Upper semicontinuous decompositions of convex metric spaces *

by

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Abstract. All decompositions in this paper are upper semicontinuous.

THEOREM A. *If G is a locally null, properly starlike-equivalent decomposition of a locally compact*, SC–WR–CE *metric space* (X, d), *then G is radially-shrinkable in* (X, d) *and* $X/G \approx X$.

COROLLARY. *If G is a locally null, starlike-equivalent decomposition of E^n, then* $E^n/G \approx E^n$.

THEOREM B. *If G is a star-0-dimensional decomposition of a locally compact*, SC–WR–CE *metric space* (X, d), *then G is shrinkable and* $X/G \approx X$.

1. Introduction. All decompositions in this paper are upper semicontinuous. The famous "dogbone" space of R. H. Bing [5] has spawned an amazing array of results and questions. In [4], Bing showed that if G is a decomposition of E^3 into at most countably many tame arcs and points, then $E^3/G \approx E^3$. This raised the following question (see S. Armentrout [2], Question 1): Suppose G is a decomposition of E^3 into tame 3-cells and points; is $E^3/G \approx E^3$? A partial answer was given by D. V. Meyer [11]: A null decomposition of E^3 into tame 3-cells and points is E^3. This result was improved by R. J. Bean [3]: Null, starlike-equivalent decompositions of E^3 yield E^3. This led J. W. Lamoreaux in [8] to ask whether locally null, starlike-equivalent decompositions of a SC-WR metric space (X, d) yield X. In this paper we show the answer is no (see Example 1 of Section 2) yet obtain the following theorem.

THEOREM A. *If G is a locally null, properly starlike-equivalent decomposition of a locally compact*, SC-WR-CE *metric space* (X, d), *then G is radially-shrinkable in* (X, d) *and* $X/G \approx X$.

T. M. Price [13] has proved that if G is a decomposition of E^n such that for each $g \in H(G)$ and for each open set V containing g there is an n-cell B such that $g \subset \operatorname{Int} B \subset V$ and $\operatorname{Bd} B \cap [\bigcup H(G)] = \emptyset$, then $E^n/G \approx E^n$. The condition that B is an n-cell is weakened in this paper. We strengthen Price's theorem and extend it to SC-WR-CE metric spaces in the following theorem.

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THEOREM B. *If G is a star-0-dimensional decomposition of a locally compact,* SC-WR-CE *metric space (X, d), then G is shrinkable and $X/G \approx X$.*

To illustrate Theorems A and B, we give three examples; Examples 1 and 2 are consequences of Theorem A, Example 3 of Theorem B.

EXAMPLE 1. If G is a locally null, starlike-equivalent decomposition of E^n, then $E^n/G \approx E^n$. In particular, we can choose G to be a locally null decomposition of E^n into tame cells (dimension $\leqslant n$) and points such that given $\varepsilon > 0$, infinitely many of the cells have diameter $\geqslant \varepsilon$.

EXAMPLE 2. Let $X(n) = \{(x_1, \ldots, x_{n+1}): \sum_{i=1}^{n+1} x_i^2 = 1, x_1 \geqslant 0, \ldots, x_n \geqslant 0,$ and $x_{n+1} \geqslant \frac{1}{2}\}$, and let $X(n)$ be topologized by d_n, the "great S^{n-1}" metric of S^n. Let G be a locally null decomposition of $X(n)$ such that $\bigcup H(G)$ is contained in the manifold interior of $X(n)$, $H(G)$ is a collection of tame cells (dimension $\leqslant n$), tame whisk-brooms, tame fan-spaces, etc., and given $\varepsilon > 0$, infinitely many members of $H(G)$ have diameter $\geqslant \varepsilon$. Then G is radially-shrinkable in $(X(n), d_n)$ and $X(n)/G \approx X(n)$.

EXAMPLE 3. Let G be a decomposition of E^n such that each $g \in G$ possesses a neighborhood base $\{U_n\}$ such that $\mathrm{Bd}\, U_n \cap [\bigcup H(G)] = \emptyset$, $U_n \supset \mathrm{Cl}\, U_{n+1}$, U_n is an open n-cell, and $\mathrm{Cl}\, U_n$ is starlike but not an n-cell ($\mathrm{Bd}\, U_n$ could have a "sin $(1/x)$ configuration"). Then G is shrinkable and $E^n/G \approx E^n$.

Example 3 can be modified for non-Euclidean spaces ala Example 2. It is the principal goal of this paper to prove Theorems A and B: Theorem A is established in Section 5 and Theorem B in Section 6. In Section 2 we give preliminaries and in Section 3 we develop the machinery used in Sections 4, 5, and 6.

2. Preliminaries. We are always in a locally compact, strongly convex metric space (X, d). For the definitions of *betweeness*, *midpoint*, *convexity*, *strong convexity* (SC), and *without ramifications* (WR), see D. Rolfsen [15]. We do *not* assume that strongly convex spaces are separable or complete. Let $a, b \in X$. We say L is a *segment* between a and b (or from a to b) if $a, b \in L$, each point of L is between a and b, and L is isometric to a real line interval of length $d(a, b)$. If L is the unique segment from a to b we write $L = [ab]$. A segment L from p to y is *maximal* if there is no $x \in X$ such that some segment from p to x properly contains L. It is well known (see [15]) that in a complete, convex metric space, there is a segment between each two points. In the presence of local compactness and strong convexity, the requirement of completeness may be dropped.

PROPOSITION 2.1. *Let (X, d) be a locally compact,* SC *metric space.*

(1) *If $a, b \in X$ then there is a segment from a to b* (see [14]).

(2) *X is arc-wise and locally arc-wise connected.*

(3) *For each $a, b \in X$, there is a unique segment joining a and b. If (X, d) is also a* WR *space, $y \neq y'$, $[xy]$ and $[xy']$ are segments in X, $y \notin [xy']$, and $y' \notin [xy]$, then $[xy] \cap [xy'] = \{x\}$.*

(4) *Let $a, b, x \in X$ such that x is between a and b. Then $x \in [ab]$.*

Let $[ab]$ be a segment in X and let h be the isometry of $[ab]$ onto $[0, d(a, b)]$ such that $h(a) = 0$. For $x, y \in [ab]$ and $\lambda \in [0, 1]$, define $(1-\lambda)x + \lambda y$ to be $h^{-1}[(1-\lambda)h(x) + \lambda h(y)]$. This algebraic operation has many useful properties (including that it is jointly continuous in λ, x, y if x, y are contained in a compact subspace of X) which will be extensively used in this paper (see [10], [14], and [15]).

The closure of a set A is denoted by $\mathrm{Cl}\, A$ and its boundary by $\mathrm{Bd}\, A$. A collection of neighborhoods containing a set A is a neighborhood base for A if each open set containing A contains an element of the base. Neighborhoods are open. If N is a neighborhood of p, then the *edge of N w.r.t. p*, or $\mathrm{Ed}_p N$, is $\{y \in \mathrm{Cl}\, N: [py]$ is maximal$\}$. We say (X, d) has *closed edges* (or (X, d) is CE) if for each point p of X, $\mathrm{Ed}_p X \cup \{p\}$ is closed (the class of convex metric spaces satisfying the closed edge property strictly contains the class of normed linear spaces). A set A is *starlike w.r.t. p* if for each $x \in A$, $[px] \subset A$; the point p is called a *reference point* of A. A starlike w.r.t. p set A is *properly starlike w.r.t. p* if for each $x \in A - p$, the segment $[px]$ is not maximal. A neighborhood N of p is *ideally starlike w.r.t. p* if N is starlike w.r.t. p and for each $x \in X - N$, $[px]$ intersects $\mathrm{Bd}\, N$ in at most one point. A set A is *radially pointlike w.r.t. p* if A is starlike w.r.t. p and for each neighborhood U of A, there is an ideally starlike w.r.t. p neighborhood V of A and homeomorphism H from $X - A$ onto $X - p$ such that (1) $\mathrm{Cl}\, V \subset U$, (2) H takes $\mathrm{Cl}\, V - A$ onto $\mathrm{Cl}\, V - p$, and (3) for each $x \in X - A$, $H(x) \in [px]$. A collection J of subsets of X is *locally null* if for each $x \in X$, there is an open set U containing x such that the collection of all sets of J that intersect U is a null collection.

