frac23 & \text{if } k \text{ is even},\\ v(k)+(-1)^{k+1}l-\frac13 & \text{if } k \text{ is odd};\end{cases}$$

$$r'_x(w(1)) = \tfrac56,\qquad r'_x(w(2)) = \tfrac56,$$

$$r'_x(w(\eta(k)+l)) = \begin{cases} v(k)+(-1)^{k+1}l+\frac56 & \text{if } k \text{ is even},\\ v(k)+(-1)^{k+1}l-\frac16 & \text{if } k \text{ is odd}.\end{cases}$$

Extend r'_x linearly to the rest of $T_x\setminus Y$; then r'_x is continuous. It may be checked that $p\circ r'_x = r_x$. If $\theta\colon (I,0)\to(T_x\setminus Y,\, q(0))$ is continuous, then θ' will mean $r'_x\circ\theta$.

By critical points, we mean the points $(1,0)$, $(1,\frac23\pi)$, $(1,\frac43\pi)$.

LEMMA 1. *Consider $p\colon (\mathbf{R},0)\to(Y,(1,0))$. If $z_1\in\mathbf{R}$ is such that $p(z_1)$ is a critical point and if $\frac56\geqslant|z_1-z_2|\geqslant\frac1{10}$, then $d(p(z_1),p(z_2))>\frac1{30}$.*

Proof. Suppose $p(z_1)=(1,\frac23\pi)$, then $z_1=n+\frac13$ for some integer n. Since $\frac56\geqslant|z_1-z_2|\geqslant\frac1{10}$, then $n-\frac12\leqslant z_2\leqslant n+\frac7{30}$ or $n+\frac{13}{30}\leqslant z_2\leqslant n+1+\frac16$. Thus, $p(z_2)=(a,0)$ for $0\leqslant a\leqslant 1$ or $p(z_2)=(a,\frac43\pi)$ for $0\leqslant a\leqslant 1$ or $p(z_2)=(a,\frac23\pi)$ for $0\leqslant a\leqslant\frac25$. In any case, $d(p(z_1),p(z_2))\geqslant\frac35>\frac1{30}$. The cases where $p(z_1)=(1,0)$ or $p(z_1)=(1,\frac43\pi)$ are similar. ■

THEOREM 3. *Let $\varphi\colon (I,0)\to(T_x\setminus Y,\, q(0))$ and $\psi\colon (I,0)\to(T_y\setminus Y,\, q(0))$ be two maps, where $T_x, T_y\in\mathscr{T}$. If $\varrho(\varphi,\psi)\leqslant\frac1{30}$, then $|\varphi'(z)-\psi'(z)|<\frac1{10}$ for $z\in I$ such that $r_x(\varphi(z))$ or $r_y(\psi(z))$ is a critical point.*

Proof. Suppose not. There is a first $z_0\in I$ such that $r_x(\varphi(z_0))$ or $r_y(\psi(z_0))$ is a critical point and $|\varphi'(z_0)-\psi'(z_0)|\geqslant\frac1{10}$. We note $0<z_0$; and, for any $z<z_0$ such that $r_x(\varphi(z))$ or $r_y(\psi(z))$ is a critical point, then $|\varphi'(z)-\psi'(z)|<\frac1{10}$. It is clear that if $d(\varphi(z),\psi(z))\leqslant\frac1{30}$, then $d(r_x(\varphi(z)),r_y(\psi(z)))\leqslant\frac1{30}$.

Case 1. $\frac56\geqslant|\varphi'(z_0)-\psi'(z_0)|\geqslant\frac1{10}$. By Lemma 1, since $p\varphi'(z_0)=p\circ r'_x\circ\varphi(z_0) = r_x\varphi(z_0)$ and $p\psi'(z_0)=r_y\psi(z_0)$, then $d(p\varphi'(z_0),p\psi'(z_0)) = d(r_x\varphi(z_0),r_y\psi(z_0))>\frac1{30}$. This is a contradiction.

Case 2. $|\varphi'(z_0)-\psi'(z_0)|>\frac56$. Let $\bar z$ be the last number less than z_0 so that $r_x\varphi(\bar z)$ or $r_y\psi(\bar z)$ is a critical point and $|\varphi'(\bar z)-\psi'(\bar z)|<\frac1{10}$. $\bar z$ clearly exists.

Suppose $r_x\varphi(z_0)=(1,0)$, so $\varphi'(z_0)=n$ for some integer n and either $\psi'(z_0)<n-\frac56$ or $\psi'(z_0)>n+\frac56$. Now if $r_x\varphi(\bar z)=(1,0)$, then $\varphi'(\bar z)=n$ (since if $\varphi'(\bar z)\leqslant n-1$ or $\varphi'(\bar z)\geqslant n+1$, then there is z' so that $\bar z<z'<z_0$ and $\varphi'(z')=n-\frac23$ or $\varphi'(z')=n+\frac13$ respectively. Thus, $r_x\varphi(z')=p\varphi'(z')=(1,\frac23\pi)$. Since $z'<z_0$, $|\varphi'(z')-\psi'(z')|<\frac1{10}$ contradicting the fact that $\bar z$ is last.) But $|\varphi'(\bar z)-\psi'(\bar z)|<\frac1{10}$ so $n-\frac1{10}<\psi'(\bar z)<n+\frac1{10}$. Since $\psi'(z_0)<n-\frac56$ or $\psi'(z_0)>n+\frac56$, then there exists z' so that $\bar z<z'<z_0$ and $\psi'(z')=n-\frac23$ or $\psi'(z')=n+\frac13$ respectively. Thus $r_y\psi(z')=p\psi'(z')=(1,\frac23\pi)$. Since $z'<z_0$, $|\varphi'(z')-\psi'(z')|<\frac1{10}$, contradicting the fact that $\bar z$ is last.

