On h-regular graded algebras of characteristic $p > 0$

by

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Abstract. A graded commutative algebra $A = \bigoplus_{i=0}^{\infty} A_i$ over a local Noetherian ring A_0 with the maximal ideal m is called $(h_1, \ldots, h_n)$-*regular* for some sequence $h_1, \ldots, h_n$, $h_i \in N \cup \{\infty\}$, N being positive rational integers, if the ideal $I = m \oplus (\bigoplus_{i>0} A_i)$ is generated by a sequence of homogeneous elements $u_1, \ldots, u_n$ such that $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k - 1})$, $k = 1, \ldots, n$, where h_k is the minimum of integers $m \geqslant 0$ such that $u_k^m = 0$ (if $u_k^m \neq 0$ for all m, then we put $h_k = \infty$ and assume that $u_k^{\infty - 1} = 0$). The paper concerns $(h_1, \ldots, h_n)$-regular graded algebras of characteristic p, where p is a prime. A criterion of $(ph_1, \ldots, ph_n)$-regularity of a graded algebra A of finite type over A_0 is proved and then some applications of that criterion to the investigation of the local structure of Hopf algebras over a perfect field of characteristic $p > 0$ is given.

Introduction. Let R be a commutative ring with identity and let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative graded R-algebra such that $A_0 = R$. A sequence $u_1, \ldots, u_n$ of homogeneous elements of A is called h-*regular in* A if $(u_1, \ldots, u_n) \neq A$ and $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k - 1})$, $k = 1, \ldots, n$, where $h_k = h(u_k)$ is the minimum of integers $m \geqslant 0$ such that $u_k^m = 0$ (if $u_k^m \neq 0$ for all m, then we put $h_k = \infty$ and $u_k^{\infty} = u_k^{\infty - 1} = 0$). A homogeneous ideal J in A is called $(h_1, \ldots, h_n)$-*regular*, for a given sequence $h_1, \ldots, h_n$, $h_i \in N^{\infty}$ (1), if there exists an h-regular (in A) sequence $u_1, \ldots, u_n$ such that $J = (u_1, \ldots, u_n)$ and $h(u_i) = h_i$ for $i = 1, \ldots, n$. The graded R-algebra A is said to be $(h_1, \ldots, h_n)$-*regular* if R is a local Noetherian ring and the unique homogeneous maximal ideal of A is $(h_1, \ldots, h_n)$-regular. The R-algebra A is said to be h-*regular* if it is $(h_1, \ldots, h_n)$-regular for some sequence $h_1, \ldots, h_n$, $h_j \in N^{\infty}$. The ring R is called $(h_1, \ldots, h_n)$-*regular* (h-*regular*) if the graded R-algebra $A = A_0 = R$ is $(h_1, \ldots, h_n)$-regular (h-regular) (2).

Let p be a fixed prime. The present paper concerns h-regular graded algebras A of characteristic p (i.e. such that $pA = 0$) and contains as the main result the following

(1) N denotes the set of all positive integers, $N^{\infty} = N \cup \{\infty\}$.

(2) The definition of an h-regular sequence as well as the definition of h-regular graded algebra come from [7].

THEOREM. *Suppose that R is a commutative local Noetherian ring of characteristic p with the maximal ideal m and that J is a homogeneous ideal in a commutative graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ satisfying the following conditions:*

(i) $J^{[p]} = \{x^p; x \in J\}$ *is an* $(l_1, \ldots, l_n)$*-regular ideal in the graded algebra* $A^{[p]} = \{a^p; a \in A\}$, *where* $n = \dim_{R/m}(J/MJ)$, $M = m \oplus (\bigoplus_{i>0} A_i)$,

(ii) $J/J^{[p]}A$ *is a* $(p_1, \ldots, p_n)$*-regular ideal in the graded algebra* $A/J^{[p]}A$ *with* $p_i = p$, $i = 1, \ldots, n$,

(iii) $\mathrm{Tor}_i^{A^{[p]}}(A^{[p]}/J^{[p]}, A) = 0$ *for* $i = 1, 2$.

Then J is a $(pl_1, \ldots, pl_n)$*-regular ideal in A* (1).

Making use of this theorem, one can prove (see Theorem 2.4) that a commutative graded R-algebra A (for R as above) is $(pl_1, \ldots, pl_n)$-regular, where $n = \dim_{R/m}(M/M^2)$, if and only if $A^{[p]}$ is an $(l_1, \ldots, l_n)$-regular graded algebra, $A/M^{[p]}A$ is a $(p_1, \ldots, p_n)$-regular graded algebra with $p_i = p$, $i = 1, \ldots, n$ and A is a flat graded $A^{[p]}$-module. This permits us to give in § 3 a new proof of the well-known Borel Theorem stating that a finitely generated commutative graded Hopf algebra over a perfect field k of characteristic p is, as a graded k-algebra, of the form $\bigotimes_{i=1}^{r} k[X_i]/(X_i^{h_i})$ where $h_i = p^{n_i}$ for some $n_i \in N^{\infty}$, $i = 1, \ldots, r$.

The theorem plays also an essential role in the proofs of Theorems A and B below, established in § 3.

THEOREM A. *If H is a commutative Hopf algebra over a perfect field k of characteristic p (for the definition see § 3) and $\mathfrak{R}$ is a prime ideal in H, then there exist $n_1, \ldots, n_r$, $n_i \in N^{\infty}$, such that the localization $H_{\mathfrak{R}}$ is a $(p^{n_1}, \ldots, p^{n_r})$-regular local ring whenever $H_{\mathfrak{R}}$ is a Noetherian ring.*

THEOREM B. *If H is a commutative Noetherian profinite Hopf k-algebra over a perfect field of characteristic p* (for def. see § 3) *and $\mathfrak{R}$ is a closed prime ideal in H, then there exist $n_1, \ldots, n_r$ in N^{∞} such that the completion of the local ring $H_{\mathfrak{R}}$ is a $(p^{n_1}, \ldots, p^{n_r})$-regular local ring.*

Theorem A is closely related to [1], Chap. III, § 3, Cor. 6.5, and Theorem B to the Dieudonné–Cartier–Gabriel Theorem [6], p. 556.

Throughout the paper all rings are assumed to be commutative (with identity) and all local rings are assumed (unless otherwise stated) to be Noetherian.

§ 1. Definitions, revision of known results and a preliminary lemma. By a graded algebra we shall mean in this paper a commutative (2) graded algebra $A = \bigoplus_{i=0}^{\infty} A_i$ such that A_0 is a local ring and A_i are finitely generated A_0-modules. If A is a graded algebra, then the unique maximal homogeneous ideal of A, i.e. the ideal $m \oplus (\bigoplus_{i>0} A_i)$

(1) The theorem is patterned on [1], Lemma in III, § 3 n° 6, p. 350.