For the definitions of *upper semicontinuous* (u.s.c.) *decomposition*, *decomposition space* (X/G), *monotone*, *pointlike*, *0-dimensional*, and *shrinkable* (or Condition B) see [1], [9], or [17]. Let G be a decomposition of X. Let $H(G)$ denote the collection of nondegenerate elements of G and let $G(\delta) = \{g \in H(G): \mathrm{diam}\, g \geqslant \delta\}$ where $\delta > 0$. We say G is *null* (*locally null*) if $H(G)$ is a null (locally null) collection.

PROPOSITION 2.2. *Let G be an u.s.c., monotone decomposition of (X, d). Then G is locally null if and only if for each $\delta > 0$, every subcollection of $G(\delta)$ has a closed point-set union. In either case, $H(G)$ is countable and hence G is 0-dimensional* (see [14]).

We say K is an *open covering of $H(G)$* if K is a collection of open sets such that each element of $H(G)$ is contained in some element of K. We say G is *starlike* if each $g \in H(G)$ is compact and starlike. We say G is *starlike-equivalent* (*properly starlike-equivalent*; *radially-pointlike*) if each $g \in H(G)$ is equivalent under a space homeomorphism to a compact, starlike set (compact, properly starlike set; compact, radially-pointlike set). Often when showing a decomposition to be shrinkable, the non-degenerate elements are shrunk along arcs (e.g. see [3] and [11]). We isolate this property, calling it radial-shrinkability. Intuitively, a decomposition is radially-shrinkable if for each $g \in H(G)$ we can choose a space homeomorphism H, a compact, starlike set q, and a reference point p of q such that H takes g onto q, and q can be shrunk along segments toward p in such a way that g is shrunk along arcs toward $H^{-1}(p)$.

Let $H(G) = \{g_\alpha: \alpha \in \mathfrak{A}\}$. We say G is *radially-shrinkable* in (X, d) if there are collections of maps $\{h_\alpha\}$, compact, starlike sets $\{q_\alpha\}$, and points $\{p_\alpha\}$ such that for each $g_\alpha \in H(G)$, h_α is a space homeomorphism taking g_α onto q_α and q_α is starlike w.r.t. p_α, and such that for each $\varepsilon>0$ and for each open set U containing $\bigcup H(G)$, there is h such that

(1) h is a homeomorphism from X onto X and $h|(X-U)$ is the identity;

(2) $\operatorname{diam} h(g_\alpha)<\varepsilon$ for each $g_\alpha \in H(G)$; and

(3) if $g_\alpha \in G(\varepsilon)$, there is a neighborhood V_α and there is a map f_α such that $g_\alpha \subset V_\alpha \subset \operatorname{Cl} V_\alpha$ and $h_\alpha[\operatorname{Cl} V_\alpha]$ is starlike w.r.t. p_α, f_α is an embedding of $h_\alpha[\operatorname{Cl} V_\alpha]$ into $h_\alpha[\operatorname{Cl} V_\alpha]$ and $f_\alpha|\operatorname{Ed}_{p_\alpha} h_\alpha(V_\alpha)$ is the identity, $f_\alpha(h_\alpha(x)) \in [p_\alpha h_\alpha(x)]$ for each $x \in \operatorname{Cl} V_\alpha$, and $h|\operatorname{Cl} V_\alpha = h_\alpha^{-1} \circ f_\alpha \circ h_\alpha$.

If $B \subset X$, then let $G(B)$ be the decomposition of X such that $H(G(B)) = \{g \in H(G): g \subset B\}$. We say G is *shrinkable* (*radially-shrinkable*) *at* $g \in H(G)$ if there is an open set U containing g such that $\operatorname{Bd} U \cap [\bigcup H(G)] = \emptyset$ and $G(U)$ is shrinkable (radially-shrinkable). We say G is *star-0-dimensional* if for each $g \in H(G)$, there is a neighborhood base $\{U_n\}$ for g such that for each n $\operatorname{Bd} U_n \cap [\bigcup H(G)] = \emptyset$, $U_n \supset \operatorname{Cl} U_{n+1}$, and $\operatorname{Cl} U_n$ is compact and homeomorphic to the closure of an open, starlike w.r.t. p_n set with empty edge w.r.t. p_n.

EXAMPLES. Let $(X(2), d_2)$ be as defined in Example 2 of Section 1. Then $(X(2), d_2)$ is a compact, SC-WR-CE metric space (which is not the linear subspace of any normed linear space). Let $p = (\frac{1}{2}\sqrt{3}, 0, \frac{1}{2})$ and let $N(p, \varepsilon)$ be the neighborhood of p with radius ε. Then $p \in \operatorname{Cl}(\operatorname{Ed}_p N(p, \varepsilon))$. Circumstances like this will force us to be careful when constructing shrinkings which move points radially toward a given point.

EXAMPLE 1. Let G be the decomposition of $X(2)$ such that $H(G_1) = \{g\}$, where $g = \{(x, y, z): (x, y, z) \in X(2) \text{ and } y = x\}$. Then G is a null, starlike decomposition of $(X(2), d_2)$ and $X(2)/G \not\approx X(2)$.

EXAMPLE 2. Let G_2 be the decomposition of $X(2)$ such that $H(G_2) = \{g\}$, where $g = \{(x, y, z): (x, y, z) \in X(2) \text{ and } y = 0\}$. Then G_2 is a null, starlike decomposition of $X(2)$ which is shrinkable and pointlike but neither radially-shrinkable nor radially-pointlike in $(X(2), d_2)$.

3. Neighborhood bases for starlike sets. One key to constructing the shrinkings used by Bing [4], Meyer [11], and Bean [3] is the fact that in E^3 starlike sets possess neighborhood bases of ideally starlike sets. In a convex metric setting, showing the existence of such neighborhood bases is non-trivial. In this section we show that in locally compact, SC-WR metric spaces, compact, starlike sets have neighborhood bases of ideally starlike sets. Using this result, we establish two results needed to construct the shrinkings of Sections 5 and 6.

LEMMA 3.1. *Let (X, d) be a locally compact,* SC-WR *metric space. Let $p \in N \subset X$ and let h be a continuous map of $N-p$ into N such that $h(x) \in [px]$ for each $x \in N$. Then h is extended continuously to N by letting $h(p) = p$, and h, thus extended, is one-to-one if and only if $y \in [px]-\{x, p\}$ implies $h(y) \in [ph(x)]-h(x)$ for each $x \in N-p$.*

LEMMA 3.2. *Let A and N be subsets of a locally compact,* SC-WR *metric space (X, d) such that N is a neighborhood of A, and each of A and $\operatorname{Cl} N$ is a compact, starlike w.r.t. p set. Let h be an embedding of $\operatorname{Cl} N$ into $\operatorname{Cl} N$ such that for each $x \in \operatorname{Cl} N$, $h(x) \in [px]$. Then there is an embedding H of $\operatorname{Cl} N$ into $\operatorname{Cl} N$ such that $H(x) \in [px]$ for each $x \in \operatorname{Cl} N$, $H|A = h$, $H|\operatorname{Bd} N$ is the identity, and if $h|\operatorname{Ed}_p N$ is the identity, then H is a homeomorphism of $\operatorname{Cl} N$ onto $\operatorname{Cl} N$.*