If $r_x\varphi(\bar z)=(1,\frac23\pi)$, then $\varphi'(\bar z)=n+\frac13$ (since if $\varphi'(\bar z)\leqslant n-1+\frac13$ or $\varphi'(\bar z)\geqslant n+1+\frac13$ we can find z' such that $\bar z<z'<z_0$ and $\varphi'(z')=n-1+\frac23$ or $\varphi'(z')=n+\frac23$ respectively. Thus, $r_x\varphi(z')=p\varphi'(z')=(1,\frac43\pi)$ and $|\varphi'(z')-\psi'(z')|<\frac1{10}$ which is a contradiction.) But $|\varphi'(\bar z)-\psi'(\bar z)|<\frac1{10}$ so $n+\frac7{30}<\psi'(\bar z)<n+\frac{13}{30}$. Since $\psi'(z_0)<n-\frac56$ or $\psi'(z_0)>n+\frac56$, there exists z' so that $\bar z<z'<z_0$ and $\psi'(z')=n-\frac13$ or $\psi'(z')=n+\frac23$ respectively. Thus, $r_y\psi(z')=p\psi'(z')=(1,\frac43\pi)$ and $|\varphi'(z')-\psi'(z')|<\frac1{10}$, a contradiction.

If $r_x\varphi(\bar z)=(1,\frac43\pi)$, the proof is similar.

If $r_y\psi(\bar z)=(1,0)$, then $\psi'(\bar z)\leqslant n-1$ or $\psi'(\bar z)\geqslant n+1$ (since if $n-1<\psi'(\bar z)<n+1$, then $\psi'(\bar z)=n$. But $\psi'(z_0)<n-\frac56$ or $\psi'(z_0)>n+\frac56$ so that there is z' with $\bar z<z'<z_0$ and $\psi'(z')=n-\frac23$ or $\psi'(z')=n+\frac13$ respectively. Thus, $r_y\psi(z')=p\psi'(z')=(1,\frac23\pi)$ and $|\varphi'(z')-\psi'(z')|<\frac1{10}$, a contradiction.) But $|\varphi'(\bar z)-\psi'(\bar z)|<\frac1{10}$ so $\varphi'(\bar z)<n-1+\frac1{10}$ or $\varphi'(\bar z)>n+1-\frac1{10}$. Since $\varphi'(z_0)=n$, there is z' with $\bar z<z'<z_0$ and $\varphi'(z')=n-\frac23$ or $\varphi'(z')=n+\frac13$ respectively. Thus, $r_x\varphi(z')=p\varphi'(z')=(1,\frac23\pi)$ and $|\varphi'(z')-\psi'(z')|<\frac1{10}$, a contradiction.

If $r_y\psi(\bar z)=(1,\frac23\pi)$ or $(1,\frac43\pi)$, the proof is similar.

The other two cases of values of $r_x\varphi(z_0)$ and three cases of values of $r_y\psi(z_0)$, each with six subcases of values of $r_x\varphi(\bar z)$ and $r_y\psi(\bar z)$, are also similar. ■

THEOREM 4. *Let $x,y\in[1,2)$ with $x\neq y$. Let*

$$\operatorname{seq} x = 1, n_1, n_2, \dots, n_k, n_{k+1}, n_{k+2}, \dots$$

and

$$\operatorname{seq} y = 1, n_1, n_2, \dots, n_k, m_{k+1}, m_{k+2}, \dots,$$

where $n_{k+1}>m_{k+1}$, i.e. $x>y$ and x and y first disagree at the $(k+1)$-st decimal place. Let $\varphi\colon (I,0)\to(T_x\setminus Y,\, q(0))$ and $\psi\colon (I,0)\to(T_y\setminus Y,\, q(0))$ be maps such that $\varphi(I)\subseteq q(0)q(\eta(k,x)+m_{k+1}+1), T_x$ and $\psi(I) = q(0)q(\eta(k+2,y)), T_y$ with $\psi(1) = q(\eta(k+2,y))$. Then there exists a $z_0\in I$ such that $d(\varphi(z_0),\psi(z_0))>\frac1{30}$.

Proof. Suppose for every $z\in I$ that $d(\varphi(z),\psi(z))\leqslant\frac1{30}$, then by Theorem 3, $|\varphi'(z)-\psi'(z)|<\frac1{10}$ for every $z\in I$. In particular, $|\varphi'(1)-\psi'(1)|<\frac1{10}$, but $\psi'(1) = r'_y(\psi(1)) = r'_y(q(\eta(k+2,y))) = v(k+2,y)$. However, $v(k,x)=v(k,y)$ is the smallest negative value $\varphi'(1)$ can assume if k is odd and is the largest positive value

$\varphi'(1)$ can assume if k is even. Now, if k is even, $\psi'(1)$ is positive so $|\psi'(1)-\varphi'(1)|$ is minimized when $\varphi'(1)$ is the largest positive value it can assume. If k is odd, $\psi'(1)$ is negative so $|\psi'(1)-\varphi'(1)|$ is minimized when $\varphi'(1)$ is the smallest negative value it can assume. Therefore, in either case

$$|\varphi'(1)-\psi'(1)| = |\psi'(1)-\varphi'(1)\geqslant|(-1)^{k+1}m_{k+1}+(-1)^{k+2}m_{k+2}|>1>\tfrac{1}{10}$$

which is a contradiction. So there is a $z_0\in I$ such that $d(\varphi(z_0),\psi(z_0))>\frac{1}{30}$. ∎

THEOREM 5. *Let* x, y, seq$x$, seq$y$ *be as above. Let* $\varphi\colon (I,0)\to(T_x\setminus Y, q(0))$ *and* $\psi\colon (I,0)\to(T_y\setminus Y, q(0))$ *be maps such that* $\varphi(I)=\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x}$ *with* $\varphi(1)=q(\eta(k,x)+m_{k+1}+1)$. *Then there exists a* $z_0\in I$ *so that*

$$d(\varphi(z_0),\psi(z_0))>\tfrac{1}{30}.$$

Proof. Suppose not. Then $d(\varphi(z),\psi(z))\leqslant\frac{1}{30}$ for every $z\in I$. But $\varphi(1)=q(\eta(k,x)+m_{k+1}+1)$ and thus $r_x\varphi(1)=1$. Thus, by Theorem 3, $|\varphi'(1)-\psi'(1)|<\frac{1}{10}$. Now, $\varphi'(1)=\nu(k,x)+(-1)^{k+1}(m_{k+1}+1)=\nu(k+1,y)+(-1)^{k+1}$.