(2) I.e. $ab = (-1)^{\deg a \cdot \deg b}$, $a^2 = 0$ if $\deg a$ is odd.

where m is the maximal ideal of A_0 will be denoted by $M(A)$. For a given homogeneous finitely generated ideal J in A, $e-\dim J$ will denote the dimension of the $A/M(A)$-vector space J/MJ. A map $f\colon A \to A'$ of graded algebras is, by definition, a grading preserving homomorphism of rings such that $f(M(A)) \subset M(A')$. Such a map is called *flat* if A', as a graded A-module, is flat. A graded algebra $A = \bigoplus A_i$ is said to be of *finite type* if the ideal $M(A)$ is finitely generated, or equivalently, if A, as an A_0-algebra, is of finite type. Finally A is said to be reduced if it has no non-zero nilpotent elements. If $u \in A$, then $h(u)$ will denote the minimum of integers $m \geq 0$ such that $u^m = 0$. If $u^m \neq 0$ for all m, then we put $h(u) = \infty$ and assume that $u^{\infty - i} = 0$ if $i < \infty$.

1.1. DEFINITION. Let A be a graded algebra. A sequence $u_1, \ldots, u_n$ of elements of A is called *h-regular in A* if u's are homogeneous and the following conditions hold:

1° $(u_1, \ldots, u_n) \neq A$,

2° $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k - 1})$, where $h_k = h(u_k)$ for $k = 1, \ldots, n$ and $(u_1, \ldots, u_{k-1}) = 0$ for $k = 1$.

A sequence $u_1, \ldots, u_n$ is called *$(h_1, \ldots, h_n)$-regular in A* if it is h-regular and $h(u_i) = h_i$, $i = 1, \ldots, n$.

1.2. DEFINITION. A homogeneous ideal J in a graded algebra A is said to be *h-regular* (*$(h_1, \ldots, h_n)$-regular*) if it is generated by an h-regular $(h_1, \ldots, h_n)$-regular sequence.

A graded algebra A is said to be *h-regular* (*$(h_1, \ldots, h_n)$-regular*) if the ideal $M(A)$ is h-regular $((h_1, \ldots, h_n)$-regular). A local ring R is said to be *h-regular* (*$(h_1, \ldots, h_n)$-regular*) if the graded algebra $A = A_0 = R$ is h-regular $((h_1, \ldots, h_n)$-regular).

Now we give a list of (more or less) known results that will be needed later.

(1.3) If A is a graded algebra and $K = \bigoplus_{i \geq 0} K_i$ is a graded A-module such that K_i are finitely generated A_0-modules, then $M(A)K = K$ implies $K = 0$.

(1.4) ([7], Lemma 1.6). If $u_1, \ldots, u_n$ is an h-regular sequence in a graded algebra A, then $u_{\sigma(1)}, \ldots, u_{\sigma(n)}$ is an h-regular sequence in A for each permutation σ of the set $\{1, \ldots, n\}$. Moreover, if $u_i \neq 0$ for all i, then $u_1, \ldots, u_n$ is a minimal sequence of generator of the ideal $J = (u_1, \ldots, u_n)$, i.e. $u_1 + MJ, \ldots, u_n + MJ$ is a base of the $A/M(A)$-vector space $J/M(A)J$.

(1.5) A map of graded algebras $f\colon A \to A'$ is flat if and only if $\mathrm{Tor}_1^A(A/M(A), A') = 0$. Furthermore, if $u_1, \ldots, u_n$ is an $(h_1, \ldots, h_n)$-regular sequence in A such that $h(f(u_i)) = h_i$ for $i = 1, \ldots, n$ and $\mathrm{Tor}_1^A(A/(u_1, \ldots, u_n), A') = 0$ for $i = 1, 2$, then $f(u_1), \ldots, f(u_n)$ is an $(h_1, \ldots, h_n)$-regular sequence in A'.

The first assertion can be proved in the same way as [8], Cor. 31, by using the equality $\bigcap_n M(A)^n = 0$ instead of the Krull Intersection Theorem. The second one is a consequence of [7], Prop. 2.6.

(1.6) Let $u_1, \ldots, u_n$ be a sequence of homogeneous elements of a graded algebra A with $h(u_i) \leqslant h_i$ for some $h_i \in N^\infty$, $i = 1, \ldots, n$. Then $u_1, \ldots, u_n$ is an $(h_1, \ldots, h_n)$-regular sequence in A if and only if there exists a j, $1 \leqslant j \leqslant n$, such that $u_1, \ldots, u_j$ is an $(h_1, \ldots, h_j)$-regular sequence in A and the images $\bar{u}_{j+1}, \ldots, \bar{u}_n$ of u's in the graded algebra $\bar{A} = A/(u_1, \ldots, u_j)$ form an $(h_{j+1}, \ldots, h_n)$-regular sequence in $\bar{A}$.

This is a slight modification of [7], Lemma 1.4.

(1.7) If $u_1, \ldots, u_n$ is an $(h_1, \ldots, h_n)$-regular sequence in a graded algebra A, then for each sequence $k_1, \ldots, k_n$, $k_i \in N^\infty$, the sequence $\bar{u}_1 = u_1 + (u_1^{k_1}, \ldots, u_n^{k_n}), \ldots, \bar{u}_n = u_n + (u_1^{k_1}, \ldots, u_n^{k_n})$ is an $(h'_1, \ldots, h'_n)$-regular sequence in the graded algebra $A/(u_1^{k_1}, \ldots, u_n^{k_n})$, where $h'_j = \min(k_j, h_j)$ for $j = 1, \ldots, n$.

(1.7) is a simple consequence of [8], Lemma 20 (see the proof of [8], Th. 19).

(1.8) Let p be a prime. If $u_1, \ldots, u_n$ is a $(p, \ldots, p)$-regular sequence in a graded algebra A of characteristic p and $v_1, \ldots, v_n$ is a minimal sequence of generators of the ideal $(u_1, \ldots, u_n)$, then $v_1, \ldots, v_n$ is also a $(p, \ldots, p)$-regular sequence in A.

The statement follows by induction on n by using (1.6).

(1.9) If A is a reduced graded algebra of finite type and of characteristic p (p as above is a prime) such that the natural inclusion $A^{[p]} = \{a^p;\, a \in A\} \hookrightarrow A$ is a flat map of graded algebras, then A_0 is a regular local ring and $A = A_0[X_1, \ldots, X_n]$ for a certain $n \in N$.