Proof. Let f be a continuous function of $\operatorname{Cl} N$ onto $[0, 1]$ such that $f(A) = 0$ and $f(\operatorname{Bd} N) = 1$ and let $F(x) = \max\{f(y): y \in [px]\}$ for each $x \in \operatorname{Cl} N$. It follows that F is a continuous function of $\operatorname{Cl} N$ onto $[0, 1]$ such that $F(A) = 0$ and $F(\operatorname{Bd} N) = 1$. Furthermore, if $y \in [px]$ then $F(y) \leqslant F(x)$. Now for each $x \in \operatorname{Cl} N-p$, define

$$G(x) = F(x)\left[1-\frac{d(p, h(x))}{d(p, x)}\right]+\frac{d(p, h(x))}{d(p, x)}.$$

It follows that G is a continuous function of $\operatorname{Cl} N-p$ into $[0, 1]$. We now construct H. For each $x \in \operatorname{Cl} N$, define

$$H(x) = \begin{cases} G(x)x+(1-G(x))p & \text{for } x \neq p, \\ p & \text{for } x = p. \end{cases}$$

Clearly H satisfies the requirements of the conclusion providing H is an embedding. The continuity of H follows from Lemma 3.1. We need only show H is one-to-one, and this is done by satisfying Lemma 3.1. Let $x \in \operatorname{Cl} N-p$ and let $y \in [px]-\{x, p\}$. It is not hard to show that $H(y) \in [pH(x)]-H(x)$ if and only if $G(y)d(p, y) < G(x)d(p, x)$. We establish this inequality by considering, three cases: $F(y) = 1$, $F(y) = 0$, and $0<F(y)<1$. The inequality holds trivially in the first two cases. Now suppose $0<F(y)<1$. Observing that $h(y) \in [ph(x)]-h(x)$ by Lemma 3.1 and hence $d(p, h(y))<d(p, h(x))$, it follows that

$$d(p, y)-d(p, x)<[d(p, h(x))-d(p, h(y))][(1/F(y))-1].$$

Manipulating algebraicly, we have

$$\begin{aligned} F(y)[d(p, y)-d(p, h(y))]+d(p, h(y)) &< F(y)[d(p, x)-d(p, h(x))]+d(p, h(x)) \\ &\leqslant F(x)[d(p, x)-d(p, h(x))]+d(p, h(x)). \end{aligned}$$

This completes the proof.

LEMMA 3.3. *Let A be a compact, starlike w.r.t. p set in a locally compact,* SC-WR *metric space (X, d) and let U be an open set containing A. Then there is a neighborhood N of A such that $\operatorname{Cl} N$ is compact, $\operatorname{Cl} N \subset U$, and N is ideally starlike w.r.t. p.*

Proof. Let $\delta>0$ such that $\operatorname{Cl} N(A, \delta)$ is compact and contained in U. Define N^* to be $\{y: [py] \cap \operatorname{Bd} N(A, \delta) = \emptyset\}$. It follows that $\operatorname{Cl} N^*$ is compact, $\operatorname{Cl} N^* \subset U$, and N^* is starlike w.r.t. p. It follows from Proposition 2.1 (4) that N^* is a neighborhood of A. Let $\Delta = \operatorname{diam} A$. Choose a circular neighborhood S of p such that $\mathrm{Cl}\, S \subset N^*$. Let $\varepsilon > 0$ such that $0 < \varepsilon < \Delta$ and $\mathrm{Cl}\, N(p, \varepsilon) \subset S$, and let $\lambda = \varepsilon/\Delta$. Define h by

$$h(x) = \begin{cases} \lambda x + (1-\lambda)p & \text{for} \quad x \neq p, \\ p & \text{for} \quad x = p. \end{cases}$$

As in the proof of Lemma 3.2, it can be shown that h is an embedding of $\mathrm{Cl}\, N^*$ into $\mathrm{Cl}\, N^*$. By Lemma 3.2, there is an embedding H of $\mathrm{Cl}\, N^*$ into $\mathrm{Cl}\, N^*$ such that $H(x) \in [px]$ for each $x \in \mathrm{Cl}\, N^*$, $H|A = h$, and $H|\mathrm{Bd}\, N^*$ is the identity. It follows that $H(A) \subset S$. We choose N to be $H^{-1}(S)$. It is straightforward to show that N is the required neighborhood of A.

LEMMA 3.4. *Let A be a compact, properly starlike w.r.t. p set in a locally compact,* SC-WR *metric space (X, d) and let U be an open set containing A. Then there is an ideally starlike w.r.t. p neighborhood N of A such that $\mathrm{Cl}\, N$ is compact, $\mathrm{Cl}\, N \subset U$, and no nondegenerate segment from p in A has its terminal point in $\mathrm{Ed}_p N$.*

LEMMA 3.5. *Let A be a compact, properly starlike w.r.t. p set in a locally compact,* SC-WR-CE *metric space (X, d), let U be an open set containing A, and let $\varepsilon > 0$. Then there is an ideally starlike w.r.t. p neighborhood N of A such that $\mathrm{Cl}\, N$ is compact, $N \subset U$, and $\mathrm{Ed}_p N \subset N(p, \varepsilon)$.*

Proof. Let S denote the collection of segments in A from p which cannot be extended in A. Let $s \in S$ and suppose the conclusion is false for s as a properly starlike w.r.t. p set. Let $\{N_n(s)\}$ be a nested neighborhood base of ideally starlike w.r.t. p sets for s such that $\mathrm{Cl}\, N_1(s)$ is compact and $s \cap \mathrm{Ed}_p N_n(s) = \emptyset$ for each n (Lemma 3.4). We choose $x_n \in \mathrm{Ed}_p N_n(s) - N(p, \varepsilon)$ for each n. Then $\{x_n\}$ is contained in the compact set $\mathrm{Ed}_p N_1(s) - N(p, \varepsilon)$.

We may assume $x_n \to x$, where $x \in \mathrm{Ed}_p N_1(s) - N(p, \varepsilon)$. But $x \in [\bigcap \mathrm{Cl}\, N_n(s)] - N(p, \varepsilon)$; this implies $x \in s - p$, a contradiction. Thus for each $s \in S$, we have a neighborhood $N(s)$ of s such that $N(s) \subset U$, $\mathrm{Cl}\, N$ is compact, and $\mathrm{Ed}_p N(s) \subset N(p, \varepsilon)$. Since A is covered by $\{N(s)\colon s \in S\}$, we may choose an ideally starlike w.r.t. p neighborhood N of A such that $\mathrm{Cl}\, N$ is compact and $\mathrm{Cl}\, N \subset \bigcup N(s)$ (Lemma 3.3). Since $\mathrm{Ed}_p N \subset \bigcup \mathrm{Ed}_p N(s)$, we have $\mathrm{Ed}_p N \subset N(p, \varepsilon)$.

