Remarks of the elementary theories of formal and convergent power series

by

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Abstract. In § 1 an example is given of two fields F_1, F_2 of characteristic 0 such that $F_1 \equiv F_2$ but $F_1[[x_1, x_2]] \not\equiv F_2[[x_1, x_2]]$. In § 2 it is shown that $\langle C\{x, y\}, C\{x\}\rangle \prec \langle C[[x, y]], C[[x]]\rangle$, where $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3, y_4)$.

In [3] and [4] Ax and Kochen and Eršov showed among other things that the ring of convergent power series $\boldsymbol{C}\{x\}$, over the complex numbers $\boldsymbol{C}$, is an elementary subring of the ring of formal power series $\boldsymbol{C}[[x]]$ over $\boldsymbol{C}$. This means that the same first order statements (in the language of valued rings) with constants from $\boldsymbol{C}\{x\}$, are true in both rings. (This is denoted $\boldsymbol{C}\{x\} \prec \boldsymbol{C}[[x]]$.) Also they showed that if fields F_1 and F_2 of characteristic 0 are elementarily equivalent, denoted $F_1 \equiv F_2$ (i.e. the same first order statements in the language of fields are true of F_1 and F_2) then $F_1[[x]] \equiv F_2[[x]]$ as valued rings (i.e. the same first order statements, in the language of valued rings, are true about $F_1[[x]]$ and $F_2[[x]]$). It is natural to ask whether these results extend to power series rings in several variables. In Section 1, we show that one can have fields $F_1 \equiv F_2$ but $F_1[[x_1, x_2]] \not\equiv F_2[[x_1, x_2]]$. In Section 2 we show that a slightly stronger statement than $\boldsymbol{C}\{x_1, \ldots, x_6\} \prec \boldsymbol{C}[[x_1, \ldots, x_6]]$ is false (¹). These remarks contradict some results claimed in [7].

Section 1. Eršov [4] showed that for any field F and for $n \geqslant 2$, $F[[x_1, \ldots, x_n]]$ is undecidable. We shall give a slightly different proof of this for the case that F has characteristic zero and use this proof to show that we can have $F_1 \equiv F_2$ of characteristic 0 but $F_1[[x_1, \ldots, x_n]] \not\equiv F_2[[x_1, \ldots, x_n]]$ ($n \geqslant 2$) as rings. Let F be a field of characteristic zero.

For the sake of clarity, we begin by showing that $\mathscr{F} = F[[x_1, \ldots, x_n]]$ is undecidable as an F algebra with x_1 and x_2 picked out, i.e., that $\mathscr{F}$ as a ring under the operations of addition and multiplication, with constants for x_1 and x_2, and with an additional predicate which picks out a particular lifting of the residue field F

(¹) (Added in proof) Some of the results of this paper and some extensions have been discoverd independently by F. Delon, *Résultats d'indécidabilité dans les anneaux de séries formelles* (to appear).

in $\mathscr{F}$, is undecidable. We do this by showing how to pick out N from F in a first order way, and applying the well known result of Gödel that the natural numbers with addition and multiplication are undecidable.

For $\alpha \in F$, let $N(\alpha)$ denote the statement:

$$\exists f \in \mathscr{F}\{(f \neq 0) \wedge (x_1 | f) \wedge$$
$$\wedge \left[\forall \beta \in F\{(x_1 - \beta x_2 | f) \Rightarrow [(x_1 - (\beta+1)x_2 | f) \vee (\beta = \alpha)]\}\right]\}.$$

It is trivial to check that α is natural number if and only if $N(\alpha)$ is true. (If α is an integer, let $f = \prod_{i=0}^{\alpha}(x_1 - ix_2)$. If α is not an integer, the statement implies that f is divisible by $x_1 - nx_2$ for each $n \in N$; since char $F = 0$, $\{x_1 - nx_2\}_{n \in N}$ are all inequivalent primes; since $F[[x_1, x_2]]$ is a unique factorization domain it follows that f has infinite order, and hence $f = 0$.)

Remark. It is interesting to notice that since $\prod_{i=0}^{n}(x_1 - ix_2)$ is a Weierstrass polynomial in x_1, that $\mathscr{F}$ is undecidable even if we restrict quantification to Weierstrass polynomials.