If $A = A_0$ then (1.9) coincides with the Kunz Theorem ([4], Th. 2.1). In the general case it can by proved in the same manner as [3], Cor. 3.3, because the assertion of (1.9) is equivalent to the finiteness of the global projective homological dimension of A.

1.10. Remark. If J is an h-regular ideal in a graded algebra A and $n = e - \dim J$, then there exists a sequence $h_1 \leqslant \ldots \leqslant h_n$ in N^∞ such that J is an $(h_1, \ldots, h_n)$-regular ideal in A. In fact, if $u_1, \ldots, u_m$ is an h-regular sequence of generators of the ideal J, then dropping zeros out of it and, possibly changing the order, we obtain an h-regular sequence of generators $v_1, \ldots, v_r$ such that $v_j \neq 0$ for all j and $h(v_1) \leqslant \ldots \leqslant h(v_r)$. From (1.4) it follows that $r = n$. Putting $h_i = h(v_i)$, $i = 1, \ldots, n$, we get what we wanted.

Now we prove a lemma which plays an important role in the proof of the main theorem.

1.11. Lemma. *Let p be a prime and let $l_1, \ldots, l_n$ be a sequence of elements from the set N^∞. A sequence $u_1, \ldots, u_n$ of homogeneous elements of a graded algebra A is $(pl_1, \ldots, pl_n)$-regular in A if and only if*

(a) *$u_1^p, \ldots, u_n^p$ is an $(l_1, \ldots, l_n)$-regular sequence in A,*

(b) *$\bar{u}_1 = u_1 + (u_1^p, \ldots, u_n^p), \ldots, \bar{u}_n = u_n + (u_1^p, \ldots, u_n^p)$ is a $(p, \ldots, p)$-regular sequence in the graded algebra $\bar{A} = A/(u_1^p, \ldots, u_n^p)$.*

Proof. $\Rightarrow$ Assertion (b) follows from (1.7). For the proof of assertion (a) we apply induction on n. If $n = 1$, then (a) holds because if $au_1^p = 0$ implies $a = bu_1^{pl_1 - p} = bu_1^{p(l_1-1)}$ by [7], Lemma 1.2. Suppose that $n > 1$ and that our assertion is true for all sequences of length $< n$. Further, consider the graded algebra $\bar{A} = A/(u_1)$ and the sequence $\bar{u}_2, \ldots, \bar{u}_n$, $\bar{u}_j = u_j + (u_1)$, which is $(pl_2, \ldots, pl_n)$-regular in $\bar{A}$ by (1.6). In virtue of the induction assumption it follows that $\bar{u}_2^p, \ldots, \bar{u}_n^p$ is an $(l_2, \ldots, l_n)$-regular sequence in $\bar{A}$. Therefore, $u_1, u_2^p, \ldots, u_n^p$ is a $(pl_1, l_2, \ldots, l_n)$-regular sequence in A again by (1.6). Hence, in particular, $\tilde{u}_1 = u_1 + (u_2^p, \ldots, u_n^p)$ is a (pl_1)-regular element in $\tilde{A} = A/(u_2^p, \ldots, u_n^p)$ and the induction step $n = 1$ shows that $\tilde{u}_1^n$ is an (l_1)-regular element in $\tilde{A}$. Consequently, $u_2^p, \ldots, u_n^p, u_1^p$ is an $(l_2, \ldots, l_n, l_1)$-regular sequence in A. The conclusion now follows from (1.4) and thus the implication $\Rightarrow$ is proved.

For the implication $\Leftarrow$ we proceed also by induction on n. If $n = 1$, $u = u_1$, $l = l_1$ and $au = 0$, then $a = a_1 u^{p(l-1)}$ because $au^p = 0$ and u^p is an (l)-regular element in A. Hence $a = 0$ if $l = \infty$, i.e. u is $(\infty = pl)$-regular element in A. If $l < \infty$, then the equality $0 = au = a_1 u^{lp-p+1} = a_1 u^{(l-2)p+1} u^p$ implies $a_1 u^{(l-2)+1} = a_2 u^{(l-1)p}$, i.e. $(a_1 - a_2 u^{p+1}) u^{(l-3)p+1} u^p = 0$. As before, this gives

$$(a_1 - a_2 u^{p-1}) u^{(l-3)p+1} = a_3 u^{(l-1)p}$$

whence $(a_1 - a_2 u^{p-1} - a_3 u^{2p-1}) u^{(l-3)p+1} = 0$. Continuing this procedure we find elements $a_1, \ldots, a_l$ in A such that

$$(a_1 - \sum_{i=2}^{l} a_i u^{(i-1)p-1}) u = 0 .$$

Since $u + (u^p)$ is (p)-regular in $A/(u^p)$, it follows that

$$a_1 - \sum_{i=2}^{l} a_i u^{(i-1)p-1} = bu^{p-1} + cu^p \quad \text{for some } b, c \in A .$$

Therefore $a_1 = a' u^{p-1}$ and consequently $a = a_1 u^{(l-1)p} = a' u^{lp-1}$. Thus we see that $(0)\colon (u) = (u^{lp-1})$, which together with the inequality $h(u) \leqslant lp$ proves that u is a (pl)-regular element in A. Now suppose $n > 1$ and the implication $\Leftarrow$ holds for all sequences of length $< n$. If $\bar{A} = A/(u_1^p)$ and $\bar{u}_i = u_i + (u_1^p)$ for $i = 2, \ldots, n$ then it is easy to see, by using (1.6), that $\bar{u}_2, \ldots, \bar{u}_n$ satisfies conditions (a) and (b) of the lemma with $h(\bar{u}_j) = l_j$, $j = 2, \ldots, n$. Hence $\bar{u}_2, \ldots, \bar{u}_n$ is a $(pl_2, \ldots, pl_n)$-regular in $\bar{A}$ in view of the induction hypothesis. Using again (1.6) we infer from it that $u_1^p, u_2, \ldots, u_n$ is an $(l_1, pl_2, \ldots, pl_n)$-regular sequence because $h(u_i) = pl_i$ (otherwise $h(\bar{u}_i) < l_i p$!). This means that $\tilde{u}_1^p$, where $\tilde{u}_1 = u_1 + (u_2, \ldots, u_n)$, is an (l_1)-regular element in $\tilde{A} = A/(u_2, \ldots, u_n)$. Moreover, there exists a natural isomorphism of graded algebras $g\colon \tilde{A}/(\tilde{u}_1^p) \to A^*/(u_2^*, \ldots, u_n^*)$, $A^* = A/(u_1^p, \ldots, u_n^p)$, $u_i^* = u_i + (u_2^p, \ldots, u_n^p)$, such that $g(\tilde{u}_1 + (\tilde{u}_1^p)) = u_1^{**} = u_1^* + (u_2^*, \ldots, u_n^*)$. Since u_1^{**} is a (p)-regular element, $\tilde{u}_1 + (\tilde{u}_1^p)$ is a (p)-regular element in $\tilde{A}/(\tilde{u}_1^p)$ and so we know that the element $\tilde{u}_1$ satisfies conditions (a) and (b) of the lemma. Now making use of the induction step $n = 1$, we conclude that $\tilde{u}_1$ is a (pl_1)-regular element in $\tilde{A}$, whence $u_1, \ldots, u_n$ is a $(pl_1, \ldots, pl_n)$-regular sequence in A. The lemma is proved.