LEMMA 3.6. *Let (X, d) be a locally compact,* SC-WR *metric space and let U be an open set in X containing p such that $\mathrm{Cl}\, U$ is compact and $\mathrm{Ed}_p U = \emptyset$. Then U is starlike w.r.t. p if and only if $U = \bigcup V_n$ where each V_n is ideally starlike w.r.t. p neighborhood and $\mathrm{Cl}\, V_n \subset V_{n+1}$ for each n.*

Proof. Sufficiency is straightforward. As for necessity, let s be a segment from p to $\mathrm{Bd}\, U$ and let p be considered the first point of s. Let $x(s)$ be the first point on s where s hits $\mathrm{Bd}\, U$. Then U is starlike w.r.t. p implies $U = \bigcup([px(s)] - x(s))$. Now let $\varepsilon_1 > 0$ such that $\varepsilon_1 < \dfrac{d(p, \mathrm{Bd}\, U)}{2}$. It can be shown that there is $\delta_1 > 0$ such that $y \notin N(\mathrm{Bd}\, U, \varepsilon_1)$ implies $[py] \cap N(\mathrm{Bd}\, U, \delta_1) = \emptyset$. Fix a segment s. With respect to the linear ordering on $[px(s)]$, let $y_1(s) = \sup\{y \in [px(s)]\colon$ there is $y' \in [yx(s)]$ such that $d(y', \mathrm{Bd}\, U) > \varepsilon_1\}$. Let $A_1 = \bigcup[py_1(s)]$. Then $\mathrm{Cl}\, A_1 \subset U$. Since A_1 is starlike w.r.t. p and $\mathrm{Cl}\, A_1$ is compact, $\mathrm{Cl}\, A_1$ is starlike w.r.t. p. By Lemma 3.3 we obtain an ideally starlike w.r.t. p neighborhood V_1 such that $C(A_1) \subset V_1 \subset \mathrm{Cl}\, V_1 \subset U$. It follows that $d(\mathrm{Bd}\, V_1, \mathrm{Bd}\, U) < \varepsilon_1$. Let $\varepsilon_2 > 0$ such that $\varepsilon_2 < \frac{1}{2} d(\mathrm{Bd}\, V_1, \mathrm{Bd}\, U)$. As above, we obtain an ideally starlike w.r.t. p neighborhood V_2 such that $\mathrm{Cl}\, V_1 \subset V_2 \subset \mathrm{Cl}\, V_2 \subset U$, and $d(\mathrm{Bd}\, V_2, \mathrm{Bd}\, U) \leqslant \varepsilon_2$. Necessity now follows by induction.

4. Radially-shrinkable and radially-pointlike decompositions. All spaces in this section are locally compact SC-WR metric spaces. We show that radially-shrinkable decompositions are radially-pointlike (Theorem 4.3); this result is an important cog of Section 5. We also establish two results for radially-shrinkable decompositions previously established for shrinkable decompositions (Theorems 7 and 10 of [9]).

LEMMA 4.1. *Let U be an open set in (X, d) containing a compact, starlike w.r.t. p set A and let f be an embedding of $\mathrm{Cl}\, U$ into $\mathrm{Cl}\, U$ such that $f(x) \in [px]$ for each $x \in \mathrm{Cl}\, U$. Then (1) if V is a starlike w.r.t. p neighborhood of A such that $\mathrm{Cl}\, V$ is compact and $V \subset U$, then $f(\mathrm{Cl}\, V) \subset \mathrm{Cl}\, V$, and (2) if $f \mid \mathrm{Ed}_p U$ is the identity, then for each neighborhood V of A such that $V \subset U$, there is a homeomorphism F of $\mathrm{Cl}\, V$ onto $\mathrm{Cl}\, V$ such that $F(x) \in [px]$ for each $x \in \mathrm{Cl}\, V$, $F|A = f$, and $F|\mathrm{Bd}\, V$ is the identity.*

Proof. (1) follows from the fact that $\mathrm{Cl}\, V$ is starlike w.r.t. p. (2) follows from (1), Lemma 3.3, and Lemma 3.2.

THEOREM 4.1. *Let G and G' be 0-dimensional decompositions of (X, d) such that $H(G) \supset H(G')$. If G is radially shrinkable in (X, d), then G' is radially-shrinkable in (X, d).*

Proof. Some details are the same as in Theorem 7 of [9]; we sketch the differences. Let $H(G) = \{g_\alpha\colon \alpha \in \mathfrak{A}\}$ and let $\{h_\alpha\}$, $\{q_\alpha\}$, and $\{p_\alpha\}$ be the collections of maps, compact, starlike sets, and points, respectively, given us by the radial-shrinkability of G. We claim that $\{h_\alpha\colon g_\alpha \in H(G')\}$, $\{q_\alpha\colon g_\alpha \in H(G')\}$, and $\{p_\alpha\colon g_\alpha \in H(G')\}$ are the required collections for G'. Let $\varepsilon > 0$ and let U be an open set containing $\bigcup H(G')$. Then $\{U, X - \bigcup G'(\varepsilon)\}$ is an open cover of $H(G)$ and is refined by K, a disjoint collection of open sets ([9], Theorem 1). Let U' be the union of all components of $\bigcup K$ which intersect $\bigcup G'(\varepsilon)$. Then U' is an open subset of U (Proposition 2.1(2)). Since G is radially-shrinkable, there is a homeomorphism h of X onto X such that $h|(X - \bigcup K)$ is the identity, $\operatorname{diam} h(g_\alpha) < \varepsilon$ for each $g_\alpha \in H(G)$, and for each $g_\alpha \in G(\varepsilon)$ there are V_α and f_α such that h_α, q_α, p_α, U, h, V_α, and f_α satisfy the remaining radial-shrinkability conditions at g_α for G. Define

$$H(x) = \begin{cases} x & \text{for} \quad x \in X - U', \\ h(x) & \text{for} \quad x \in U'; \end{cases}$$

then H is a homeomorphism of X onto X such that $H|(X - U)$ is the identity and $\operatorname{diam} H(g) < \varepsilon$ for each $g \in H(G')$. Let $g_\alpha \in G'(\varepsilon)$. Choose a starlike w.r.t. p_α neighborhood W_α such that $q_\alpha \subset W_\alpha \subset h_\alpha(V_\alpha \cap U')$ and $\mathrm{Cl}\, W_\alpha$ is compact (Lemma 3.3). Letting $F_\alpha = f_\alpha | \mathrm{Cl}\, W_\alpha$, it is easy to verify using Lemma 4.1(1) that h_α, q_α, p_α, U, H, $h_\alpha^{-1}(W_\alpha)$, and F_α satisfy the remaining radial-shrinkability conditions at g_α for G'.

THEOREM 4.2. *Let G be a 0-dimensional decomposition of (X, d). Then G is radially-shrinkable in (X, d) if and only if G is radially-shrinkable in (X, d) at each element of $H(G)$.*

Proof. The proof of Theorem 10 of [9] may be modified to obtain this proof in virtually the same way the proof of Theorem 7 of [9] is modified to obtain the proof of Theorem 4.1 (see [14]).

THEOREM 4.3. *Let G be a 0-dimensional radially-shrinkable decomposition of (X, d). Then G is radially-pointlike in (X, d).*

Proof. Let $g \in H(G)$ and let h, q, p be such that h is a space homeomorphism taking g onto q and q is a compact, starlike w.r.t. p set, given us by the radial-shrinkability of G. Let U be an open set containing q. We must construct V and H satisfying the radially-pointlike conditions for q in order to conclude G is radially pointlike. The rest of the proof is divided into several parts.

(i) For each $\varepsilon>0$, there is an open set O containing g such that for each open subset W of O containing g, there are homeomorphisms H_ε and F_ε such that H_ε is a homeomorphism of X onto X, $H_\varepsilon|(X-h^{-1}(U))$ is the identity, $\operatorname{diam} H_\varepsilon(g)<\varepsilon$, F_ε is a homeomorphism of $h(\operatorname{Cl} W)$ onto $h(\operatorname{Cl} W)$, $F_\varepsilon|\operatorname{Bd} h(W)$ is the identity, $F_\varepsilon(h(x)) \in [ph(x)]$ for each $x \in \operatorname{Cl} W$, and $H_\varepsilon|\operatorname{Cl} W = h^{-1}\circ F_\varepsilon \circ h$.