CLAIM. $\psi(1)\notin\overline{q(0)q(\eta(k+2,y)), T_y}$.

Case 1. Suppose $\psi(1)\in\overline{q(0)q(\eta(k+1,y)), T_y}$, then $\nu(k+1,y)$ is the smallest negative value $\psi'(1)$ can assume if $k+1$ is odd and is the largest positive value it can assume if $k+1$ is even. If $k+1$ is even, $\varphi'(1)$ is positive, so $|\varphi'(1)-\psi'(1)|$ is minimized when $\psi'(1)$ is the largest positive value it can assume. If $k+1$ is odd, $\varphi'(1)$ is negative, so $|\varphi'(1)-\psi'(1)|$ is minimized when $\psi'(1)$ is the smallest negative value it can assume. Thus, in either case, $|\varphi'(1)-\psi'(1)|=|(-1)^{k+1}|=1>\frac{1}{10}$ which is a contradiction.

Case 2. Suppose $\psi(1)\in\overline{q(\eta(k+1,y))q(\eta(k+2,y)), T_y}$, then $|\psi'(1)-\varphi'(1)|=|\nu(k+1,y)+(-1)^{k+2}l-\nu(k+1,y)+(-1)^{k+2}|=|(-1)^{k+2}(l+1)|=l+1>1>\frac{1}{10}$ where $0\leqslant l\leqslant m_{k+2}$ and l is a real number, not necessarily an integer. But this is a contradiction. Hence $\psi(0)=q(0)$ and

$$\psi(1)\in T_y\setminus\overline{q(0)q(\eta(k+2,y)), T_y}.$$

Since ψ is continuous and I is a continuum, there is a first $u\in I$ so that $\psi(u)=q(\eta(k+2,y))$. Let $h\colon I\to[0,u]$ be defined by $h(x)=x\cdot u$, then h is continuous and $h(0)=0$, $h(1)=u$. Let $\varphi''=\varphi\circ h$ and $\psi''=\psi\circ h$, then $\varphi''\colon(I,0)\to(T_x\setminus Y, q(0))$ and $\psi''\colon(I,0)\to(T_y\setminus Y, q(0))$ are continuous and

$$\varphi''(I)\subseteq\varphi(I)=\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x},$$
$$\psi''(I)=\psi([0,u])=\overline{q(0)q(\eta(k+2,y)), T_y}$$

with $\psi''(1)=\psi(u)=q(\eta(k+2,y))$. Hence, by Theorem 4, there is a $z_1\in I$ so that $d(\varphi''(z_1),\psi''(z_1))>\frac{1}{30}$. Let $z_0=h(z_1)$, then $z_0\in I$ and $\varphi(z_0)=\varphi(h(z_1))=\varphi''(z_1)$, $\psi(z_0)=\psi(h(z_1))=\psi''(z_1)$. Thus $d(\varphi(z_0),\psi(z_0))>\frac{1}{30}$ which is a contradiction. ∎

THEOREM 6. *Let* H *be a continuum. Let* $T_x, T_y\in\mathscr{I}$ *and* $f\colon H\to T_x$ *and* $g\colon H\to T_y$ *be onto maps. If* $\varrho(f,g)<\frac{1}{1500}$, *then* $x=y$, *i.e.*, $T_x=T_y$.

Proof. Suppose $x\neq y$. We assume without loss of generality that $x>y$. Let seq$x=1, n_1, \ldots, n_k, n_{k+1}, n_{k+2}, \ldots$ and seq$y=1, n_1, \ldots, n_k, m_{k+1}, m_{k+2}, \ldots$ where $n_{k+1}>m_{k+1}$ since $x>y$. By Theorem 1, there is a subcontinuum H_1 of H so that $f(H_1)=\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x}$.

Case 1. $g(H_1)\subseteq T_y\setminus Y$. We apply Theorem 2 with H_1 for H, $q(0)$ for a, $q(\eta(k,x)+m_{k+1}+1)=b$, $g(H_1)=\overline{ce, T_y}$, where c precedes e. Note $|\overline{q(0)a, T_x}|=0<\frac{1}{150}$ and $r\in g(f^{-1}(a))$ implies $d(q(0),r)<\frac{1}{1500}$ which in turn implies $|\overline{q(0)r, T_y}|<\frac{1}{150}$. Also, $\varrho(f,g)<\frac{1}{1500}<\frac{1}{150}$. Thus, there are maps $\varphi\colon(I,0)\to(T_x\setminus Y, q(0))$ and $\psi\colon(I,0)\to(T_y\setminus Y, q(0))$ with $\varphi(I)=\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x}$ and $\varphi(1)=q(\eta(k,x)+m_{k+1}+1)$ and $\varrho(\varphi,\psi)\leqslant\frac{1}{30}$. But this contradicts Theorem 5.