§ 2. Main results. From now on all graded algebras A under consideration are assumed to be of characteristic p (i.e. $pA = 0$), where p is a fixed prime.

If A is a graded algebra and S is a subset of A, then $S^{[p^i]}$ will denote the set $\{a^{[p^i]} \in A;\ a \in S\}$. It is easy to check that $A^{[p]}$ is a graded subalgebra of A and $J^{[p]}$ is a homogeneous ideal in $A^{[p]}$ for any homogeneous ideal J in A.

2.1. THEOREM. *Let A be a graded algebra and let J be a homogeneous ideal in A satisfying the following conditions:*

(i) *$J^{[p]}$ is an $(l_1, \ldots, l_n)$-regular ideal in $A^{[p]}$ for $n = e-\dim J$ and some sequence $l_1 \leqslant \ldots \leqslant l_n$ in N^∞,*

(ii) *$J/J^{[p]}A$ is a $(p_1, \ldots, p_n)$-regular ideal in $A/J^{[p]}A$, where $p_1 = \ldots = p_n = p$,*

(iii) *$\mathrm{Tor}_i^{A^{[p]}}(A^{[p]}/J^{[p]}, A) = 0$ for $i = 1, 2$.*

Then J is a $(pl_1, \ldots, pl_n)$-regular ideal in A.

Proof. Let $1 = l_1 = \ldots = l_{s-1} < l_s$ and let $u_s^p, \ldots u_n^p$ be a sequence of generators of the ideal $J^{[p]}$, which is $(l_s, \ldots, l_n)$-regular in $A^{[p]}$. Moreover, let $v_1, \ldots, v_r$ be a sequence of homogeneous elements of J such that $v_i^p = 0$, $i = 1, \ldots, r$, and $v_1+MJ, \ldots, v_r+MJ$, $M = M(A)$, is a base of the A/M-vector space J_p+MJ/MJ where $J_p = \{a \in J, a^p = 0\}$. We show that $v_1, \ldots, v_r, u_s, \ldots, u_n$ is a minimal sequence of homogeneous generators of the ideal J. It is clear that $(v_1, \ldots, v_r, u_s, \ldots, u_n) \subset J$. If $a \in J$, then $a^p = \Sigma a_i^p u_i^p$, i.e. $(a - \Sigma a_i u_i)^p = 0$ whence $a - \Sigma a_i u_i = \Sigma b_j v_j + y$ for some $y \in MJ$. This means that $J = (v_1, \ldots, v_r, u_s, \ldots, u_n) + MJ$ and consequently $J = (v_1, \ldots, v_r, u_s, \ldots, u_n)$ in virtue of (1.3). Now, let $\Sigma b_j v_j + \Sigma a_i u_i = 0$ for some b_j, $a_i \in A$. Then $\Sigma a_i^p u_i^p = 0$, which gives $a_i^p \in M^{[p]}$ because in view of (1.4) $u_s^p, \ldots, u_n^p$ is a minimal set of generators of the ideal $J^{[p]}$. Therefore $a_i \in M$ and $\Sigma b_j v_j \in MJ$. Hence we infer that $b_j \in M$ since as $\{v_j + MJ\}$ for $j = 1, \ldots, r$ is a base of the A/M-vector space J_p+MJ/MJ. Thus we have shown that $v_1, \ldots, v_r, u_s, \ldots, u_n$ is a minimal set of generators of the ideal J. Since $n = e-\dim J (= \dim_{A/M}(J/MJ)$, it follows that $r = s-1$. By definition, let $u_1 = v_1, \ldots, u_{s-1} = v_{s-1}$. For the proof of the theorem it is sufficient to show that $u_1, \ldots, u_n$ is a $(pl_1, \ldots, pl_n)$-regular sequence in A. First observe that by (1.5) $u_1^p, \ldots, u_n^p$ is an $(l_1, \ldots, l_n)$-regular sequence in A because it is such a sequence in $A^{[p]}$ ($(u_1^p, \ldots, u_n^p) = (0, \ldots, 0, u_s^p, \ldots, u_n^p)$) and $\mathrm{Tor}_i^{A^{[p]}}(A^{[p]}/(u_1^p, \ldots, u_n^p), A) = 0$ for $i = 1, 2$ by condition (iii). Next (1.8) implies that $\bar{u}_1 = u_1 + (u_1^p, \ldots, u_n^p), \ldots, \bar{u}_n = u_n + (u_1^p, \ldots, u_n^p)$ is a $(p, \ldots, p)$-regular sequence in $A/(u_1^p, \ldots, u_n^p) = A/J^{[p]}A$ as $\bar{u}_1, \ldots, \bar{u}_n$ is a minimal set of generators of the ideal $J/J^{[p]}A$ in $A/J^{[p]}A$. The conclusion now follows from Lemma 1.11. The proof of the theorem is completed.

2.2. Remark. It would be interesting to know if conditions (i)–(iii) in the above theorem are independent on each other. It is not difficult to prove that condition (ii) is independent of conditions (i) and (iii). The same refers to condition (iii) and conditions (i) and (ii). We are unable to show that condition (i) is independent of conditions (ii) and (iii).

2.3. Remark. In Theorem 2.1 the converse implication is not true, although condition (ii) holds whenever A is a $(pl_1, \ldots, pl_n)$-regular graded algebra by (1.7). In fact, if k is a field of characteristic 2 and $A = A_0 = k[X, Y, Z]/(X^2 - YZ^2, Y^2, Z^4)$, then the ideal $J = (u)$, $u = Z + (X^2 - YZ^2, Y^2, Z^4)$ is (4)-regular in A while $J^2 = (u^2)$ is not (2)-regular in $A^2 = k^2[X, Z]/(X^2, XZ, Z^2)$. This means that condition (i) is not satisfied. One can also verify that in this case condition (iii) is not satisfied ($\mathrm{Tor}_2^{A^{[2]}}(A^{[2]}/J^{[2]}, A) \neq 0$!). However, as is shown by the next theorem, such an example does not exist if $J = M(A)$.