Let G_1 be the decomposition of X such that $H(G_1) = \{g\}$. Then by Theorem 4.1 G_1 is a radially-shrinkable decomposition. Thus we have a homeomorphism h_ε from X onto X such that $h_\varepsilon|(X-h^{-1}(U))$ is the identity $\operatorname{diam} h_\varepsilon(g)<\varepsilon$, and there are O and f such that h, q, p, $h^{-1}(U)$, h_ε, O, and f satisfy the remaining radial-shrinkability conditions at g for G_1. Let W be any open set containing g such that $W \subset O$. By Lemma 4.1 there is a homeomorphism F_ε taking $h(\operatorname{Cl} W)$ onto $h(\operatorname{Cl} W)$ such that $F_\varepsilon(h(x)) \in [ph(x)]$ for each $x \in \operatorname{Cl} W$, $F_\varepsilon|q = f$, and $F_\varepsilon|\operatorname{Bd} h(W)$ is the identity. Define

$$H_\varepsilon(x) = \begin{cases} x & \text{for } x \in X-\operatorname{Cl} W, \\ h^{-1}(F_\varepsilon(h(x))) & \text{for } x \in \operatorname{Cl} W. \end{cases}$$

It follows that H_ε is a homeomorphism of X onto X and $H_\varepsilon|(X-h^{-1}(U))$ is the identity. It also follows that $H_\varepsilon(g) = h_\varepsilon(g)$; hence $\operatorname{diam} H_\varepsilon(g)<\varepsilon$.

(ii) Construction of V of H.

Let $\{N_k\}$ be a neighborhood base for g such that $N_1 \supset \operatorname{Cl} N_2 \supset N_2 \supset \ldots$, $\operatorname{Cl} N_1$ is compact, $h(\operatorname{Cl} N_1) \subset U$, and $h(N_k)$ is ideally starlike w.r.t. p for each k (Lemma 3.3). Choose M_1 to be N_1 and let $V = h(N_1)$. By the uniform continuity of h on $\operatorname{Cl} M_1$, there is a sequence of positive numbers $\{\delta_n\}$ such that $d(x, y)<2\delta_n$ implies $d(h(x), h(y))<1/n$ for each $x, y \in \operatorname{Cl} M_1$. For each positive ingeter n, let O_n be the open set containing g given us by (i) for δ_n and $h^{-1}(U)$. Assuming $M_n \in \{N_k\}$ has been chosen, choose $M_{n+1} \in \{N_k\}$ such that $\operatorname{Cl} M_{n+1} \subset M_n \cap O_{n+1}$. Then by (i) we have collections of homeomorphisms $\{H_n\}$ and $\{F_n\}$ such that for each n the following hold: H_n is a homeomorphism of X onto X, $H_n|(X-h^{-1}(U))$ is the identity, $\operatorname{diam} H_n(g)<\delta_n$, F_n is a homeomorphism of $h(\operatorname{Cl} M_n)$ onto $h(\operatorname{Cl} M_n)$, $F_n|\operatorname{Bd}(h(M_n))$ is the identity, $F_n(h(x)) \in [ph(x)]$ for each $x \in \operatorname{Cl} M_n$, and $H_n|\operatorname{Cl} M_n = h^{-1}\circ F_n \circ h$. For each n define $h_n(x) = F_n(\ldots(F_1(x))\ldots)$ for $x \in \operatorname{Cl}(h(M_n)-h(\operatorname{Cl} M_{n+1}))$. Define

$$H^*(x) = \begin{cases} x & \text{for } x \in X-h(M_1), \\ h_n(x) & \text{for } x \in \operatorname{Bd}(h(M_{n+1})) \end{cases}$$

and

$$H(x) = \begin{cases} H^*(x) & \text{for } x \in X-\bigcup[h(M_n)-h(\operatorname{Cl} M_{n+1})], \\ h_n(x) & \text{for } x \in h(M_n)-h(\operatorname{Cl} M_{n+1}). \end{cases}$$

Clearly H is well defined on $X-q$.

(iii) $x_m\to q$ implies $H(x_m)\to p$.

Let $y_m = h^{-1}(x_m)$, let $\varepsilon>0$, and let N be so large that $2/N<\varepsilon$. Since H_N is uniformly continuous on $\operatorname{Cl} M_N$, there is $\xi>0$ such that $\xi<\operatorname{diam} g$ and $d(x, y)<\xi$ implies $d(H_N(x), H_N(y))<\delta_N$ for $x, y \in \operatorname{Cl} M_N$. Let J be so large that $m\geqslant J$ implies $\{x_m\}\subset \operatorname{Cl} M_N$ and $d(y_m, g)<\operatorname{diam} g$. Then $m\geqslant J$ implies $d(H_N(y_m), H_N(q))<2\delta_N$ which implies

$$d(F_N(x_m), F_N(q)) = d(h(H_N(y_m)), h(H_N(g)))<1/N.$$

So $d(F_N(x_m), p)<2/N<\varepsilon$. Now suppose $x_m \in h(M_n)-h(\operatorname{Cl} M_{n+1})$, where $m\geqslant J$ and $n\geqslant N$. Since $H(x_m) = F_n(\ldots(F_1(x_m))\ldots)$, it follows that

$$d(H(x_m), p)\leqslant d(F_N(\ldots(F_1(x_m))\ldots), p)\leqslant d(F_N(x_m), p)<2/N<\varepsilon.$$

It follows that $x_m\to q$ implies $H(x_m)\to p$.

(iv) H is continuous on $X-q$.

Since $\{h(M_n)\}$ is a neighborhood base for q, then $\{h(M_n)-h(\operatorname{Cl} M_{n+1})\}$ is a locally null collection of disjoint, open sets. Since each of H^* and h_n is continuous on its domain, it follows from Theorem 2 of [9], that H is continuous on $X-q$.

(v) $H(x) \in [px]$ for $x \in X-q$.

If $x \in X-V$, then $H(x) = x$. If $x \in V$, assume x is in some $h(M_n)-h(\operatorname{Cl} M_{n+1})$. Then

$$H(x) = F_n(\ldots(F_1(x))\ldots) \in [pF_{n-1}(\ldots(F_1(x))\ldots)]\subset\ldots\subset[pF_1(x)]\subset[px]$$

by applying Lemma 3.1 inductively to $\{F_1, \ldots, F_n\}$.

(vi) $H(X-q) = X-p$ and $H(\operatorname{Cl} V-q) = \operatorname{Cl} V-p$.

Since H is the identity on $X-V$, we need only show $H(V-q) = V-p$. Let $x \in X-q$ and assume $x \in h(M_n)-h(\operatorname{Cl} M_{n+1})$. For each k, $F_k(x) = p$ if and only if $x = p$. Since $x \in V-q$ implies $x \neq p$, then none of $F_1(x), \ldots, F_n(x)$ equals p. By induction, $H(x) \neq p$. Thus $H(V-q)\subset V-p$. Now let $y \in V-p$ and suppose $y \in [pz]-z$ where we assume $H(z) = z$. From (v) we have $H^{-1}(y)\subset[pz]$. We suppose there is no preimage of y on $[pz]$; then by the continuity of H, $H([pz]-q)\subset[yz]$. From (iii) we have $d(x, q)\to 0$ implies $d(H(x), p)\to 0$. This contradicts the fact that $d(y, p)>0$. Thus $H(V-q)\supset V-p$.

(vii) H^{-1} is continuous on $X-p$.

It is straightforward to show (using (v), the properties of the F_n's, and Lemma 3.1) that H is one-to-one. It is also straightforward to show that for each n, F_n takes open sets of $h(M_k)$ onto open sets of $h(M_k)$. Hence it follows from (iii) that $\{h_n[h(M_n)-h(\mathrm{Cl}M_{n+1})]\}$ is a locally null collection of disjoint, open sets. We now apply Theorem 2 of [9], to obtain the continuity of H^{-1} on $X-p$.

5. Properly starlike-equivalent decompositions. In this section (X, d) is a locally compact, SC-WR-CE metric space. We show that locally null, properly starlike-equivalent decompositions of (X, d) are radially-shrinkable. We then show that for locally null decompositions of (X, d), some of the properties studied in this paper are equivalent. These two results include Theorem A as stated in Section 1. We must first establish a result (Lemma 5.1) in which we construct a preliminary shrinking which moves along segments and moves any edge points; if a map moves along segments and moves edge points, it cannot be an onto map and hence cannot be a shrinking. The reader might find it helpful to refer to the space $(X(2), d_2)$ of the examples of Section 2 while working the proof of Lemma 5.1; he may also wish to consult [14].