We now show $\mathscr{F}$ is undecidable as a ring by coding up the same idea in the residue field of $\mathscr{F}$ rather than in its lifting. In 4) below we give a first order statement which says the constant term of a unit is a natural number. In the above we made use of the specific elements 0, 1, x_1, and x_2 of the ring $\mathscr{F}$. Zero and one are, of course, first order definable over the ring, but x_1 and x_2 are not. In line 3, we eliminate the usage of x_1 and x_2 by coding up a property of x_1 and x_2 which is sufficient for the application in 4).

Let $\mathscr{F} = F[[x_1, \ldots, x_n]]$ and let m be the maximal ideal of $\mathscr{F}$. Then m is the set of non-invertable elements of $\mathscr{F}$ and hence is first order definable in the ring $\mathscr{F}$. Hence $\mathscr{F}/m (\simeq F)$ is first order definable from $\mathscr{F}$. We shall show how to define the natural numbers N in $\mathscr{F}/m$ in a first order way. Notice that the units in $\mathscr{F}$ are the power series with nonzero constant terms.

We begin by listing some first order statements:

1) $U(t)$ denotes the statement that t is a unit.

$$U(t) \leftrightarrow \exists v (tv = 1).$$

2) $P(t)$ denotes the statement that t is prime.

$$P(t) \leftrightarrow [\forall q, r[t = qr \rightarrow U(q) \text{ or } U(r)]] \text{ and } \neg U(t).$$

3) $P(y_1, y_2)$ says that different linear combinations of y_1 and y_2 give inequivalent primes.

$$P(y_1, y_2) \leftrightarrow P(y_1) \wedge P(y_2) \wedge \forall u_1, u_2[\{U(u_1) \wedge U(u_2) \wedge U(u_1 - u_2)\}$$
$$\rightarrow \{P(y_1 - u_1 y_2) \wedge P(y_1 - u_2 y_2) \wedge (\text{non } \exists v[U(v) \wedge (v(y_1 - u_1 y_2) = y_1 - u_2 y_2)])\}].$$

Note that $P(x_1, x_2)$ is true.

4) Define

$$UN(t) \leftrightarrow U(t) \wedge \exists y_1, y_2, f[P(y_1, y_2) \wedge (f \neq 0) \wedge (y_2 - y_1 | f) \wedge$$
$$\wedge \left[\forall s[\{U(s) \wedge (y_1 - sy_2) | f\} \rightarrow \{(\exists r[U(r) \wedge \neg U(r-s-1) \wedge\right.$$
$$\left. \wedge (y_1 - ry_2 | f)]) \vee (\neg U(s-t))\}]\right].$$

LEMMA. *In $\mathscr{F}$, $UN(t) \leftrightarrow t$ is a unit and the constant term of t is in N.*

Proof. If the constant term of $t = n \in N$ take $y_1 = x_1$, $y_2 = x_2$ and $f = \prod_{i=1}^{n}(x_1 - ix_2)$. It is trivial to check that $UN(t)$ is true. Conversely suppose that $UN(t)$ is true but that the constant term of t is $\notin N$. Then from the truth of $UN(t)$ it follows that there exist y_1 and y_2 and f such that $P(y_1, y_2)$ and there exist units s_i, $i \in N$, such that for all i, $y_1 - s_i y_2 | f$, and the constant term of $s_{i+1} = i+1$. But from $P(y_1, y_2)$ it would then follow that all the $y_1 - s_i y_2$ $(i \in N)$ are inequivalent primes and hence that f has infinite order. This contradicts $f \neq 0$.

Remark. Clearly the same argument also shows the ring of convergent power series in $n \geqslant 2$ variables is undecidable.

Let $N(t)$ denote the image of $UN(t)$ in $\mathscr{F}/m$, with the induced operations of addition and multiplication from $\mathscr{F}$. It is clear that $N(t)$ is first order definable over $\mathscr{F}$ by the formula $N(t)$ say, and that $N(t)$ is true iff $t \in N \subset \mathscr{F}/m$. Let $\boldsymbol{R}$ denote the real numbers and let R^* denote any nonarchimedean real closed field, for instance an ultrapower of $\boldsymbol{R}$. (For such a field F, the order is first order definable in terms of the field operations because a number $y \in F$ is positive iff there exist $x \in F$ with $y = x^2$.) Let φ be the statement which says that $N(t)$ is cofinal in $\mathscr{F}/m$. Then φ is true in $\boldsymbol{R}[[x_1, \ldots, x_n]]$ and false in $\boldsymbol{R}^*[[x_1, \ldots, x_n]]$. Hence we have:

THEOREM 1. *There exist fields F_1 and F_2 of characteristic 0 such that $F_1 \equiv F_2$ but for $n \geqslant 2$, $F_1[[x_1, \ldots, x_n]] \not\equiv F_2[[x_1, \ldots, x_n]]$ as rings.*

In the spirit of the above one can write down a formula $A(t)$ such that $A(t)$ picks out the elements of $\mathscr{F}/m$ which are algebraic (over $\boldsymbol{Q}$) and hence one can conclude that for $n \geqslant 2$, $\overline{\boldsymbol{Q}}[[x_1, \ldots, x_n]] \not\equiv \boldsymbol{C}[[x_1, \ldots, x_n]]$ where $\overline{\boldsymbol{Q}}$ denotes the algebraic closure of $\boldsymbol{Q}$. The details are long and tedious so we defer them to an appendix.

At this point, one should point out that a transfinite counting argument shows that there exists nonisomorphic fields F_1 and F_2 such that

$$F_1[[x_1, \ldots, x_n]] \equiv F_2[[x_1, \ldots, x_n]].$$

(There are more than $2^{\aleph_0}$ isomorphism classes of fields, but at most $2^{\aleph_0}$ elementary equivalence classes since there are only countable many first order statements and an elementary equivalence class is determined by the set of all first order statements which hold in that class.) However, if F_1 and F_2 are countable fields it is tempting to conjecture that $F_1[[x_1, x_2]] \equiv F_2[[x_1, x_2]] \Rightarrow F_1$ is isomorphic to F_2. The opinion of the authors on this question is somewhat less than unanimous.

We now recall Artin's theorem [2]: There is an integer valued function $\beta = \beta(n, N, d, \alpha)$ so that for any field F and polynomials $f = (f_1, \ldots, f_m)$ in $F[x, y]$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_N)$ with Σ degree $(f_i) \leqslant d$, if $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N) \in k[x]$ satisfy $f(x, \bar{y}) = 0 \bmod m(x)^\beta$ then there exist

$$y = (y_1, \ldots, y_N) \in k[[x]]$$

so that $f(x, y) = 0$ and $y = \bar{y} \bmod m(x)^\alpha$. A similar result holds for polynomials in y with coefficients in the power series ring $F[[x]]$, without any explicit bound on β.

If one restricts consideration to only existential statements, then we get some positive results. By an existential statement, we mean a first order statement using only existential quantifiers. By a positive existential statement we mean an existential statement containing equalities, but no inequalities. $R_1 \equiv_\exists R_2$ means that the same existential statements are true of R_1 and R_2. $R_1 \prec_\exists R_2$ means the same existential statements with constants from R_1 are true in R_1 and R_2. $R_1 \equiv_{p\exists} R_2$ and $R_1 \prec_{p\exists} R_2$ are defined in a similar manner using positive existential statements.

Proposition 1. *Let x denote $(x_1, \ldots, x_n)$*

(i) *If F_1 and F_2 are fields and $F_1 \equiv_\exists F_2$, then $F_1[[x]] \equiv_\exists F_2[[x]]$ as rings.*

(ii) *If $F_1 \subseteq F_2$ are fields and $F_1 \prec_\exists F_2$, then $F_1[[x]] \prec_\exists F_2[[x]]$ as rings and as F_1 algebras.*

(iii) *If the existential theory of the field F is decidable, then the positive existential theory of $F[[x]]$ as a ring and as an F algebra is decidable (i.e. positive existential statements from the ring are decidable), where F is of characteristic zero.*

(iv) *For any field F, the existential theory of $F[[x_1, x_2]]$ with two predicates P_1 and P_2, picking out the subrings $F[[x_1]]$ and $F[[x_2]]$ is undecidable i.e., formula of the form*

$$\exists y_1 \ldots \exists y_n (y_{i_j} \in F[[x_{\varepsilon_j}]] \wedge \bigwedge_{i=1}^{k} p_i(x, y) = 0 \wedge q(x, y) \neq 0)$$

$(\varepsilon_j = 1, 2)$ are undecidable. This statement will be abreviated by saying that the system $(F[[x_1, x_2]], F[[x_1]], F[[x_2]])$ is existentially undecidable.