2.4. THEOREM. *Let $l_1 \leqslant \ldots \leqslant l_n$ be a sequence of elements of the set N^∞. A graded algebra A with $e-\dim A = n$ is $(pl_1, \ldots, pl_n)$-regular if and only if*

(1) *$A^{[p]}$ is an $(l_1, \ldots, l_n)$-regular graded algebra,*

(2) *$A^{(p)} = A/M^{[p]}A$, $M = M(A)$, is a $(p_1, \ldots, p_n)$-regular graded algebra with $p_1 = \ldots = p_n = p$,*

(3) *$A^{[p]} \hookrightarrow A$ is a flat map of graded algebras.*

Proof. The implication $\Leftarrow$ follows from Theorem 2.1 applied to $J = M$. In order to show the implication $\Rightarrow$ we prove successively that conditions (1)–(3) are satisfied for a $(pl_1, \ldots, pl_n)$-regular graded algebra A.

Let $F\colon A \to A$ denote the Frobenius homomorphism of A, i.e. the homomorphism of rings defined by the formula: $f(a) = a^p$ for $a \in A$. To establish (1) it is enough to show, by using (1.7), that $\mathrm{Ker} F = (u_1^{l_1}, \ldots, u_n^{l_n})$ for a given $(pl_1, \ldots, pl_n)$-regular sequence $u_1, \ldots, u_n$ of generators of the ideal $M = M(A)$. We shall do it by induction on n. If $n = 0$, there is nothing to do. Suppose that $n > 0$ and (having in mind the induction assumption) that $F(a) = 0$ for a certain $a \in A$. Then $\bar{F}(a + (u_1)) = 0$ where $\bar{F}\colon \bar{A} = A/(u_1) \to A/(u_1)$ is the Frobenius homomorphism of the graded algebra $\bar{A}$. Since $e-\dim \bar{A} = n-1$ and $\bar{u}_2, \ldots, \bar{u}_n$, $\bar{u}_i = u_i + (u_1)$, is a $(pl_2, \ldots, pl_n)$-regular sequence of generators of the ideal $M(\bar{A})$, the induction assumption implies that $a + (u_1) \in (\bar{u}_2^{l_2}, \ldots, \bar{u}_n^{l_n})$, i.e. $a = \sum_{i=2}^{n} a_i u_i^{l_i} + a_1 u_1$. If $l_1 = 1$ we have obtained what we wanted. If $l_1 > 1$, then $0 = a^p = a_1^p u_1^p$ gives $a_1^p = b u_1^{l_p - p} \in (u_1)$ by [7], Lemma 1.2, whence $\bar{F}(a_1 + (u_1)) = 0$. As before, it follows that $a_1 \in (u_1, u_2^{l_2}, \ldots, u_n^{l_n})$. Therefore $a \in (u_1^2, u_2^{l_2}, \ldots, u_n^{l_n})$. Repeating the reasoning, we conclude that $a \in (u_1^{l_1}, \ldots, u_n^{l_n})$ provided that $l_1 < \infty$. If $l_1 = \infty$, then the equality $a_1^p u_1^p = 0$ implies $a_1^p = 0$, which means that $a \in (u_2^{l_2}, \ldots, u_n^{l_n}) + MJ$ where $J = \mathrm{Ker} F$. Since $(u_2^{l_2}, \ldots, u_n^{l_n}) \subset J$, we have $J = (u_2^{l_2}, \ldots, u_n^{l_n}) + MJ$ and finally $J = \mathrm{Ker} F = (u_2^{l_2}, \ldots, u_n^{l_n}) = (u_1^{l_1}, \ldots, u_n^{l_n})$ by (1.3). Thus condition (1) is proved.

Condition (2) is a consequence of (1.7).

For condition (3), by (1.5), we need only to verify that $\mathrm{Tor}_1^{A^{[p]}}(A^{[p]}/M^{[p]}, A) = 0$. In the proof of condition (1) we showed that $F\colon A \to A$ induces an isomorphism of rings $f\colon \bar{A} = A/(u_1^{l_1}, \ldots, u_n^{l_n}) \to A^{[p]}$. Hence $u_1^p, \ldots, u_n^p$ is an $(l_1, \ldots, l_n)$-regular sequence in $A^{[p]}$ because, by (1.7), $u_1 + (u_1^{l_1}, \ldots, u_n^{l_n}), \ldots, u_n + (u_1^{l_1}, \ldots, u_n^{l_n})$ is an $(l_1, \ldots, l_n)$-regular sequence in $\bar{A}$ and $f(u_j + (u_1^{l_1}, \ldots, u_n^{l_n})) = u_j^p$, $j = 1, \ldots, n$. Furthermore, $M^{[p]} = M(A)^{[p]} = (u_1^p, \ldots, u_n^p)$. Now, if $j\colon A^p \hookrightarrow A$ is the natural inclusion, then $j(u_1^p), \ldots, j(u_n^p)$ is an $(l_1, \ldots, l_n)$-regular sequence in A by Lemma 1.11 and the required result follows from [7], Cor. 2.9. The theorem is proved.

§ 3. Applications of the main resuls in the theory of Hopf algebras over a perfect field. Let k be a fixed perfect field of characteristic p (i.e. $k^p = k$). If A is a graded algebra, then $N(A)$ will denote the nilradical of A.

Case I. Graded Hopf algebras over k.

3.1. DEFINITION. A graded Hopf k-algebra is (in this paper) a *graded algebra H over k* such that $H_0 = k$ together with a map of graded k-algebras

$$\Delta: A \to A \otimes A \qquad (\otimes = \otimes_k)$$

such that

1) $\Delta(x) = 1\otimes x + x \otimes 1 + \sum x_i \otimes y_i$, $0 < \deg(y_i) < n$, if $x \in A_n$,

2) $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$.

If H and H' are graded Hopf k-algebras, then a map $g\colon H \to H'$ of graded k-algebras is called a *morphism of graded Hopf k-algebras* if $\Delta \cdot f = (f \otimes f)\Delta$. A graded Hopf k-algebra is called of *finite type* if it is of finite type as a k-algebra. If H is a Hopf k-algebra, then it is easy to see that $H^{[p]}$ is a graded Hopf k-algebra.