LEMMA 5.1. *Let G be a monotone, locally null decomposition of (X, d), let $g \in H(G)$ be a compact, properly starlike w.r.t. p set, let W be an open set containing g, and let $\varepsilon>0$. Then there is an open set U, an open set M, and a homeomorphism h from X onto X satisfying.*

(1) $g\subset M\subset \mathrm{Cl}M\subset U\subset W$;

(2) $U\cap \mathrm{Bd}[\bigcup H(G)] = \emptyset$;

(3) *M is ideally starlike w.r.t. p;*

(4) *$h(x)\in [px]$ for each $x\in \mathrm{Cl}M$;*

(5) *$h|(X-M)$ is the identity; and*

(6) *$\mathrm{diam}\,h(g')<\varepsilon$ for each $g'\in H(G(U))$.*

Proof. The proof is given in three parts.

(i) Construction of the "controls" and the open sets U and M.

Because of Proposition 2.2 we may choose a neighborhood base $\{U_n\}$ for g such that $\mathrm{Bd}\,U_n\cap[\bigcup H(G)] = \emptyset$ and $U_n\subset W$ for each n. Let G' be the decomposition of X such that $H(G') = H(G)-\{g\}$. Because of Proposition 2.2 we may choose a neighborhood base $\{V_n\}$ for p such that $\mathrm{Bd}\,V_n\cap[\bigcup H(G')] = \emptyset$ for each n. There is a nested neighborhood base $\{N_n\}$ for g such that $\mathrm{Cl}N_1$ is compact, each N_n is ideally starlike w.r.t. p, and no nondegenerate segment from p in g has its terminal point on the edge w.r.t. p of any N_n (Lemma 3.4). Choose $V\in\{V_n\}$ such that $\mathrm{Cl}\,V\subset N_1\cap N(p, \frac{1}{3}\varepsilon)$. Let $\delta_1>0$ such that if $a, b\in \mathrm{Bd}N_1\cap \mathrm{Ed}_pN_1$ and $d(a, b)<\delta_1$, then for every ideally starlike w.r.t. p neighborhood N with $N\subset N_1$,

$$d([pa]\cap[\mathrm{Bd}N\cup \mathrm{Ed}_pN],\ [pb]\cap[\mathrm{Bd}N\cup \mathrm{Ed}_pN])<\tfrac{1}{6}\varepsilon\,.$$

Let $\delta_2>0$ such that if $a, b\in \mathrm{Bd}N_1\cup \mathrm{Ed}_pN_1$, $c\in[pa]-V$, $d\in[pb]-V$, and $d(c, d)<\delta_2$, then $d(a, b)<\delta_1$. Let $\delta_3 = d(\mathrm{Bd}N_1, g)$ and let $\delta>0$ such that $\delta<\min\{\delta_1, \delta_2, \delta_3, \frac{1}{6}\varepsilon\}$. Since G is locally null and g is compact, we may choose $U\in\{U_n\}$ such that $U\subset N_1$ and if $g'\in H(G)-\{g\}$, then $g'\subset U$ only if $\mathrm{diam}\,g'<\delta$. Let $R_i = N(p, i\delta)$, $i = 1, 2, 3, \ldots$ We now choose $M_1, \ldots, M_k$ members of $\{N_n\}$ such that

(1) R_k contains N_1;

(2) $g\subset M_1\subset \mathrm{Cl}M_1\subset M_2\subset\ldots\subset M_k\subset \mathrm{Cl}M_k\subset U\cap N(g, \delta)$;

(3) if $g'\in H(G)-\{g\}$, and $g'\cap \mathrm{Bd}M_i\neq\emptyset$, then $g'\cap \mathrm{Bd}M_{i-1} = \emptyset$, $i\neq k$ implies $g'\subset M_{i+1}$, and in any case $\mathrm{diam}\,g'<\delta$;

(4) for each i, $g\cap \mathrm{Ed}_pM_i = \emptyset$ and $\mathrm{Ed}_pM_i\subset R_i$ (Lemma 3.5); and

(5) if $a\in[\mathrm{Bd}N_1\cup \mathrm{Ed}_pN_1]-R_i$ and $i, j\in\{1, \ldots, k\}$, then

$$d([pa]\cap[\mathrm{Bd}M_i\cup \mathrm{Ed}_pM_i],\quad [pa]\cap[\mathrm{Bd}M_j\cup \mathrm{Ed}_pM_j])<\delta\,.$$

We choose M to be M_k.

(ii) Construction of the shrinking h satisfying conditions (4) and (5) of the conclusion.

Let $x\in \mathrm{Cl}M_k-p$ and let s be the segment from p to a point on $\mathrm{Bd}N_1\cup \mathrm{Ed}_pN_1$ so that $[px]\subset s$. For $i\in\{1, \ldots, k\}$, let

$$m_i(x) = s\cap[\mathrm{Bd}M_i\cup \mathrm{Ed}_pM_i] \quad\text{and}\quad r_i(x) = s\cap[\mathrm{Bd}R_i\cup \mathrm{Ed}_pR_i]\,.$$

It follows that each of $m_i(x)$ and $r_i(x)$ is continuous on $\mathrm{Cl}M_k-p$. We now define a map H from $\bigcup[\mathrm{Bd}M_i\cup \mathrm{Ed}_pM_i]$ into $\mathrm{Cl}M_k$ by

$$H(m_i) = \begin{cases} m_i & \text{for } d(p, m_i)\leqslant d(p, r_i(m_i))\,,\\ r_i(m_i), & \text{otherwise}\,,\end{cases}$$

where $m_i\in[\mathrm{Bd}M_i\cup \mathrm{Ed}_pM_i]$ and $i\in\{1, \ldots, k\}$. It follows from properties (1) and (4) of $\{M_1, \ldots, M_k\}$ that $H|[\mathrm{Bd}M_k\cup \mathrm{Ed}_pM_k]$ is the identity. It can be shown that H is continuous on its domain. We now define h from $\mathrm{Cl}M_k$ into $\mathrm{Cl}M_k$ by

$$h(x) = \begin{cases} \dfrac{d(m_{i+1}(x), x)}{d(m_{i+1}(x), m_i(x))}H(m_i(x))+\dfrac{d(x, m_i(x))}{d(m_{i+1}(x), m_i(x))}H(m_{i-1}(x)) \\ \qquad\text{for}\quad x\in \mathrm{Cl}M_{i+1}-\mathrm{Cl}M_i \text{ and } 1\leqslant i\leqslant k-1\,,\\ \dfrac{d(m_1(x), x)}{d(m_1(x), p)}p+\dfrac{d(x, p)}{d(m_1(x), p)}H(m_1(x)) \quad \text{for}\quad x\in \mathrm{Cl}M_k-p\,,\\ p \quad \text{for}\quad x = p\,.\end{cases}$$

It follows that $h(x)\in[px]$ for each $x\in \mathrm{Cl}M_k$ and that h is continuous on $\mathrm{Cl}M_k-p$ and hence, by Lemma 3.1, on $\mathrm{Cl}M_k$. It can be shown that h is one-to-one on $\mathrm{Cl}M_k$ by satisfying Lemma 3.1. Now let $x\in \mathrm{Cl}M_i-p$. From the definition and continuity of h we have $h([pm_k(x)]) = [pm_k(x)]$. It can now be shown that $h(\mathrm{Cl}M_k) = \mathrm{Cl}M_k$.

If we extend h to $X-\mathrm{Cl}M_k$ by $h(x)=x$, then h is a homeomorphism of X onto X such that $h|(X-M)$ is the identity.