Proof of (i) and (ii). Since $\exists$ distributes over $\vee$, it clearly suffices to consider just formulas of the form:

$$\exists y_1 \ldots \exists y_N (\bigwedge_{i=1}^{k} p_i(x, y) = 0 \wedge q(x, y) \neq 0), \quad \text{where} \quad p_i, q \in F_j[[x]][y].$$

By Artin's theorem this can be reduced to a existential statement about a finite dimensional vector space over F_j.

Proof of (iii). By the above argument, it suffices to give an algorithm for computing the β in Artin's theorem. This can be done by a careful chase through its proof, using the fact that the γ of Theorem 6.5 is computable because its statement is a first order statement in logic.

Proof of (iv). We show how to pick out N with existential quantifiers, by associating n to the powers x_1^n and x_2^n. Addition in N corresponds to multiplication in $F[[x_1, x_2]]$. Multiplication in N is defined as follows:

$$\operatorname{ord} f(x_1) = n, \quad \text{and} \quad \operatorname{ord} g(x_2) = k,$$

and

$$\operatorname{ord} h(x_1) = nk \leftrightarrow (g(x_2), x_2 - f(x_1)) F[[x_1, x_2]] = (h(x_1), x_2 - f(x_1)) F[[x_1, x_2]],$$

which is an existential statement. In this statement we only need inequalities to say that the elements $f(x_1)$ and $g(x_2)$ are nonzero. For additional details of this argument, see [5] where N is coded up using the full first order theory, but no subrings.

It is interesting to note the following corollary of the above, pointed out by Jan Denef. Let F be a finite field or an uncountable algebraically closed field. In the existential theory of $(F[[x_1, x_2]], F[[x_1]], F[[x_2]])$, it is possible to code up N with the operations of addition and multiplication, but it is not possible to code it up in the positive existential theory of $(F[[x_1, x_2]], F[[x_1]], F[[x_2]])$. We see this by considering the following statement which codes up an arbitrary diophantine equation.

$$\exists Y_1, \ldots, Y_m \in F[[x_1]], \exists Y_{m+1}, \ldots, Y_l \in F[[x_2]],$$
$$\exists Y_{l+1}, \ldots, Y_N \in F[[x_1, x_2]] (\bigwedge_{i=1}^{k} p_i(x, y) = 0, \text{ where } p_i, q \in F[x, y]).$$

By a standard iterated projection argument, if this has a solution $\bmod m(x)^n$ for all n, then it has a solution. It follows immediately that there exist β such that if it has a nonzero solution $\bmod m(x)^\beta$, then it has a nonzero solution. Since F is decidable, by checking if there is no solution $\bmod m(x)^n$ for each n, we give an algorithm which will detect if there is no solution. On the other hand, we can check for solutions of a diophantine equation by substituting integers inductively. This gives an algorithm for deciding the positive existential theory of N, a contradiction.

Section 2. Let $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3, y_4)$. In this section we shall show that $C\{x, y\} \prec C[[x, y]]$ as rings with a designated subring $C\{y\}$ (resp. $C[[y]]$) picked out. We do not know what happens without the designated subrings. The counterexample of Gabrielov in [6] shows that there exists $\varphi(x) = (\varphi_1(x), \ldots, \varphi_4(x))$, $\varphi_i(x) \in C\{x\}$ such that mapping $f(y) \mapsto f(\varphi)$ is one-to-one from $C\{y\} \to C\{x\}$ but is not one-to-one from $C[[y]]$ to $C[[x]]$, i.e., in $C\{x, y\}$, $\exists f(y)(f(y) \neq 0 \wedge f(\varphi) = 0)$ is false, and in $C[[x, y]]$, $\exists f(y)(f(y) \neq 0 \wedge f(\varphi) = 0)$ is true. Notice that in this statement we have used the subring to talk about $f(y)$ (i.e. $f \in C\{y\}$, resp. $C[[y]]$). We have also used composition of power series. Next we show how to make an equivalent statement without using composition.