3.2. THEOREM (Borel) [5], Th. 7.11. *If H is a graded Hopf k-algebra of finite type, then there exist a sequence $n_1, \ldots, n_r$, $n_i \in N^\infty$, and a sequence $k_1, \ldots, k_r$, $k_i \in N$, such that H, as a graded k-algebra, is isomorphic to $\bigotimes_{i=1}^{r} k[X_i]/(X^{p^{n_i}})$, where $\deg X_i = k_i$, $i = 1, \ldots, r$.*

Proof. In view of [7], Prop. 1.14, we have to show that H is a $(p^{n_1}, \ldots, p^{n_r})$-regular graded k-algebra for some sequence $n_1, \ldots, n_r$. This will be done by induction on $h(H) = \min\{i;\ N(H)^{[p^i]} = 0\}$. If $h(H) = 0$, i.e. $N(H) = 0$, then the assertion is a consequence of (1.9) because by [5], Prop. 1.7, $H^{[p]} \hookrightarrow H$, being a monomorphism of graded Hopf k-algebras, is a flat map of graded k-algebras. Suppose that $h(H) > 0$ and that our assertion is true for all graded Hopf k-algebras H' of finite type with $h(H') < h(H)$. To prove that H is a $(p^{n_1}, \ldots, p^{n_r})$-regular graded k-algebra we apply Theorem 2.4. Since $h(H^{[p]}) = h(H) - 1$, condition (1) of this theorem is satisfied in virtue of the induction hypothesis. Condition (2) follows from Lemma 3.3 below since $M(A^{(p)})^{[p]} = 0$. Finally, as we mentioned before, $H^{[p]} \hookrightarrow H$ is a flat map of graded k-algebras. The theorem is proved.

3.3. LEMMA. *If H is a graded Hopf k-algebra of finite type such that $M(A)^{[p]} = 0$, then H is a $(p_1, \ldots, p_n)$-regular graded algebra, where $n = e\text{-}\dim H$ and $p_1 = \ldots = p_n = p$.*

Proof. Induction on $n = e\text{-}\dim H$. If $n = 0$, them $H = k$ and the lemma is true. Assume that $n > 0$ and that the lemma holds for all graded Hopf k-algebras H' such that $M(H')^{[p]} = 0$ and $e\text{-}\dim H' < n$. If $u_1, \ldots, u_n$ is a minimal sequence of homogeneous generators of the ideal $M = M(H)$ such that $\deg u_1 \leqslant \ldots \leqslant \deg u_n$, then clearly u_1 is a primitive element of H, i.e. $\Delta(u_1) = 1 \otimes u_1 + u_1 \otimes 1$. It is easy to see that for each primitive element $u \in H$ $h(u) = p^s$ for a certain $s \in N$. Since $M^{[p]} = 0$, it gives $h(u_1) = p$. Now let $H_1 = k[X]/(X^p)$, $\deg X = \deg u_1$, be a graded Hopf k-algebra with $\Delta\colon H_1 \to H_1 \otimes H_1$ given by $\Delta(\bar{X}) = 1 \otimes \bar{X} + \bar{X} \otimes 1$ and let $f\colon H_1 \to H$ be a monomorphism of graded Hopf k-algebras defined by $f(\bar{X}) = u_1$.

Then by [5], Prop. 1.7, f is a flat map of graded algebras, whence u_1 is a (p)-regular element in H by (1.5). Furthermore $H' = H/(u_1)$ is a graded Hopf k-algebra of finite type with $M(H')^{[p]} = 0$ and $e\text{-}\dim H' = n-1$. So, in virtue of the induction assumption, H' is a $(p_1, \ldots, p_{n-1})$-regular graded algebra where $p_i = p$ for $i = 1, \ldots, n-1$. This means that there exists a sequence $\bar{v}_2, \ldots, \bar{v}_n$, $\bar{v}_j = v_j + (u_1)$, of generators of the ideal $M(H') = M/(u_1)$ which is $(p, \ldots, p)$-regular in H'. In particular, $h(v_i) = p$ because otherwise $h(\bar{v}_i) < p$ ($h(v_i) \leqslant p$ as $M^{[p]} = 0$!). Hence $u_1, v_2, \ldots, v_n$ is a $(p, \ldots, p)$-regular sequence of generators of the ideal M by (1.6) and consequently H is a $(p, \ldots, p)$-regular graded k-algebra, as was to be shown.

Case II. Hopf algebras over k.

3.4. DEFINITION. By a *Hopf k-algebra* we mean in this paper a commutative k-algebra H together with maps of k-algebra $\Delta\colon H \to H \otimes H$ ($\otimes = \otimes_k$), $S\colon H \to H$, $\varepsilon\colon H \to k$ satisfying the following conditions:

1° $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$, $m(S \otimes 1)\Delta = \eta\varepsilon$ where $m\colon H \otimes H \to H$ is the multiplication and $\eta\colon k \to H$ is the structural map,

2° $(\varepsilon \otimes 1)\Delta(a) = 1 \otimes a$, $(1 \otimes \varepsilon)\Delta(a) = a \otimes 1$ for $a \in H$.

If H and H' are Hopf k-algebras, then a homomorphism of k-algebras $g\colon H \to H'$ is called a *morphism of Hopf k-algebras* if $\Delta g = (g \otimes g)\Delta$, $S'g = gS$, $\varepsilon' g = \varepsilon$. It is easy to see that for each morphism of Hopf k-algebras $g\colon H \to H'$ the set $\{a \in H,\ (g \otimes 1)\Delta(a) = 1 \otimes a\}$ is a subalgebra of H with augmentation being the restriction of $\varepsilon\colon H \to k$. This k-algebra will be denoted by H-$\mathrm{Ker}\, g$.

3.5. LEMMA. *If $g\colon H \to H'$ is a surjective morphism of Hopf k-algebras and $L = H$-$\mathrm{Ker}\, g$, then $\mathrm{Ker}\, g = L^+ H$, where $L^+ = \mathrm{Ker}\, \varepsilon_L$ and $\varepsilon_L\colon L \to k$ is the augumentation of L. Moreover, $L \hookrightarrow H$ is a flat homomorphism of k-algebras, i.e. H is flat as an L-module.*

Proof. The lemma is a consequence of [6], VI_B, Prop. 9.2 (ii), (xii).