Case 2. $g(H_1)\cap Y\neq\emptyset$. Now, there is an $h_1\in H_1$ such that $f(h_1)=q(0)$, but $d(f(h_1),g(h_1))=d(q(0),g(h_1))<\frac{1}{1500}$. Also, there exists an $h_0\in H_1$ so that $g(h_0)\in Y$. But $g(H_1)$ is connected, so $T_y\setminus(Y\cup\overline{q(0)g(h_1), T_y})\subseteq g(H_1)$. Let $a=g(h_1)$, then applying Theorem 1 to H_1 and g, there is a subcontinuum H_2 of H_1 so that $g(H_2)=\overline{aq(\eta(k+2,y)), T_y}$. We know $f(H_2)\subseteq f(H_1)=\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x}$. We apply Theorem 2 with H_2 for H, g for f, f for g, x for y, y for x, a as above, $b=q(\eta(k+2,y))$, and $f(H_2)=\overline{ce, T_x}$ where c precedes e. We note $d(q(0),a)<\frac{1}{1500}$, so $|\overline{q(0)a, T_y}|<\frac{1}{150}$. Also, if $r\in f(g^{-1}(a))$, then $d(q(0),r)\leqslant d(q(0),a)+d(a,r)<\frac{1}{1500}+\frac{1}{1500}=\frac{1}{750}$ which implies $|\overline{q(0)r, T_x}|<\frac{1}{150}$. Furthermore,

$$\sup_{h\in H_2} d(f(h),g(h))\leqslant\tfrac{1}{1500}<\tfrac{1}{150}.$$

Thus, by Theorem 2 there are maps $\varphi\colon(I,0)\to(T_y\setminus Y, q(0))$ and $\psi\colon(I,0)\to(T_x\setminus Y, q(0))$ with $\varphi(I)=\overline{q(0)q(\eta(k+2,y)), T_y}$ and $\varphi(1)=q(\eta(k+2,y))$ and

$$\psi(I)=\overline{q(0)e, T_x}=\overline{q(0)c, T_x}\cup\overline{ce, T_x}=\overline{q(0)c, T_x}\cup f(H_2)$$
$$\subseteq\overline{q(0)q(\eta(k,x)+m_{k+1}+1), T_x}$$

and $\varrho(\varphi,\psi)\leqslant\frac{1}{30}$. But this contradicts Theorem 4 with φ and ψ interchanged. ∎

Remarks. It is easily seen that every element of $\mathscr{I}$ is a planar λ-dendroid. So there is no model for planar λ-dendroids. Since every element of $\mathscr{I}$ is one-dimensional, we have also shown there is no model for (planar) one-dimensional continua.

Note also that, for every x, $T_x\times I$ is aposyndetic. Further, $T_x\times I$ can be mapped onto T_x by projection. Thus, there can be no model for aposyndetic continua. If there were a model H, it could be mapped onto $T_x\times I$ and hence onto T_x, for every $x\in[1,2)$. This contradicts the fact that there is no model for $\mathscr{T}$. So H cannot exist. It is, however, still an open question whether there exists a model for planar aposyndetic continua.

The uncountable collection $\mathcal{S}$ of planar continua which Waraszkiewicz presented is

$$\mathcal{S} = \{S_x: x \in [1, 2)\}$$

where S_x is defined exactly as T_x except that for S_x the alternating spiral limits on a unit circle instead of on a simple triod. Each S_x can be mapped onto the corresponding T_x in an obvious fashion, so there is no model for $\mathcal{S}$.

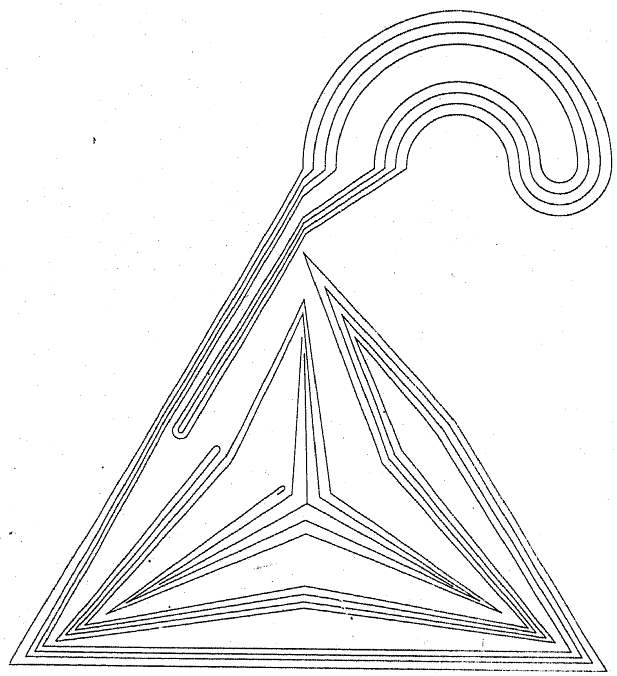

Fig. 2

Section 2. There is an uncountable collection $\mathcal{I}$ of indecomposable tree-like plane continua which has no model. We construct for every $T_x \in \mathcal{T}$ an indecomposable tree-like continuum $I(T_x)$ in the plane such that T_x is a retract of $I(T_x)$.

We construct $I(T_x)$ by "blending" Knaster's indecomposable continuum K with T_x in the manner shown in Figure 2. We define $\mathcal{I} = \{I(T_x): T_x \in \mathcal{T}\}$. If a continuum H could be mapped onto every $I(T_x) \in \mathcal{I}$, then H could be mapped onto every $T_x \in \mathcal{T}$. But we have just shown this is impossible. So $\mathcal{I}$ has no model.

Each continuum in $\mathcal{I}$ contains a simple triod. Since this paper was written, Ingram has constructed an uncountable collection of planar atriodic indecomposable tree-like continua without a model [5] and an uncountable collection of planar hereditarily indecomposable tree-like continua without a model [6].