Lemma. *Let $x = (x_1, \ldots, x_n)$, $y = y_1, \ldots, y_N$ and let*

$$f(x, y), \quad \varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x))$$

be fixed elements of $K\{x,y\}$ *or* $K[[x,y]]$ (*K is a (valued) field*). *Then for both formal and convergent power series*:

$$f(x,\varphi(x))=0\leftrightarrow \exists A(x,y)[f(x,y)=A(x,y)(y-\varphi(x))].$$

Proof. One direction is trivial by substituting $y=\varphi(x)$. The other direction follows from Taylor's Theorem as follows: $f(x,\varphi(x))-f(x,y)$ is an element of the ideal generated by $y-\varphi(x)$. Hence $f(x,\varphi(x))-f(x,y)=A(x,y)(y-\varphi(x))$, for some $A(x,y)=(A_1(x,y),\dots,A_N(x,y))$.

Let χ denote the formula $\exists f, A\,[f$ depends only on $y\wedge(f\neq 0)\wedge f=A(y-\varphi(x))]$. Then χ is true in $C[[x,y]]$ and false in $C\{x,y\}$. By $\mathfrak{A}\prec_{\exists}\mathfrak{B}$ we mean that $\mathfrak{A}\subseteq\mathfrak{B}$ and that every existential statement with constants from $\mathfrak{A}$ which is true in $\mathfrak{B}$ is true in $\mathfrak{A}$. Then we have:

THEOREM 2. *Let* $x=(x_1,x_2)$, $y=(y_1,y_2,y_3,y_4)$. *Then* $C\{x,y\}\prec_{\exists}C[[x,y]]$ *as rings with designated subrings corresponding to the power series which depend only on* y.

The authors do not know in general if $C\{z\}\prec C[[z]]$, or $C\{z\}\equiv C[[z]]$, where $z=(z_1,\dots,z_n)$. However, the theorem of Artin in [1] is equivalent to the statement that $C\{z\}\prec_{\exists}C[[z]]$ as rings (or rings with the order valuation into N).

Proof. Artin's theorem says that a convergent power series $f(x,y)$ has a nonzero solution $y(x)$ in $C\{x\}$ if and only if it has a nonzero solution in $C[[x]]$. We show that it suffices to consider only polynomial equations with coefficients in $C\{x,y\}$. Let $f(x,y)$ be a fixed element of $C\{x,y\}$. Then

$$\exists\varphi(x)[f(x,\varphi(x))=0,\ \text{with } \varphi \bmod m^{\nu}\ \text{specified}]$$
$$\leftrightarrow \exists A(x,y),\varphi(x)[f(x,y)=A(x,y)(y-\varphi(x)),\ \text{with } \varphi \bmod m^{\nu}\ \text{specified}]$$
$$\leftrightarrow \exists A,\psi[f(x,y)=A(x,y)(y-\psi(x,y))\ \text{with } \psi \bmod m^{\operatorname{ord}\varphi+2}\ \text{specified}].$$

Note that we may also restate the conditions without order by introducing universal quantifiers:

$$\exists\varphi(x)\neq 0(f(x,\varphi(x))=0)$$
$$\leftrightarrow \exists A,\psi,\psi_1,\psi_2[f(x,y)=A(x,y)(y-\psi(x,y))\wedge(y\nmid\psi)\wedge$$
$$\wedge(\psi(x,y)=x\psi_1(x,y)+y^2\psi_2(x,y))]$$

where by y^2 we mean the set of all monomials of degree 2 in the y_i's.

We now improve Theorem 2 in a manner which is probably of more interest to algebraists.

THEOREM 3. *Let* $x=(x_1,x_2)$, $y=(y_1,y_2,y_3,y_4)$. *Then* $C\{x,y\}\not\equiv C[[x,y]]$ *as rings with designated subrings corresponding to the power series which depend only on* y.