3.6. LEMMA. *If H is a Hopf k-algebra with $N(H)^{[p]} = 0$ and $\mathfrak{R}$ is a prime ideal in H such that the localization $H_{\mathfrak{R}}$ is a Noetherian ring, then $N(H_{\mathfrak{R}})$ is a $(p, \ldots, p)$-regular ideal in $H_{\mathfrak{R}}$.*

Proof. Let $H_{\mathrm{red}} = H/N$, where $N = N(H)$. It is well known [1], II, § 5, n° 2, Cor. 2.3, that H_{red} is a Hopf k-algebra and the natural projection $q\colon H \to H_{\mathrm{red}}$ is a morphism of Hopf k-algebras. Denote by K the algebra with augmentation H-$\mathrm{Ker}\, q$. By Lemma 3.5, $K^+ H = N$ and H is a flat K-module. Let $\{u_i + (K^+)^2;\ i \in I\}$ be a base of the k-vector space $K^+/(K^+)^2$ and let $U = \{u_i;\ i \in I\}$. We are going to show that $h(u_i) = p$ for all $i \in I$ and that U is an h-regular set in H in the sense of [7], Def. 1.7, i.e. that each finite subset $\{v_1 = u_{i_1}, \ldots, v_n = u_{i_n}\}$ of the set U is h-regular in H. Let H' be a subHopf k-algebra of H which contains all v_i and is of finite type as a k-algebra (such a subHopf k-algebra always exists). Then we have the following commutative diagram:

$$\begin{array}{ccccc} K & \hookrightarrow & H & \xrightarrow{q} & H_{\mathrm{red}} \\ \uparrow & & \uparrow & & \uparrow i \\ L & \hookrightarrow & H' & \xrightarrow{q'} & H'_{\mathrm{red}} \end{array}$$

where q' is the natural projection, $L = H\text{-}\mathrm{Ker}\, q'$ and i is the morphism of Hopf k-algebras induced by the inclusion $H' \hookrightarrow H$. It is easy to check, by using the injectivity of i, that $L = H' \cap K$. In particular, $v_i \in L$ for $i = 1, \ldots, n$. Moreover, the inclusion $L \hookrightarrow H$ is a flat homomorphism of k-algebras because so are $L \hookrightarrow H'$ and $H' \hookrightarrow H$ by Lemma 3.5 and [1], p. 353, respectively. Hence in order to prove that $v_1, \ldots, v_n$ is a $(p, \ldots, p)$-regular sequence in H it is sufficient to show (see (1.5)) that it is $(p, \ldots, p)$-regular in L. However, $L = k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$ in virtue of [1], III, § 3, Lemma 6.9, and consequently, using (1.6), we conclude that $v_1, \ldots, v_n$ is a $(p, \ldots, p)$-regular sequence in L, whenever it is a part of a minimal set of generators of the ideal L^+. But this easily follows from the fact that $\{v_i + (K^+)^2;\ i = 1, \ldots, n\}$ is a part of a base of the k-vector space $K^+/(K^+)^2$. Thus we have proved that $U = \{u_i\colon i \in I\}$ is an h-regular set in H and $h(u_i) = p$ for all $i \in I$.

Now if $\mathfrak{N}$ is a prime ideal in H, then by (1.5) $\{u_i' = u_i/1 \in H_{\mathfrak{N}};\ i \in I\}$ is an h-regular set in $H_{\mathfrak{N}}$ contained obviously in $N_{\mathfrak{N}} = N(H_{\mathfrak{N}})$ and $h(u_i') = p$ for $i \in I$. Furthermore, if J is the ideal in $H_{\mathfrak{N}}$ generated by $\{u_i';\ i \in I\}$, then it is easy to verify that $N_{\mathfrak{N}} = J + N_{\mathfrak{N}}^2$. Hence $J = N_{\mathfrak{N}}$ if $H_{\mathfrak{N}}$ is Noetherian because $N_{\mathfrak{N}}$ being finitely generated is a nilpotent ideal. This is possible only when I is a finite set, since each h-regular set of generators is a minimal set of generators by (1.4). Consequently, $N_{\mathfrak{N}} = N(H_{\mathfrak{N}})$ is a $(p, \ldots, p)$-regular ideal in $H_{\mathfrak{N}}$, as was to be proved.

3.7. Theorem. *If H is a Hopf k-algebra and $\mathfrak{N}$ is a prime ideal in H such that $H_{\mathfrak{N}}$ is a Noetherian ring, then there exist $n_1, \ldots, n_r$, $n_i \in N^\infty$ such that $H_{\mathfrak{N}}$ is a $(p^{n_1}, \ldots, p^{n_r})$-regular local ring.*

Proof. If, as above, $N = N(H)$ and $H_{\mathrm{red}} = H/N$, then we have the exact sequence

$$0 \to N_{\mathfrak{N}} \to H_{\mathfrak{N}} \xrightarrow{\bar{q}} (H_{\mathrm{red}})_{\bar{\mathfrak{N}}} \to 0$$

where $\bar{\mathfrak{N}} = q(\mathfrak{N})$, $\bar{q}(a/s) = q(a)/q(s)$ and $N_{\mathfrak{N}} = \{a/s \in H_{\mathfrak{N}};\ a \in N\}$ is the ideal in $H_{\mathfrak{N}}$ equal to $N(H_{\mathfrak{N}})$. Since H_{red} is a reduced Hopf k-algebra and $(H_{\mathrm{red}})^{[p]} \hookrightarrow H_{\mathrm{red}}$ (whence $((H_{\mathrm{red}})_{\bar{\mathfrak{N}}})^{[p]} \hookrightarrow (H_{\mathrm{red}})_{\bar{\mathfrak{N}}}$) is a flat homomorphism of rings by [1], p. 353 $(H_{\mathrm{red}})_{\bar{\mathfrak{N}}}$ is a regular local ring in view of (1.9). Hence we deduce, by applying (1.6), that it remains to show that $N_{\mathfrak{N}}$ is a $(p^{n_1}, \ldots, p^{n_l})$-regular ideal in $H_{\mathfrak{N}}$ for some l. We shall do it by induction on $h(\mathfrak{N}) = \min\{i,\ (N_{\mathfrak{N}})^{[p^i]} = 0\}$. If $h(\mathfrak{N}) = 0$, i.e. $N_{\mathfrak{N}} = 0$, then there is nothing to do. Suppose that $h(\mathfrak{N}) > 0$ and that the assertion is true for all pairs $(H', \mathfrak{N}')$, where H' is a Hopf k-algebra and $\mathfrak{N}'$ is a prime ideal in H' such that $H'_{\mathfrak{N}'}$ is a Noetherian ring and $h(\mathfrak{N}') < h(\mathfrak{N})$. To prove that $N_{\mathfrak{N}}$ is $(p^{n_1}, \ldots, p^{n_l})$-regular ideal in $H_{\mathfrak{N}}$ we apply Theorem 2.1. Consider the Hopf k-algebras $H^{[p]}$ and $H^{(p)} = H/N^{[p]}H$ with the prime ideals $\mathfrak{N}^{[p]}$ and $\tilde{\mathfrak{N}} = \mathfrak{N}/N^{[p]}H$, respectively. It is obvious that $h(\mathfrak{N}^{[p]}) = h(\mathfrak{N}) - 1$ and $N(H^{(p)})^{[p]} = 0$. Furthermore, one can easily verify that $N(H^{[p]}_{\mathfrak{N}^{[p]}}) = N((H_{\mathfrak{N}})^{[p]}) \simeq (N_{\mathfrak{N}})^{[p]}$ and $N(H^{(p)}_{\tilde{\mathfrak{N}}}) \simeq N_{\mathfrak{N}}/N_{\mathfrak{N}}^{[p]}H_{\mathfrak{N}}$ because $H^{(p)}_{\tilde{\mathfrak{N}}} \simeq H_{\mathfrak{N}}/N_{\mathfrak{N}}^{[p]}H_{\mathfrak{N}}$. Using the induction assumption, Lemma 3.6 and Theorem 2.1, we obtain the required result. The theorem is proved.