(iii) The shrinking h satisfies condition (6) of the conclusion.

Let $g'\in H(G(U))$. If $g'\subset U-\mathrm{Cl}M_k$, then $\operatorname{diam} h(g')<\delta<\varepsilon$. Let $g'\cap \mathrm{Cl}M_k\neq \emptyset$. Then $g'\subset U$ and hence $\operatorname{diam} g'<\delta$. We assume for the time being that

$$g'\cap[\mathrm{Cl}R_1\cup \mathrm{Cl}M_1\cup V]=\emptyset .$$

Now suppose $g'\subset \mathrm{Cl}M_k$. Then $g'\subset M_{i+1}-\mathrm{Cl}M_{i-1}$ for some $i>1$. Let $x,y\in g'$. We distinguish three cases.

(1) $d(m_{i-1}(x),p)>d(r_{i-1}(x),p)$ and $d(m_{i-1}(y),p)>d(r_{i-1}(y),p)$; it follows that $d(h(x),h(y))<\delta+\frac{1}{6}\varepsilon+\delta<\frac{1}{2}\varepsilon$.

(2) $d(m_{i-1}(x),p)\leqslant d(r_{i-1}(x),p)$ and $d(m_{i-1}(y),p)\leqslant d(r_{i-1}(y),p)$; it follows that $d(h(x),h(y))<\delta+\frac{1}{6}\varepsilon+\delta<\frac{1}{2}\varepsilon$.

(3) $d(m_{i-1}(x),p)\leqslant d(r_{i-1}(x),p)$ and $d(m_{i-1}(y),p)>d(r_{i-1}(y),p)$; it follows that $d(h(x),h(y))<\delta+\frac{1}{6}\varepsilon+\frac{1}{6}\varepsilon+2\delta<\frac{5}{6}\varepsilon$.

Thus if $g'\subset \mathrm{Cl}M_k$, $\operatorname{diam} h(g')\leqslant\frac{5}{6}\varepsilon$. Now suppose $g'\cap(X-\mathrm{Cl}M_k)\neq\emptyset$. From the connectedness of g' and the above three cases it follows that

$$\operatorname{diam} h(g')\leqslant \operatorname{diam} h(g'-\mathrm{Cl}M_k)+\operatorname{diam} h(g'\cap \mathrm{Cl}M_k)\leqslant \delta+\tfrac{5}{6}\varepsilon<\varepsilon .$$

Now let $g'\cap[\mathrm{Cl}R_1\cup \mathrm{Cl}M_1\cup V]\neq\emptyset$. Then it follows that $\operatorname{diam} h(g')\leqslant 4\delta<\varepsilon$, $\operatorname{diam} h(g')\leqslant 4\delta<\varepsilon$, and $\operatorname{diam} h(g')<\frac{2}{3}\varepsilon$, respectively. Thus h satisfies condition (6) of the conclusion.

THEOREM 5.1. *Let G be a locally null, properly starlike-equivalent decomposition of (X,d). Then G is radially-shrinkable in (X,d).*

Proof. For each $g_n\in H(G)$, we have h_n, q_n, and p_n such that h_n is a space homeomorphism taking g_n onto q_n and q_n is properly starlike w.r.t. p_n. We claim $\{h_n\}$, $\{q_n\}$, and $\{p_n\}$ are the required collections for radial-shrinkability. Let $\varepsilon>0$ and let U be an open set containing $\bigcup H(G)$. We set $G(\varepsilon)=\{g_1,g_2,\ldots\}$. Using Proposition 2.2 we obtain a locally null, open covering $\{O_n\}$ of $G(\varepsilon)$ such that $g_n\subset \mathrm{Cl}\,O_n$ is compact and $O_n\cap \mathrm{Bd}[\bigcup H(G)]=\emptyset$ for each n, and $\mathrm{Cl}\,O_n\cap \mathrm{Cl}\,O_m=\emptyset$ if $n\neq m$. For each n let G_n be the decomposition of X such that $H(G_n)=\{h_n(g)\colon g\subset O_n\}$. Then each G_n is an u.s.c., monotone, locally null decomposition of X with q_n as a nondegenerate element. We choose $\delta_n>0$ such that $d(h_n(x),h_n(y))<\delta_n$ implies $d(x,y)<\frac{1}{2}\varepsilon$ for $x,y\in \mathrm{Cl}\,O_n$. From Lemma 5.1 we have U_n, V_n, f_n for each n such that

(1) $g_n\subset V_n\subset \mathrm{Cl}\,V_n\subset U_n\subset h_n(O_n)$, $U_n\cap \mathrm{Bd}[\bigcup H(G_n)]=\emptyset$, and V_n is ideally starlike w.r.t. p_n;

(2) f_n is a homeomorphism of X onto X such that $f_n(x)\in[px]$ for each $x\in \mathrm{Cl}\,V_n$ and $f_n|(X-V_n)$ is the identity; and

(3) $\operatorname{diam} f_n(q)<\delta_n$ for each $q\in H(G_n(U_n))$.

We now define h from X onto X by

$$h(x)=\begin{cases} h_n^{-1}\big(f_n(h_n(x))\big) & \text{for } x\in f_n^{-1}(U_n) \text{ and } n\geqslant 1,\\ x & \text{for } x\in X-[\bigcup f_n^{-1}(U_n)]. \end{cases}$$

It can be shown that h and h^{-1} satisfy the conditions of Theorem 2 of [9] and thus h is a homeomorphism. It is not difficult to show that $\{h_n\}$, $\{q_n\}$, $\{p_n\}$, U, h, $\{f_n^{-1}(V_n)\}$, and $\{f_n\}$ satisfy the conditions of radial-shrinkability for G.

PROPOSITION 5.1. *Let G be a decomposition of a locally compact,* SC *metric space (Y,e), where G need not be u.s.c. If G is radially-pointlike in (Y,e), then G is properly starlike-equivalent in (Y,e).*

Proof. The proof follows by contradiction.

THEOREM 5.2 (Theorem A). *Let G be a locally null decomposition of (X,d). Then the following are equivalent:*

(1) *G is properly starlike-equivalent in (X,d);*

(2) *G is radially shrinkable in (X,d);*

(3) *G is radially shrinkable in (X,d) at each element of $H(G)$; and*

(4) *G is radially-pointlike in (X,d).*

If any of the above hold, $X/G\approx X$.

Proof. The circle of implications follows from the theorems of Sections 4 and 5. That $X/G\approx X$ follows from (2) and Theorem 4 of [9].

COROLLARY 5.1. *Each compact starlike subset of E^n is radially-pointlike.*

Other consequences of Theorem 5.2 are given in Examples 1 and 2 of Section 1.

6. Star-0-dimensional decompositions. In this section (X,d) is a locally compact, SC-WR-CE metric space. We recall from Section 1 that Price has shown [13] that a decomposition G of E^n yields E^n if for each $g\in H(G)$, there is a collection of n-cells $\{B_k\}$ such that $\{\mathrm{Int}\,B_k\}$ is a neighborhood base for g and $\mathrm{Bd}\,B_k\cap[\bigcup H(G)]=\emptyset$ for each k. Such a decomposition is star-0-dimensional, but star-0-dimensional decompositions may not satisfy Price's conditions because there are open n-cells which are starlike but whose closures are not n-cells. In this section we prove Theorem B after first proving Lemma 6.1, a result analogous to Lemma 5.1.