Proof. We will show how to remove the reference to the specific power series $\varphi(x)$. We begin by showing that the order valuation is first order definable in both of the rings $R=C[[x_1,\dots,x_n]]$, $C\{x_1,\dots,x_n\}$. (One does not need to add it to the language.) We begin by listing some first order statements:

a) Let $z\in R$ be prime. The statement that f is a power of z, i.e., $f=\text{unit}\cdot z^l$, for some $l\in N$:

$$\operatorname{Pow}_z(f)\leftrightarrow(z|f)\wedge\forall\ \text{prime}\ p\in R(p|f\to p\ \text{is equivalent to}\ z).$$

Clearly the set of powers of z is isomorphic to N, and are ordered by divisibility.

b) $f\in m^2\leftrightarrow\exists f_1,\dots,f_k,\ g_1,\dots,g_k\in m,\ k=\binom{n}{2}$ so $f=\sum f_i g_i$.

c) $S(z_1,\dots,z_n)\leftrightarrow(z_i\in m)\wedge\forall\alpha_i\in R(\sum_{i=1}^{n}\alpha_i z_i\in m^2\to\text{all}\ \alpha_i\in m)$.

d) $g\in C_{z_i}(f)\leftrightarrow\exists\alpha_i(S(z_1,\dots,z_n)$, and $\alpha_2,\dots,\alpha_n$ are units, and $\operatorname{Pow}_{z_1}(g)$ and, g is in ideal generated by $z_2-\alpha_2 z_1,\dots,z_n-\alpha_n z_1,f)$.

e) $\operatorname{ord} f=\min_{g\in C_{z_i}(f)}\operatorname{ord}_{z_1}(g)$.

It is trivial to check that the above is independent of the choice of the z_i and defines the usual order.

Now $\varphi(x)$ induces a mapping φ_* of local Noetherian rings $C\{y\}\to C\{x\}$, $C[[y]]\to C[[x]]$ and $\hat{\varphi}_*$ is injective, if and only if the two topologies on ($C\{y\}$, $C[[y]]$ respectively) agree. (See the Chevalley subspace theorem [8, Vol. 2, page 270].) Hence the statement

$$\forall\varphi(x)\in m\ [\varphi_*\ \text{injective}\to\text{topologies agree}]$$

is true formally but false convergently. It remains to make this into a first order statement.

Note that $v(f(\varphi))=\max\{v(h): f(y)\in(h(x,y),y-\varphi(x))\}$. Hence

$$\text{topologies agree}\leftrightarrow\forall k\exists l\forall f(y)[v(f(\varphi(x)))\geqslant l\to v(f(y))\geqslant k]$$
$$\leftrightarrow\forall k\exists l\forall f(y)[\exists h(x,y)(f\in(h,y-\varphi)\wedge v(h)\geqslant l)\to v(f)\geqslant k].$$

For $\psi(x,y)$, let $TA(\psi)$ be the statement:

$$\forall k\in N\,\exists l\in N\,\forall f(y)[\exists h(x,y)(f\in(h,y-\psi)\wedge v(h)\geqslant l)\to v(f)\geqslant k].$$

Also φ_* injective $\leftrightarrow\forall f(y)\exists k\,\forall h(x,y)[f\in(h,y-\varphi)\to v(h)\leqslant k]$. For $\psi(x,y)$, let $\operatorname{In}(\psi)$ be the statement:

$$\forall f(y)\exists k\in N\,\forall h(x,y)[f\in(h,y-\psi)\to v(h)\leqslant k].$$

Then the following is our required first order statement:

$$\forall\psi(x,y)\in m[(y\nmid\psi)\wedge(\exists\psi_1(x,y),\psi_2(x,y)(\psi=x\psi_1+y^2\psi_2))\wedge\operatorname{In}(\psi)\to TA(\psi)].$$

Note that this statement contains the symbols y_1, y_2, y_3, y_4 which are in $C\{y\}$. However we may remove the reference to these in the same manner as in defining order. Details are left to the reader.

Section 3. If we consider $C\{x\}$ and $C[[x]]$ (one variable) as rings with the extra operation of composition for elements of the maximal ideal, m, then we have:

PROPOSITION. (i) *$C[[x]]$ and $C\{x\}$ are undecidable.*

(ii) $C[[x]] \not\equiv C\{x\}$.