There is no model for arc-connected continua. We follow the same plan as in Section 1. We construct an uncountable collection $\mathcal{F}$ of arc-connected continua, all subcontinua of a continuum N. We show $\mathcal{F}$ has the property that if H is a continuum and f and g are two maps of H into N such that $f(H)$, $g(H) \in \mathcal{F}$ and $\varrho(f, g) < \frac{1}{1500}$, then $f(H) = g(H)$. Again this is sufficient to show that H is not a model for $\mathcal{F}$.

Refer back to Section 1. Define

$$M_0 = M = \{(x, y, 0): (x, y) \in M\},$$

i.e., M_0 is M considered embedded in $\mathbf{R}^3$. Define

$$M_n = \left\{\left(x, y, \frac{1}{n}\right): (x, y) \in M\right\} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

$$Z_1 = \{(2, 0, z): 0 \leqslant z \leqslant 1\},$$
$$Z_2 = \{(2, 0, z): -1 \leqslant z \leqslant 0\},$$
$$Z_3 = \{(1, 0, z): -1 \leqslant z \leqslant 0\},$$
$$Z_4 = \{(x, 0, -1): 1 \leqslant x \leqslant 2\}.$$

Define $N = (\bigcup_{n=0}^{\infty} M_n) \cup (\bigcup_{n=1}^{4} Z_n)$, then N is a continuum. Note that $M = M_0$ is a retract of $N \setminus Z_4$ simply by defining $r: N \setminus Z_4 \to M$ by $r(x, y, z) = (x, y, 0)$. Recall from Section 1 how seq$x$ and T_x are defined for $x \in [1, 2)$. Define

$$T_x(0) = T_x = \{(u, v, 0): (u, v) \in T_x\},$$

$$T_x(n) = \left\{\left(u, v, \frac{1}{n}\right): (u, v) \in \overline{q(0)q(n), T_x}\right\} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

We associate with x a subcontinuum F_x of N defined by

$$F_x = (\bigcup_{n=0}^{\infty} T_x(n)) \cup (\bigcup_{n=1}^{4} Z_n).$$

It is easily seen that F_x is an arc-connected continuum. Define $\mathcal{F} = \{F_x: x \in [1, 2)\}$. Note that $T_x = T_x(0)$ is a retract of $F_x \setminus Z_4$ where the retraction map is $r|F_x \setminus Z_4$, written hereafter simply as r.

THEOREM 7. *Let H be a continuum and $F_x \in \mathcal{F}$. If $f: H \to F_x$ is an onto map, then for any $n \geqslant 1$, there exists a subcontinuum H_1 of H so that $f(H_1) = T_x(n)$.*

Proof. The proof is exactly the same as that of Theorem 1 with a few changes in notation. ■

THEOREM 8. *Let H be a continuum. Let $F_x, F_y \in \mathcal{F}$ and $f: H \to F_x$ and $g: H \to F_y$ be onto maps. If $\varrho(f, g) < \frac{1}{1500}$, then $x = y$, i.e., $F_x = F_y$.*

Proof. Suppose $x \neq y$. We assume without loss of generality that $x > y$. Let $\operatorname{seq} x = 1, n_1, \ldots, n_k, n_{k+1}, n_{k+2}, \ldots$ and $\operatorname{seq} y = 1, n_1, \ldots, n_k, m_{k+1}, m_{k+2}, \ldots$ where $n_{k+1} > m_{k+1}$ since $x > y$. By Theorem 7, there is a subcontinuum H_1 of H so that $f(H_1) = T_x(\eta(k, x) + m_{k+1} + 1)$. Now, $\varrho(f, g) < \frac{1}{1500}$, so $f(H_1) \subseteq F_x \setminus Z_4$ and $g(H_1) \subseteq F_y \setminus Z_4$. Thus, $rf(H_1) = \overline{q(0)q(\eta(k, x) + m_{k+1} + 1, T_x)}$ and $rg(H_1) \subseteq T_y$ and

$$\sup_{h \in H_1} d(rf(h), rg(h)) \leqslant \sup_{h \in H_1} d(f(h), g(h)) < \tfrac{1}{1500}.$$

The remainder of the proof is exactly the same as that of Theorem 6 starting from Case 1 with rf in place of f and rg in place of g. ■

Remarks. Unfortunately, the F_x in $\mathscr{F}$ are not planar. Thus, it is still an open question whether there exists a model for planar arc-connected continua.

Section 3. In this section we generalize the results of Section 1 where we showed there was no model for Y-like continua. If $\mathscr{P}$ is any collection of polyhedra other than {arc}, {circle}, or {arc, circle}, we show there is no model for $\mathscr{P}$-like continua. The method used is to construct an uncounterable collection $\mathscr{D} = \{D_x \colon x \in [1, 2)\}$ of $\mathscr{P}$-like continua with the property that D_x can be mapped onto T_x for every x, or D_x can be mapped onto S_x for every x. Since there is no model for $\mathscr{T}$ or $\mathscr{S}$, there can be no model for $\mathscr{D}$.

PROPOSITION 1. *Let $\mathscr{P}$ be a collection of polyhedra. If $\mathscr{P}$ contains a polyhedron of dimension greater than* 1, *then every tree-like continuum is $\mathscr{P}$-like.*

Proof. Let X be a tree, R be a polyhedron of dimension n greater than 1, and $\varepsilon > 0$. It suffices to show there is an ε-map (diameters of point inverses are less than ε) from X onto R. Triangulate R, and let C be a simplex of maximal dimension. Then C is homeomorphic to E^n, for some $n \geqslant 2$, where

$$E^n = \{p \in \boldsymbol{R}^n \colon |p| \leqslant 1\}.$$

We will consider R to be imbedded in $\boldsymbol{R}^{2n+1}$ and C to be E^n.

X is a contractible finite union of arcs which intersect only in their endpoints. We may assume each arc is a straight line segment of length 1. We call an endpoint p of an arc a *free* point if $X \setminus \{p\}$ is connected; otherwise, we call p a *branch* point. If an arc has a free point, we call the arc free.