LEMMA 6.1. *Let G be a monotone decomposition of an open starlike w.r.t. p set U in (X,d) such that $\mathrm{Cl}\,U$ is compact and $\mathrm{Ed}_p U=\emptyset$. Then for each $\varepsilon>0$, there is an ideally starlike w.r.t. p neighborhood V such that $\mathrm{Cl}\,V\subset U$, and there is a homeomorphism h from $\mathrm{Cl}\,U$ onto $\mathrm{Cl}\,U$ satisfying these conditions: $h(x)\in[px]$ for each $x\in \mathrm{Cl}\,U$, $h|(U-V)$ is the identity, and $\operatorname{diam} h(g)<\varepsilon$ for each $g\in H(G)$.*

Proof. Let $\varepsilon>0$. From Lemma 3.3 we may obtain an ideally starlike w.r.t. p neighborhood N containing $\mathrm{Cl}\,U$ such that $\mathrm{Cl}\,N$ is compact and $\mathrm{Ed}_p N=\emptyset$.

By Lemma 3.6, $U=\bigcup V_n$ where each V_n is ideally starlike w.r.t. p, $\mathrm{Cl}\,V_n\subset V_{n+1}$ for each n, and each $\mathrm{Ed}_p V_n=\emptyset$. Let $\delta_1>0$ such that if $a,b\in \mathrm{Bd}\,N$ and $d(a,b)<\delta_1$, then $d([pa]\cap \mathrm{Bd}\,M,[pb]\cap \mathrm{Bd}\,M)<\frac{1}{6}\varepsilon$ for each ideally starlike w.r.t. p neighborhood $M \subset N$. Let $\delta_2 > 0$ such that if $a, b \in \operatorname{Bd} N$, $c \in [pa] - V_1$ and $d \in [pb] - V_1$, and $d(c, d) < \delta_2$, then $d(a, b) < \delta_1$. Let $\delta > 0$ such that $\delta < \min\{\delta_1, \delta_2, \frac{1}{6}\varepsilon\}$. It follows that there is an integer J such that if $a \in \operatorname{Bd} N$ and $n, m \geqslant J$, then

$$d([pa] \cap \operatorname{Bd} V_n, [pa] \cap \operatorname{Bd} V_m) < \delta .$$

Let $V_{n_1} \in \{V_n\}$ such that $n_1 \geqslant J$ and if $\operatorname{diam} g \geqslant \delta$ (where $g \in H(G)$), then $g \subset \operatorname{Cl} V_{n_1}$. Let $V_{n_2} \in \{V_n\}$ such that $n_2 > n_1$ and if $g \cap \operatorname{Cl} V_{n_1} \neq \emptyset$, then $g \subset V_{n_2}$. We continue this process inductively until a V_{n_k} has been chosen and k is so large that $\operatorname{Cl} U \subset N(p, k\delta)$. We choose V to be V_{n_k}. Let $R_i = N(p, i\delta) \cap N$ for $i \in \{1, \ldots, k\}$. Each R_i is ideally starlike w.r.t. p. Let $x \in \operatorname{Cl} U - p$. Then $x \in [pa]$ where $a \in \operatorname{Bd} N$. For $i \in \{1, \ldots, k\}$, let $r_i(x) = [pa] \cap \operatorname{Bd} R_i$ and $v_i(x) = [pa] \cap \operatorname{Bd} V_n$. The procedure is now completely analogous to that of Lemma 5.1.

THEOREM 6.1 (Theorem B). *Let G be a star-o-dimensional decomposition of (X, d). Then G is shrinkable. Hence $X/G \approx X$.*

Proof. Let $g \in H(G)$, and let W be an open set about g such that $\operatorname{Cl} W$ is compact and $\operatorname{Bd} W \cap [\bigcup H(G)] = \emptyset$. Let $\varepsilon > 0$ and let U be an open set containing $\bigcup H(G(W))$. Let $G'_W(\varepsilon) = \{g' \in H(G(W)) \colon \operatorname{diam} g' \geqslant \varepsilon\}$. Then $\bigcup G'_W(\varepsilon)$ is compact. For each $g' \in G'_w(\varepsilon)$, let $O(g')$ be an open set containing g' such that $O(g') \subset U$, $\operatorname{Bd}(O(g')) \cap [\bigcup H(G)] = \emptyset$, and $\operatorname{Cl}(O(g'))$ is homeomorphic to an open, starlike set with compact closure and empty edge w.r.t. p. Using Lemma 6.1, we proceed exactly as in Lemma 1.2 of [13] to obtain a homeomorphism h of X onto X such that $h|(X-U)$ is the identity and $\operatorname{diam} h(g') < \varepsilon$ for each $g' \in H(G(W))$. Hence $G(W)$ is shrinkable, i.e. G is shrinkable at g. By Theorem 10 of [9], G is shrinkable and by Theorem 4 of [9], $X/G \approx X$.

COROLLARY 6.1. *Let G be a decomposition of E^3 such that $H(G)$ is countable.*

(1) *The following are equivalent:*

(i) *G is star-0-dimensional;*

(ii) *G is shrinkable;*

(iii) *$E^3/G \approx E^3$; and*

(iv) *G satisfies Prices condition.*

(2) *If G is starlike, then G is star-0-dimensional.*

Proof. (1) follows from Theorem 6.1 and Theorem 1.4 of [13]; (2) follows from (1) and Theorem 2 of [4].

Other consequences of Theorem 6.1 are given in Example 3 of Section 1.

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Remarks of the elementary theories of formal and convergent power series

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Abstract. In § 1 an example is given of two fields F_1, F_2 of characteristic 0 such that $F_1 \equiv F_2$ but $F_1[[x_1, x_2]] \not\equiv F_2[[x_1, x_2]]$. In § 2 it is shown that $\langle C\{x, y\}, C\{x\}\rangle \prec \langle C[[x, y]], C[[x]]\rangle$, where $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3, y_4)$.

In [3] and [4] Ax and Kochen and Eršov showed among other things that the ring of convergent power series $\boldsymbol{C}\{x\}$, over the complex numbers $\boldsymbol{C}$, is an elementary subring of the ring of formal power series $\boldsymbol{C}[[x]]$ over $\boldsymbol{C}$. This means that the same first order statements (in the language of valued rings) with constants from $\boldsymbol{C}\{x\}$, are true in both rings. (This is denoted $\boldsymbol{C}\{x\} \prec \boldsymbol{C}[[x]]$.) Also they showed that if fields F_1 and F_2 of characteristic 0 are elementarily equivalent, denoted $F_1 \equiv F_2$ (i.e. the same first order statements in the language of fields are true of F_1 and F_2) then $F_1[[x]] \equiv F_2[[x]]$ as valued rings (i.e. the same first order statements, in the language of valued rings, are true about $F_1[[x]]$ and $F_2[[x]]$). It is natural to ask whether these results extend to power series rings in several variables. In Section 1, we show that one can have fields $F_1 \equiv F_2$ but $F_1[[x_1, x_2]] \not\equiv F_2[[x_1, x_2]]$. In Section 2 we show that a slightly stronger statement than $\boldsymbol{C}\{x_1, \ldots, x_6\} \prec \boldsymbol{C}[[x_1, \ldots, x_6]]$ is false (¹). These remarks contradict some results claimed in [7].

Section 1. Eršov [4] showed that for any field F and for $n \geqslant 2$, $F[[x_1, \ldots, x_n]]$ is undecidable. We shall give a slightly different proof of this for the case that F has characteristic zero and use this proof to show that we can have $F_1 \equiv F_2$ of characteristic 0 but $F_1[[x_1, \ldots, x_n]] \not\equiv F_2[[x_1, \ldots, x_n]]$ $(n \geqslant 2)$ as rings. Let F be a field of characteristic zero.

For the sake of clarity, we begin by showing that $\mathscr{F} = F[[x_1, \ldots, x_n]]$ is undecidable as an F algebra with x_1 and x_2 picked out, i.e., that $\mathscr{F}$ as a ring under the operations of addition and multiplication, with constants for x_1 and x_2, and with an additional predicate which picks out a particular lifting of the residue field F

(¹) (Added in proof) Some of the results of this paper and some extensions have been discoverd independently by F. Delon, *Résultats d'indécidabilité dans les anneaux de séries formelles* (to appear).