3.8. Corollary. *If H is a Hopf k-algebra of finite type, then for each prime ideal $\mathfrak{N}$ in H there exist a regular local ring S and elements $n_1, \ldots, n_r$ from N^∞, $r \leqslant e - \dim H$, such that*

$$H_{\mathfrak{N}} \simeq S[[X_1, \ldots, X_r]]/(X_1^{p^{n_1}}, \ldots, X_r^{p^{n_r}}).$$

Proof. This is a consequence of [7], Th. 26.

3.9. Remark. Corollary 3.8 is also a consequence of [1], III, § 3, Cor. 6.4.

Case III. Profinite Hopf k-algebras.

3.10. Definition. A profinite k-algebra is a *topological k-algebra B* such that $B = \mathrm{Lim}\, B_i$ where B_i are finite k-algebras considered with the discrete topology (for details see [6], VII_B).

If $B = \mathrm{Lim}\, B_i$ and $C = \mathrm{Lim}\, C_j$ are profinite k-algebras, then by definition $B \hat{\otimes} C = \mathrm{Lim}(B_i \otimes_k C_j)$.

A profinite Hopf k-algebra is defined analogously to a Hopf k-algebra, the usual tensor product over k being replaced by $\hat{\otimes}$ (1).

3.11. Theorem. *If H is a profinite Hopf k-algebra, Noetherian as a ring and $\mathfrak{N}$ is a closed prime ideal in H, then there exists a sequence $n_1, \ldots, n_r$, $n_i \in N^\infty$, $r = e - \dim H_{\mathfrak{N}}$, such that the completion of the local ring $H_{\mathfrak{N}}$ is a $(p^{n_1}, \ldots, p^{n_r})$-regular local ring.*

Proof. Since the proof of this theorem is very similar to that of Theorem 3.7, we omit it. For the required results concerning profinite Hopf k-algebras see [6], VII_B, and [2], Th. on p. 48.

3.12. Corollary. *Under the assumptions of Theorem* 3.11 *there exist a complete regular local ring S and numbers $n_1, \ldots, n_s$, $n_i \in N^\infty$, such that*

$$\hat{H}_{\mathfrak{N}} \simeq S[[X_1, \ldots, X_s]]/(X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}}).$$

Proof. This follows from the previous theorem and [7], Cor. 24.

References

[1] M. Demazure and P. Gabriel, *Groupes Algébriques*, v. I, North Holland 1970.
[2] — *p-divisible groups*, Lect. Notes in Math. 302 (1972).
[3] J. Herzog, *Ringe der charakteristik p und Frobeniusfunktoren*, Math. Zeitschrift 140 (1974), pp. 67–78.
[4] E. Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. 91 (1969), pp. 772–784.
[5] J. W. Milnor and J. C. Moore, *On the structures of Hopf algebras*, Ann. of Math. 81 (1965), pp. 211–264.

(1) In [6], instead of profinite Hopf k-algebras one speaks about affine algebras of formal groups over the field k.

[6] J. W. Milnor and J. C. Moore, *Schemas en Groupes*, Lect. Notes in Math. 151 (1970).
[7] A. Tyc, *On h-regular graded algebras*, Fund. Math. 86, (1974), pp. 41–52.
[8] — *Basic properties of h-regular local Noetherian rings*, Fund. Math. 105 (1979), pp. 13–27.

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On fine shape theory

by

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Dedicated to Professor Karol Borsuk for his 70th birthday

Abstract. A fine shape category $\mathcal{C}_f$ is defined. The shape category $\mathcal{C}$ introduced by Borsuk is a quotient category of $\mathcal{C}_f$. $\mathcal{C}$ and $\mathcal{C}_f$ are not isomorphic. It is proved that $\mathcal{C}_f$ is isomorphic to the proper homotopy category of complements of compacta in the Hilbert cube and to a certain full subcategory of the proper shape category introduced by Ball and Sher.

1. Introduction. The concept of shape for compacta was first introduced by K. Borsuk [2]. T. Chapman [4] defined a weak proper homotopy category of complements of compacta in the Hilbert cube Q and proved that this category is isomorphic to Borsuk's shape category. As D. A. Edwards [6] asserted, it is natural to introduce a new shape category corresponding to a proper homotopy category of complements of compacta in Q. D. A. Edwards called it a strong shape category, but this terminology was already used by Borsuk [3] for a different concept, so we call it a fine shape category.

In this paper, first we shall define a fine shape category after a manner of Borsuk's fundamental sequences, and prove the equivalence to the proper homotopy category of complements of compacta in Q. Next, we shall give another characterization of this category in terms of the proper shape category introduced by Ball and Sher [1].

Throughout the paper all spaces are metrizable and maps are continuous. AR and ANR mean those for metric spaces.

2. Fine shape category. Let X be a compactum. We denote by $\mathscr{A}(X)$ the family of AR's M containing X as a subset. Let R_+ $(=[0,\infty))$ be the space of non negative reals. For compacta X, Y and for $M\in\mathscr{A}(X)$, $N\in\mathscr{A}(Y)$, a continuous map $F\colon M\times R_+\to N$ is said to be a *fundamental map from X to Y in M, N* if for every neighborhood V of Y in N there exist a neighborhood U of X in M and a number $t_0\in R_+$ such that

$$F(U\times[t_0,\infty))\subset V. \tag{1.1}$$