Let m be a positive integer such that $1/m < \varepsilon/2$. Divide each arc of X into m equal intervals. Pick a free arc of X. Divide C into $m-1$ concentric closed annular regions surrounding a central closed spherical region. By the Hahn–Mazurkiewicz theorem the first interval of the arc, starting at a free point, can be mapped onto $R \setminus C$ union the closed outermost annular region of C in such a way that both endpoints of the interval are mapped onto a point p_1 of the inner boundary of the annular region. The next interval of the arc can be mapped onto the next closed annular region of C so that the first endpoint of the interval is mapped onto p_1 and the other endpoint is mapped onto a point p_2 of the inner boundary of this annular region. Continue this process until the last interval of the arc. If the last endpoint is free, map the last interval onto the entire remaining closed region of C so that the first endpoint of the interval is mapped onto p_{m-1}. If the last endpoint is a branch point, divide the remaining closed region of C into a closed outer annular region surrounding an inner closed spherical region. Map the last interval onto the annular region as before with the branch point being mapped onto p_m on the common boundary of the annular and spherical regions. If there are k other arcs emanating from the branch point, then divide the remaining closed region into k closed regions each with the point p_m in its boundary. Using the above procedure, map each of these k arcs into a different one of the k closed regions. Eventually the process terminates, and all of the maps can be fitted together into one map of X onto R which is an ε-map. See Figures 3, 4, and 5 for examples of this process. ■

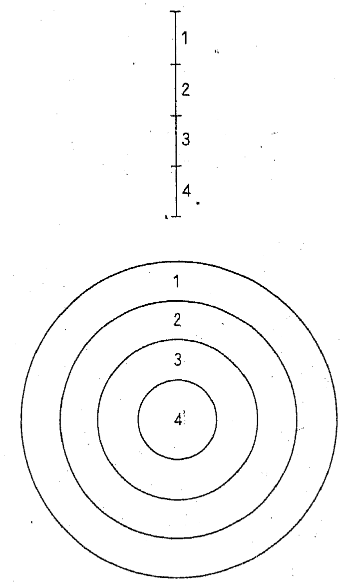

Fig. 3

THEOREM 9. *If $\mathscr{P}$ is a collection of polyhedra other than {arc}, {circle}, {arc, circle}, then there is no model for $\mathscr{P}$-like continua.*

Proof. It suffices to consider $\mathscr{P}$ which contain only one polyhedron R, and R is neither an arc nor a circle. We know from Section 1 that there is no model for Y-like continua. Suppose R has dimension greater than 1, then Proposition 1 says every Y-like continuum is R-like. Thus, there can be no model for R-like continua since such a model, if it existed, would be a model for Y-like continua. Therefore, we may assume that R has dimension 1. So R is either a tree (but not an arc) or a one dimensional continuum containing a simple closed curve as a proper subcontinuum.

Suppose R is a tree but not an arc, then R may be represented as

$$R = Y \cup A \cup B \cup C$$

where Y is the standard simple triod as defined in Section 1; A is a tree and $A \cap Y = \{(0,0)\}$ or $A = \emptyset$; B is a tree and $B \cap Y = \{(1, \frac{2}{3}\pi)\}$ or $B = \emptyset$; C is a tree and $C \cap Y = \{(1, \frac{4}{3}\pi)\}$ or $C = \emptyset$; and A, B, and C are disjoint. Define

$$\mathscr{R} = \{R(T_x)\colon\ x \in [1, 2)\}$$

where

$$R(T_x) = T_x \cup A' \cup B' \cup C'$$

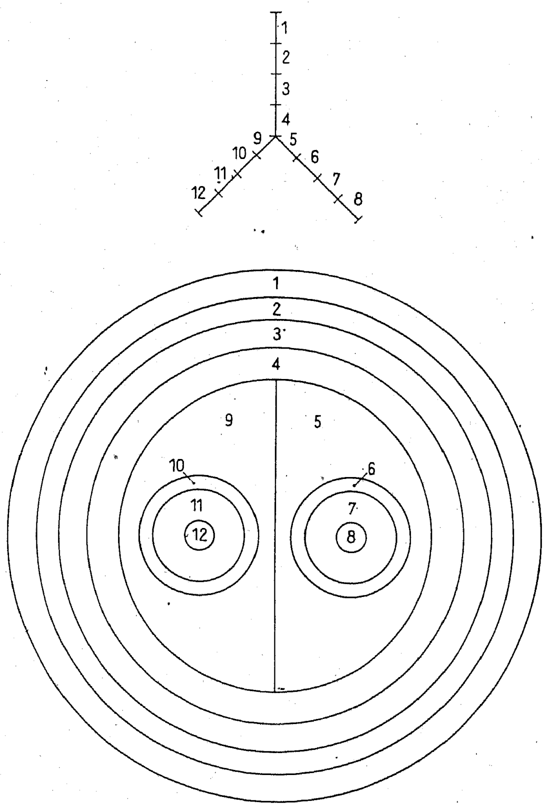

Fig. 4

and T_x is as defined in Section 1; $A' = \emptyset$ if $A = \emptyset$ or A' is is homeomorphic to A if $A \neq \emptyset$ and A' lies in $\boldsymbol{R}^3$ but not in $\boldsymbol{R}^2$ except for $(0, 0)$; $B' = \emptyset$ if $B = \emptyset$ or B' is