(iii) *There exist fields F_1 and F_2 with $F_1 \equiv F_2$, but $F_1[[x]] \not\equiv F_2[[x]]$.*

Proof. Define $f \sim g \leftrightarrow (f|g$ and $g|f)$ for $f, g \in m$. Then $m/\sim$ is isomorphic to N, and multiplication and composition of elements of m corresponds to addition and multiplication in N, respectively. Hence N is first order definable in $C[[x]]$ and $C\{x\}$. Also note that the order valuation is first order definable; just let $v(f) = [f]$, the equivalence class of f. For the proof of (ii) let R be either of the rings $C[[x]]$ or $C\{x\}$. Then the derivative f' is first order definable since

$$g = f' \leftrightarrow \forall n \in N\, \exists k \in N\, \forall h \in R\left[v(h) \geqslant k \rightarrow v\left(\frac{f(x+h)-f(x)}{h} - g(x)\right) \geqslant n\right].$$

It is trivial to check that the only power series solution to the differential equation $x^2 f'(x) + x = f(x)$, is $f(x) = \sum_{n=0}^{\infty} n!\, x^{n+1}$, which is divergent. This gives us a first order statement using x which is true in $C[[x]]$, but false in $C\{x\}$; hence $C\{x\} \not\prec C[[x]]$.

One can also show the rings are not elementarily equivalent, by eliminating the use of the symbol x, in the following manner: It is not hard to show there is a formal solution but no convergent solution to the equations:

$$(E-1)^2 f' + E - 1 = f \quad \text{and} \quad E' = E \quad \text{and} \quad \neg U(E-1).$$

Alternate proof of part (ii). If we consider the rings $C[[x]]$, $C\{x\}$ with the element x, picked out, we can show $C\{x\} \not\prec_{\exists} C[[x]]$ as follows: A straightforward calculation shows that the equation $f(x+x^2) = 2f(x) - x$ has a unique solution in $C[[x]]$ and it is not convergent.

Proof of (iii). Recall $F = \mathscr{F}/m$ is first order definable over $\mathscr{F}$. For $k \in F$, let the first order statement that k is a power of 2 be denoted:

$$\mathrm{Pow}_2(k) \leftrightarrow \exists f \in \mathscr{F}(\neg U(k - f(2x)/f(x))).$$

Then the powers of 2 are cofinal in R but not in R^*. Hence $R[[x]] \not\equiv R^*[[x]]$.

Remark. It is interesting to note that in parts (i) and (ii) we only needed existential statements, but in part (iii), we used both universal and existential quantifiers. If F_1 and F_2 are finite fields or uncountable algebraically closed fields then it is not possible to do part (iii) with just existential statements because the remark below shows

$$(F_1[[x, y]], F_1[[x]], F_1[[x]]) \equiv_{\exists} (F_2[[x, y]], F_2[[x]], F_2[[y]])$$

as rings, implies $F_1[[x]] \equiv_{\exists} F_2[[x]]$ with composition. The authors conjecture that one can drop the condition on the fields F_i.

Remark. The results of Section 3 have application to the previous two sections because any statement using composition in $F[[x]]$ can be changed into a corresponding statement using just addition and multiplication in the system $(F[[x, y]], F[[x]], F[[y]])$. This is done as follows: Let $\Phi(f, g)$ be any atomic formula over $F[[x]]$ in which the symbol $f(g(x))$ appears, where $f, g \in F[[x]]$, and Q_1, Q_2 (equal to either $\exists$ or $\forall$) be the respective quantifiers of f and g. Change $\Phi(f, g)$ to $\Psi(g, f, h, s_1, s_2, \sigma)$, where Ψ is formed from Φ by replacing $f(g(x))$ by $h(x)$ and adding two statements which insure that $h(x) = f(g(x))$. More precisely, ψ is the statement:

$$Q_2 g\, Q_1 f\, \exists h\, \exists S_1(x, y), S_2(x, y), \sigma(y)$$
$$[\Phi(f, g, h) \wedge (f(x) = S_1(x, y)(y-x) + \sigma(y)) \wedge$$
$$(h(x) = S_2(x, y)(y - g(x)) + \sigma(y))].$$

Clearly the same thing works for convergent power series. As a consequence of part (ii) of the proposition, we can conclude that

$$(C\{x, y\}, C\{x\}, C\{y\}) \not\equiv_{\exists} (C[[x, y]], C[[x]], C[[y]]),$$

as rings with the elements x and y picked out.