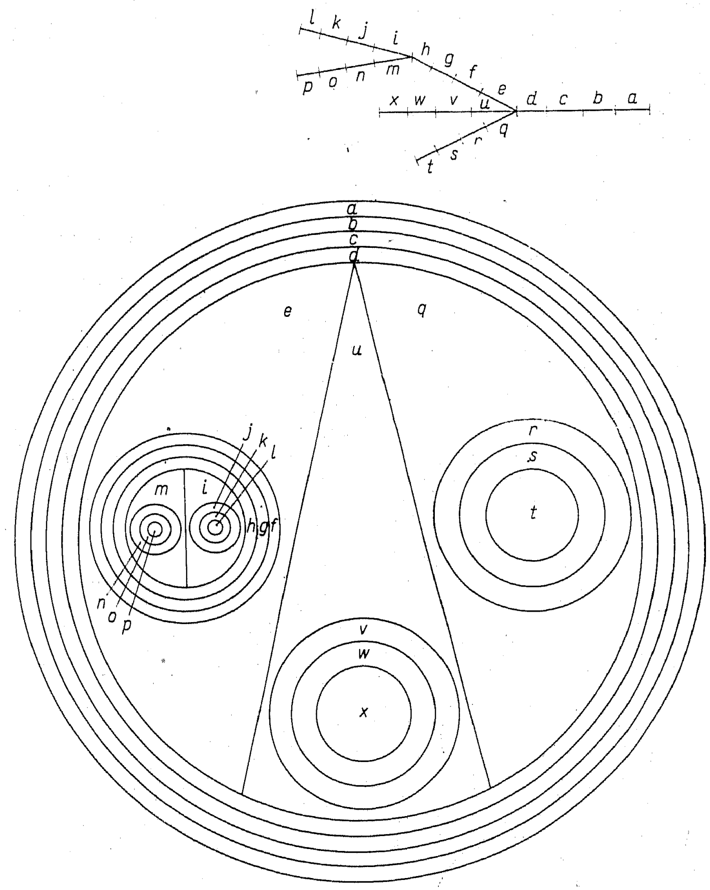

Fig. 5

homeomorphic to B if $B \neq \emptyset$ and B' lies in $\boldsymbol{R}^3$ but not in $\boldsymbol{R}^2$ except for $(1, \frac{2}{3}\pi)$; $C' = \emptyset$ if $C = \emptyset$ or C' is homeomorphic to C if $C \neq \emptyset$ and C' lies in $\boldsymbol{R}^3$ but not in $\boldsymbol{R}^2$ except for $(1, \frac{4}{3}\pi)$; and A', B' and C' are disjoint.

We can map $R(T_x)$ onto T_x by $f\colon R(T_x) \to T_x$ where

$$f(p) = \begin{cases} p & \text{if } p \in T_x, \\ (0, 0) & \text{if } p \in A', \\ (1, \frac{2}{3}\pi) & \text{if } p \in B', \\ (1, \frac{4}{3}\pi) & \text{if } p \in C'. \end{cases}$$

In fact, T_x is a retract of $R(T_x)$. We discern by the use of nerves of ε-covers that $\dot{R}(T_x)$ is R-like. But there can be no model for $\mathscr{R}$ since, if there were, it would also be a model for $\mathscr{T}$.

Finally, suppose R contains a simple closed curve, then R may be represented as

$$R = S^1 \cup G_1 \cup \ldots \cup G_k \cup A$$

where S^1 is the standard unit circle; $G_1, \ldots, G_k$ are arcs that all lie in the unbounded component of $\mathbf{R}^2 \setminus S^1$; $G_i \cap S^1 = p_i$ for $1 \leqslant i \leqslant k$ where p_i is one endpoint of G_i; $G_i \cap G_j = \emptyset$ or $G_i \cap G_j = p_i = p_j$ for $i \neq j$; A is a one-dimensional polyhedron (not necessarily connected) and $A \cap S^1 = \emptyset$; if $A \neq \emptyset$, then $A \cap (G_1 \cup \ldots \cup G_k)$ is a non-empty subset of the endpoints of the G_i's; and $k \geqslant 1$ (see Fig. 6).

Define $\mathscr{R} = \{R(S_x): x \in [1, 2)\}$ where $R(S_x)$ is the same as R except the k arcs of R are replaced by k spirals limiting on S^1 and each spiral alternates its direction