Appendix. We show how to say in a first order statement that the constant term of an element in $R = C[[x, y]]$ is an algebraic number. This will show that $A[[x, y]] \not\equiv C[[x, y]]$ as rings with the elements x and y picked out. For each integer d, the numbers A_d which are algebraic over Q of degree d are first order definable. The obvious problem is that one has no upper bound on the degree of the polynomial, the number satisfies. We begin with some observations:

a) Let $\gamma \in C$, then $\gamma \in A$ if and only if there is a nonzero polynomial $p(x, y)$ of homogeneous degree and with rational coefficients such that $p(\gamma y, y) = 0$. By passing to a polynomial of possible higher degree, we may assume that the coefficients c_{ij} of $x^i y^j$, $i+j = \deg p$, are all nonzero.

b) Recall from Sections 1 and 2 the residue field F of $F[[x, y]]$, the natural numbers N in F, and the order evaluation v are all first order definable over the ring $F[[x, y]]$. Hence Q, the rational numbers, are first order definable. If $u \in F[[x, y]]$, by the statement $u \in Q$ we will mean the first order statement that the image $\bar{u}$ of u in the residue field is rational. Also $u \in A$ will mean $\bar{u}$ is algebraic.

c) Let $\gamma \in C[[x, y]]$, then $\gamma \in A$ iff there exist $f \in C[[x, y]]$, $f = p + h$, $p =$ initial term of f, p has rational coefficients, and $v(f(\gamma y, y)) > v(f(x, y))$. Again we may assume all coefficients of p are nonzero.

d) We would like say pick out the coefficients of f via the statement

$$C(i, f, \alpha) \leftrightarrow (v(f) = n) \wedge$$
$$\wedge (\exists g_1, g_2 \in R \text{ so that } f - \alpha x^i y^{n-i} = x^{i+1} g_1 + y^{n-i+1} g_2).$$

However we do not have x^i and y^{n-i} in our language, only x and y. We can say $\varphi = x^i$ unit, via $\mathrm{Pow}_x(\varphi)$ and $v(\varphi) = i$, but we do not know the constant term of the unit. (Note if φ_1, φ_2 have the above property then the respective units have the same constant term if and only if $v(\varphi_1 - \varphi_2) > v(\varphi_1) = v(\varphi_2)$.) We get around these difficulties by saying the ratio of consecutive coefficients of p is rational. Since this condition is vacuous when one of the coefficients is zero, we require that all the coefficients be nonzero.

The statement that the initial term of f has rational coefficients becomes

$$\mathrm{IRC}(f) \leftrightarrow (v(f) = n) \wedge \forall i \in N, \quad 1 \leqslant i \leqslant n,$$

$$\exists u_i, u_{i-1}, u_{i+1}, v_{n-1}, v_{n-i+1}, v_{n-i+2} \in R$$
$$\exists g_1, g_2, g_3, g_4, \alpha, \beta \in R, \ \exists r \in Q \text{ so that}$$
$$\mathrm{Pow}_x(u_{i-1}) \wedge \mathrm{Pow}_x(u_i) \wedge \mathrm{Pow}_x(u_{i+1}) \wedge \mathrm{Pow}_y(v_{n-i}) \wedge$$
$$\wedge \mathrm{Pow}_y(v_{n-i+1}) \wedge \mathrm{Pow}_y(v_{n-i+2}) \wedge$$
$$\wedge (v(u_{i-1}) = i-1) \wedge (v(u_i) = i) \wedge (v(u_{i+1}) = i+1) \wedge$$
$$\wedge (v(v_{n-1}) = n-i) \wedge (v(v_{n-i+1}) = n-i+1) \wedge (v(v_{n-i+2}) = n-i+2) \wedge$$
$$\wedge (v(xu_{i-1} - u_i) > i) \wedge (v(yv_{n-i} - v_{n-i+1}) > n-i+1) \wedge$$
$$\wedge (f - \alpha u_i v_{n-i} = u_{i+1} g_1 + v_{n-i+1} g_2) \wedge (\alpha \neq 0) \wedge$$
$$\wedge (f - \beta u_{i-1} v_{n-i+1} = u_i g_3 + v_{n-i+2} g_4) \wedge (\beta \neq 0) \wedge$$
$$\wedge (\neg U(\alpha - r\beta)).$$

Hence for $\gamma \in R$,

$$A(\gamma) \leftrightarrow \exists f, g \in R, \ n \in N [\mathrm{IRC}(f) \wedge (v(f) = n) \wedge (v(g) > n) \wedge$$
$$\text{the ideals } (x - \gamma y, f)R \text{ and } (x - \gamma y, g)R \text{ are equal}].$$

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