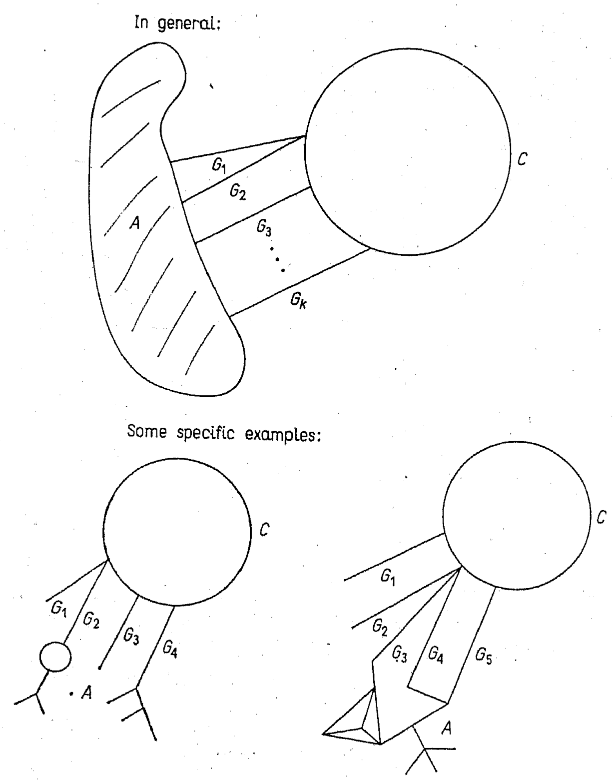

Fig. 6

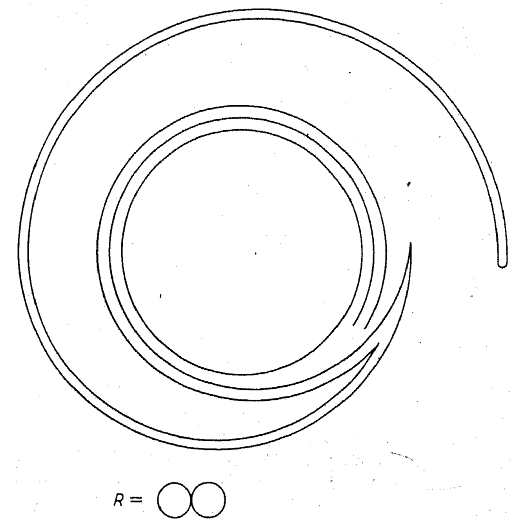

Fig. 7

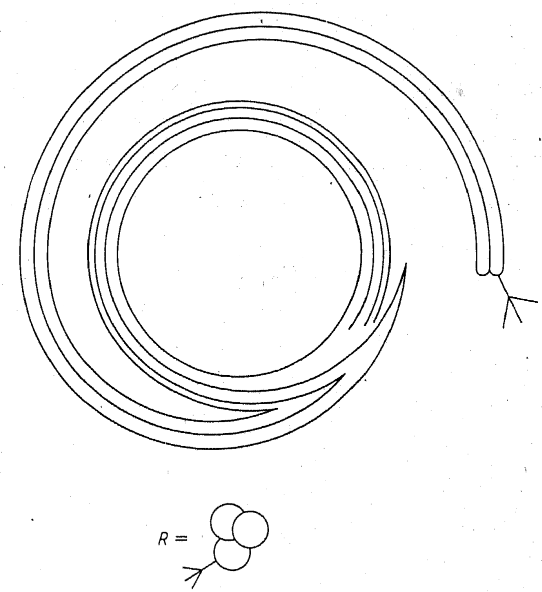

Fig. 8

as determined by x in the manner described in Section 1. See Figure 7 for a picture of $R(S_x)$ where R is a figure eight, and Figure 8 for another example.

We can map $R(S_x)$ onto S_x by mapping S^1 identically onto itself, all of the k spirals homeomorphically onto one of themselves, and A onto the endpoint of that spiral. It may be checked by means of nerves of ε-covers that $R(S_x)$ is R-like. But there can be no model for $\mathscr{R}$ since, if there were, it would also be a model for $\mathscr{S}$. ■

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Decompositions in the product of a measure space and a Polish space

by

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Abstract. Let X, $\mathcal{M}$, μ be a complete probability space and Y a Polish space with Borel field $\mathcal{B}_Y$. It is shown that if $A \in \mathcal{M}\otimes\mathcal{B}_Y$, then $\{x \in X;\ A(x) \text{ is } F_\sigma\}$ and $\{x \in X;\ A(x) \text{ is } F_{\sigma\delta}\}$ are both measurable. Furthermore, we prove the existence of "measurable decompositions". From those results, we deduce a theorem on the stability of the class of the Baire-2 functions under integration.

Introduction. Assume X, μ a probability space and let $\mathscr{M}$ be the σ-algebra of σ-measurable subsets of X. Let Y be a Polish space with Borel field $\mathscr{B}_Y$. By well known arguments, we obtain that if $A \in \mathscr{M}\otimes\mathscr{B}_Y$, then the sections $A(x)$, where x is taken in X, are of bounded Baire class. Hence $\mathscr{M}\otimes\mathscr{B}_Y$ is the union of the classes $\mathscr{S}_\alpha$ ($\alpha<\omega_1$), consisting of the sets $A \in \mathscr{M}\otimes\mathscr{B}_Y$, such that $A(x)$ is of Baire class at most α, for each $x \in X$, where the Baire class is defined with respect to the closed sets. Let $\mathscr{F}_0 = \mathscr{S}_0$, which is stable under countable intersections. Starting from $\mathscr{F}_0$, we obtain a Baire system $(\mathscr{F}_\alpha)_{\alpha<\omega_1}$. It is a natural question if $\mathscr{S}_\alpha$ and $\mathscr{F}_\alpha$ coincide for all $\alpha<\omega_1$. We will answer if affirmatively for $\alpha = 1$ and $\alpha = 2$.

Let $\mathscr{P} = \{M\times F;\ M \in \mathscr{M},\ F \text{ closed in } Y\}$. The class of the $\mathscr{P}$-analytic subsets of $X\times Y$ will be denoted by $\mathscr{A}(X, Y)$, or simply $\mathscr{A}$, if no confusion is possible. Let $A \in \mathscr{A}$ and assume $A = = \bigcup_{\nu}\bigcap_{k}(M_{\nu|k}\times F_{\nu|k})$, where ν runs over $\mathscr{N} = N^N$. In such a representation, it will be always assumed that

$$M_{\nu|k}\times F_{\nu|k} \neq \emptyset, \quad M_{\nu|k+1}\times F_{\nu|k+1} \subset M_{\nu|k}\times F_{\nu|k} \quad \text{and} \quad \operatorname{diam} F_{\nu|k} \leqslant 1/k,$$

for each $\nu \in \mathscr{N}$ and $k \in N$. It is easily seen that $\mathscr{A}$ contains $\mathscr{M}\otimes\mathscr{B}_Y$.

Definition 1. If $A \subset X\times Y$, then $\bar{A}^s \subset X\times Y$ is defined by $\bar{A}^s(x) = \overline{A(x)}$.

The following description of the set $\bar{A}^s$ will be useful. If $y \in Y$ and $\varepsilon>0$, then $B(y, \varepsilon)$ is the open ball with midpoint y and radius ε. Let $(y_n)_n$ be a dense sequence in Y. For every $n \in N$ and $k \in N$ we take $M_{nk} = \pi_1(A \cap (X\times B(y_n, 1/k)))$, where π_1 is the projection on X. Then $\bar{A}^s = \bigcap_k\bigcup_n(M_{nk}\times B(y_n, 1/k))$. From this observation, we obtain immediately: