Upper semicontinuous decompositions of convex metric spaces *

by

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Abstract. All decompositions in this paper are upper semicontinuous.

Theorem A. If $G$ is a locally null, properly starlike-equivalent decomposition of a locally compact, SC-WR-CE metric space $(X, d)$, then $G$ is radially-shrinkable in $(X, d)$ and $X/G \approx X$.

Corollary. If $G$ is a locally null, starlike-equivalent decomposition of $E^n$, then $E^n/G \approx E^n$.

Theorem B. If $G$ is a star-0-dimensional decomposition of a locally compact, SC-WR-CE metric space $(X, d)$, then $G$ is shrinkable and $X/G \approx X$.

1. Introduction. All decompositions in this paper are upper semicontinuous. The famous “dogbone” space of R. H. Bing [5] has spawned an amazing array of results and questions. In [4], Bing showed that if $G$ is a decomposition of $E^3$ into at most countably many tame arcs and points, then $E^3/G \approx E^3$. This raised the following question (see S. Armentrout [2, Question 1]): Suppose $G$ is a decomposition of $E^3$ into tame 3-cells and points; is $E^3/G \approx E^3$? A partial answer was given by D. V. Meyer [11]: A null decomposition of $E^3$ into tame 3-cells and points is $E^3$. This result was improved by R. J. Bean [3]: Null, starlike-equivalent decompositions of $E^3$ yield $E^3$. This led J. W. Lamoreaux in [8] to ask whether locally null, starlike-equivalent decompositions of a SC-WR metric space $(X, d)$ yield $X$. In this paper we show the answer is no (see Example 1 of Section 2) yet obtain the following theorem.

Theorem A. If $G$ is a locally null, properly starlike-equivalent decomposition of a locally compact, SC-WR-CE metric space $(X, d)$, then $G$ is radially-shrinkable in $(X, d)$ and $X/G \approx X$.

T. M. Price [13] has proved that if $G$ is a decomposition of $E^n$ such that for each $g \in H(G)$ and for each open set $V$ containing $g$ there is an $n$-cell $B$ such that $g \in \operatorname{Int} B \subseteq V$ and $B \cap (\bigcup H(G)) = \emptyset$, then $E^n/G \approx E^n$. The condition that $B$ is an $n$-cell is weakened in this paper. We strengthen Price’s theorem and extend it to SC-WR-CE metric spaces in the following theorem.

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THEOREM B. If \( G \) is a star-0-dimensional decomposition of a locally compact, SC-WR-CE metric space \((X, d)\), then \( G \) is shrinkable and \( X/G \approx X \).

To illustrate Theorems A and B, we give three examples; Examples 1 and 2 are consequences of Theorem A, Example 3 of Theorem B.

**Example 1.** If \( G \) is a locally null, starlike-equivalent decomposition of \( E^n \), then \( E^n/G \approx E^n \). In particular, we can choose \( G \) to be locally null deformation of \( E^n \) into tame cells (dimension \( <n \)) and points such that given \( \varepsilon > 0 \), infinitely many of the cells have diameter \( \geq \varepsilon \).

**Example 2.** Let \( X(n) = \bigcup_{i=1}^{\infty} \{x_i, \ldots, x_{n+1} \}: \sum_{i=1}^{n} x_i^2 = 1, x_i \geq 0, \ldots, x_{n+1} \geq 0 \} \), and let \( X(n) \) be topologized by \( d_2 \), the "great \( S^{n-1} \)-metric of \( S^n \). Let \( G \) be a locally null decomposition of \( X(n) \) such that \( \bigcup H(G) \) is contained in the manifold interior of \( X(n) \). \( H(G) \) is a collection of tame cells (dimension \( <n \)), tame whisk-brooms, tame fan-spaces, etc., and given \( \varepsilon > 0 \), infinitely many members of \( H(G) \) have diameter \( \geq \varepsilon \). Then \( G \) is radically-shrinkable in \( (X(n), d_2) \) and \( X(n)/G \approx X(n) \).

**Example 3.** Let \( G \) be a decomposition of \( E^n \) such that each \( g \in G \) possesses a neighborhood base \( \{U_n \} \) such that \( {\cap} H(G) = \emptyset \), \( U_n = CI U_{n+1} \), \( U_n \) is an open \( n \)-cell, and \( CI U_n \) is starlike but not an \( n \)-cell. Then \( G \) is maximal and \( E^n/G \approx E^n \).

**Example 3.** This example can be modified for non-Euclidean spaces as in Example 2. It is the principal goal of this paper to prove Theorems A and B: Theorem A is established in Section 5 and Theorem B in Section 6. In Section 2 we give preliminaries and in Section 3 we develop the machinery used in Sections 4, 5, and 6.

2. Preliminaries. We are always in a locally compact, strongly convex metric space \((X, d)\). For the definitions of betweenness, midpoint, convexity, strong convexity (SC), and without ramifications (WR), see D. Rolfsen [15]. We do not assume that strongly convex spaces are separable or complete. Let \( a, b \in X \). We say \( L \) is a segment between \( a \) and \( b \) (or from \( a \) to \( b \)) if \( a, b \in L \), each point of \( L \) is between \( a \) and \( b \), and \( L \) is isometric to a real line interval of length \( d(a, b) \). If \( L \) is the unique segment from \( a \) to \( b \) we write \( L = \{a \} \). A segment \( L \) from \( p \) to \( y \) is maximal if there is no \( x \in X \) such that some segment from \( p \) to \( x \) properly contains \( L \). It is well known (see [15]) that in a complete, convex metric space, there is a segment between each two points. In the presence of local compactness and strong convexity, the requirement of completeness may be dropped.

**Proposition 2.1.** Let \((X, d)\) be a locally compact, SC metric space.

1. If \( a, b \in X \) then there is a segment joining \( a \) to \( b \) (see [14]).
2. \( X \) is arc-wise and locally arc-wise connected.
3. For each \( a, b \in X \), there is a unique segment joining \( a \) and \( b \). If \((X, d)\) is also a WR space, \( y \neq y' \), \( [x] \) and \( [x'] \) are segments in \( X \), \( y \notin [x'] \), and \( y' \notin [x] \), then \( [x] \cap [x'] = \{x\} \).
4. Let \( a, b, x \in X \) such that \( x \) is between \( a \) and \( b \). Then \( x \in [ab] \).

Let \([ab]\) be a segment in \( X \) and let \( h \) be the isometry of \([ab]\) onto \([0, d(a, b)]\) such that \( h(a) = 0 \). For \( x, y \in [ab] \) and \( \lambda \in (0, 1) \), define \( (1 - \lambda)x + \lambda y \) to be \( h^{-1}((1 - \lambda)h(x) + \lambda h(y)) \). This algebraic operation has many useful properties (including that it is jointly continuous in \( x, y \) if \( x, y \) are contained in a compact subspace of \( X \)) which will be extensively used in this paper (see [10], [14], and [15]).

The closure of a set \( A \) is denoted by \( \overline{A} \) and its boundary by \( \partial A \). A collection of neighborhoods containing a set \( A \) is a neighborhood base for \( A \) if each open set containing \( A \) contains an element of the base. Neighborhoods are open. If \( N \) is a neighborhood of \( p \), then the edge of \( N \) w.r.t. \( p \), or \( Ed_N \), is \( \{y \in CL: [y] \) is maximal\}. We say \((X, d)\) has closed edges (or \((X, d)\) is CE) if for each point \( p \) of \( X \), \( Ed_X \cup \{p\} \) is closed (the class of convex metric spaces satisfying the closed edge property strictly contains the class of normed linear spaces). A set \( A \) is starlike w.r.t. \( p \) if for each \( x \in A \), \( [x] \subset A \); the point \( p \) is called a reference point of \( A \). A starlike w.r.t. \( p \) set \( A \) is properly starlike w.r.t. \( p \) if for each \( x \in A \), the segment \( [x] \) is not maximal. A neighborhood \( N \) of \( p \) is ideally starlike w.r.t. \( p \) if \( N \) is starlike w.r.t. \( p \) and for each \( x \in X \), \( [x] \) intersects \( N \) in at most one point. A set \( A \) is radially pointlike w.r.t. \( p \) if \( A \) is starlike w.r.t. \( p \) and for each neighborhood \( U \) of \( A \), there is an ideally starlike w.r.t. \( p \) neighborhood \( V \) of \( A \) and homeomorphism \( H \) from \( X - A \) onto \( X - p \) such that \((1) CI V = U, (2) H \) takes \( CI V - A \) onto \( CI F - p \), and \( (3) \) for each \( x \in X - A, H(x) \in [x] \). A collection \( J \) of subsets of \( X \) is locally null if for each \( x \in X \), there is an open set \( U \) containing \( x \) such that the collection of all sets of \( J \) that intersect \( U \) is a null collection.

For the definitions of upper semicontinuous (u.s.c.) decomposition, decomposition (\( X/G \)), monotone, pointlike, 0-dimensional, and shrinkable (or Condition B) see [1], [9], or [17]. Let \( G \) be a decomposition of \( X \). Let \( H(G) \) denote the collection of nondegenerate elements of \( G \) and let \( G(\delta) = \{g \in H(G): \text{diam} g \geq \delta \} \), where \( \delta > 0 \). We say \( G \) is null (locally null) if \( H(G) \) is a null (locally null) collection.

**Proposition 2.2.** Let \( G \) be an u.s.c., monotone decomposition of \( X, d \). Then \( G \) is locally null if and only if for each \( p \geq 0 \), every subcollection of \( G(\delta) \) has a closed point-set union. In either case, \( H(G) \) is countable and hence \( G \) is 0-dimensional (see [14]).

We say \( K \) is an open covering of \( H(G) \) if \( K \) is a collection of open sets such that each element of \( H(G) \) is contained in some element of \( K \). We say \( G \) is starlike if each \( g \in H(G) \) is compact and starlike. We say \( G \) is starlike-equivalent (properly starlike-equivalent; radially-pointlike) if each \( g \in H(G) \) is equivalent under a space homeomorphism to a compact, starlike set (compact, properly starlike set; compact, radially-pointlike set). Often when showing a decomposition to be shrinkable, the nondegenerate elements are shrunk along arcs (e.g. see [3] and [11]). We isolate this property, calling it radial-shrinkability. Intuitively, a decomposition is radially-shrinkable if for each \( g \in H(G) \) we can choose a space homeomorphism \( H \), a compact, starlike set \( G \), and a reference point \( p \) of \( G \) such that \( H \) takes \( g \) onto \( G \), and \( G \) can be shrunk along segments toward \( p \) in such a way that \( g \) is shrunk along arcs toward \( H^{-1}(p) \).
Let $H(G) = \{g_x : x \in \mathbb{R}\}$. We say $G$ is \textit{radially-shrinkable} in $(X, d)$ if there are collections of maps $\{h_x\}$, compact, starlike sets $\{g_x\}$, and points $\{p_x\}$ such that for each $g_x \in H(G)$, $h_x$ is a space homeomorphism taking $g_x$ onto $g_x$ and $q_x$ is starlike w.r.t. $p_x$, and such that for each $\epsilon > 0$ and for each open set $U$ containing $\bigcup H(G)$, there is $h$ such that

(1) $h$ is a homeomorphism from $X$ onto $X$ and $h(x) = x$ for each $x \in N$;

(2) diam$(g_x) < \epsilon$ for each $g_x \in H(G)$; and

(3) if $g_x \in G(0)$, there is a neighborhood $V_x$ and there is a map $h_x$ such that $g_x \in V_x \subset C_i V_x$ and $h_x : C_i V_x \to h_x : C_i V_x$, is a starlike w.r.t. $p_x$, and $h_x : C_i V_x \subset C_i V_x$ is the identity, $f_x(h_x)(x) \in \{p_x, h(x)\}$ for each $x \in C_i V_x$, and $h_x : C_i V_x \to h(x) = h_x$. If $B \subset X$, then let $G(B)$ be the decomposition of $X$ such that $H(G(B)) = \{g \in H(G) : g \subset B\}$. We say $G$ is \textit{shrinkable} (radially-shrinkable) at each $g \in H(G)$ if there is an open set $U$ containing $g$ such that $B \cup U \cap (\bigcup H(G)) = \emptyset$ and $G(U)$ is shrinkable (radially-shrinkable). We say $G$ is \textit{star-0-dimensional} if for each $g \in H(G)$, there is a neighborhood base $\langle U_n \rangle$ for $g$ such that for each $n$, $U_n \cap (\bigcup H(G)) = \emptyset$, $U_n \subset C_i U_{n+1}$, and $C_i U_n$ is compact and homeomorphic to the closure of an open, starlike w.r.t. $p_x$, set with empty edge w.r.t. $p_x$.

\textbf{Examples.} Let $(X, d)$ be as defined in Example 2 of Section 1. Then $(X, d)$ is a compact, SC-WR-CE metric space (which is not the linear subspace of any normed linear space). Let $p = (4, 3, 0, 2)$ and let $N(p, \epsilon)$ be the neighborhood of $p$ with radius $\epsilon$. Then for each $p \in C_i Ed_{n}(N(p, \epsilon))$, circumstances like this will force us to be careful when constructing starshrinkings which move points toward a given point.

\textbf{Example 1.} Let $G$ be the decomposition of $(X, d)$ such that $H(G) = \{g\}$, where $g = \{(x, y, z) : (x, y, z) \in X(2)$ and $y = x\}$. Then $G$ is a null, starlike decomposition of $(X, d)$ and $(X, G) \cong (X, d)$.

\textbf{Example 2.} Let $G_2$ be the decomposition of $(X, d)$ such that $H(G_2) = \{g\}$, where $g = \{(x, y, z) : (x, y, z) \in X(2)$ and $y = 0\}$. Then $G_2$ is a null, starlike decomposition of $(X, d)$ which is shrinkable and pointlike but neither radially-shrinkable nor radial-pointlike in $(X, d)$.

3. \textbf{Neighborhood bases for starlike sets.} One key to constructing the shrinkings used by Bing [6], Meyer [11], and Bean [3] is the fact that in $E^3$ starlike sets possess neighborhood bases of ideally starlike sets. In a convex metric setting, the existence of such neighborhood bases is non-trivial. In this section we show that in locally compact, SC-WR metric spaces, compact, starlike sets have neighborhood bases of ideally starlike sets. Using this result, we establish two results needed to construct the shrinkings of Sections 5 and 6.

\textbf{Lemma 3.1.} Let $(X, d)$ be a locally compact, SC-WR metric space. Let $p \in N \subset X$ and let $h$ be a continuous map of $N \to N$ such that $h(x) \in [p_x]$ for each $x \in N$. Then $h$ is extended continuously to $N$ by letting $h(p) = p$, and $h$, thus extended, is one-to-one if and only if $y \in [p_x] \subset \{x, p\}$ implies $h(y) \in [p_x] \subset \{x, p\}$ for each $x \in N$.

\textbf{Lemma 3.2.} Let $A$ and $B$ be subsets of a locally compact, SC-WR metric space $(X, d)$ such that $N$ is a neighborhood of $A$, and each of $A$ and $C \subset N$ is a compact, starlike w.r.t. $p$ set. Let $h$ be an embedding of $C_i N$ into $C_i N$ such that $h \subset C_i N \subset C_i N$. Then there is an embedding $H$ of $C_i N$ into $C_i N$ such that $H(x) \in [p_x]$ for each $x \in C_i N$, $H(A) = h$, and $h(B)$ is the identity, and if $H(0)$ is the identity, then $H$ is a homeomorphism of $C_i N$ onto $C_i N$.

\textbf{Proof.} Let $f$ be a continuous function of $C_i N$ onto $[0, 1]$ such that $f(0) = 0$ and $f(Bd N) = 1$ and let $F(x) = \max \{f(y) : y \in [p_x]\}$ for each $x \in C_i N$. It follows that $F$ is a continuous function of $C_i N$ onto $[0, 1]$ such that $F(0) = 0$ and $F(Bd N) = 1$. Furthermore, if $y \in [p_x]$ then $F(y) \leq F(x)$. Now for each $x \in C_i N$, define

$$G(x) = F(x) \left[ 1 - \frac{d(p, h(x))}{d(p, x)} \right] + \frac{d(p, h(x))}{d(p, x)}.$$ 

It follows that $G(x)$ is a continuous function of $C_i N \to [0, 1]$. We now construct $H$. For each $x \in C_i N$, define

$$H(x) = \begin{cases} G(x) \cdot \frac{(1 - G(x))}{p} & \text{for } x \neq p, \\ p & \text{for } x = p. \end{cases}$$

Clearly $H$ satisfies the requirements of the conclusion providing $H$ is an embedding. The continuity of $H$ follows from Lemma 3.1. We need only show $H$ is one-to-one, and this is done by satisfying Lemma 3.1. Let $x \in C_i N \setminus p$ and let $y \in [p_x] \subset \{x, p\}$. It is not hard to show that $H(y) \in [p_x] \subset \{x, p\}$ if and only if $G(x) \frac{d(p, x)}{d(p, y)} < G(x) \frac{d(p, x)}{d(p, y)}$. We establish this inequality by considering, three cases: $F(y) = 1$, $F(y) = 0$, and $0 < F(y) < 1$. The inequality holds trivially in the first two cases. Now suppose $0 < F(y) < 1$. Observing that $h(y) \in [p_x] \subset \{x, p\}$ by Lemma 3.1 and hence $d(p, h(x)) > d(p, h(x))$, it follows that

$$d(p, y) - d(p, x) < \left[ \frac{d(p, h(x))}{d(p, x)} - d(p, h(x)) \right] \frac{1}{1 - F(y)}. \tag{1}$$

Manipulating algebraically, we have

$$F(y) \left[ \frac{d(p, y) - d(p, h(x))}{d(p, y)} + d(p, h(x)) \right] + d(p, h(x)) \leq F(y) \left[ \frac{d(p, h(x))}{d(p, x)} + d(p, h(x)) \right] + d(p, h(x)).$$

This completes the proof.

\textbf{Lemma 3.3.} Let $A$ be a compact, starlike w.r.t. $p$ set in a locally compact, SC-WR metric space $(X, d)$ and let $B$ be an open set containing $A$. Then there is a neighborhood $N$ of $A$ such that $C_i N$ is compact, $C_i N \subset U$, and $N$ is ideally starlike w.r.t. $p$.

\textbf{Proof.} Let $\delta > 0$ such that $C_i N(A, \delta)$ is compact and contained in $U$. Define $N^*_\delta$ to be $\{y : [p_y] \cap Bd N(A, \delta) = \emptyset\}$. It follows that $C_i N^*_\delta$ is compact, $C_i N^*_\delta \subset U$, and $N^*_\delta$ is starlike w.r.t. $p$. It follows from Proposition 2.1(4) that $N^*_\delta$ is a neighborhood.
of $A$. Let $A = \text{diam} A$. Choose a circular neighborhood $S$ of $p$ such that $\text{CNS} \subset N$. Let $\epsilon > 0$ such that $0 < \epsilon < \delta$ and $\text{CNS} \subset S$, and let $\lambda = \frac{\epsilon}{\delta}$. Define $h$ by

$$h(x) = \begin{cases} \frac{1}{2}(x + (1 - \lambda)p) & \text{for } x \neq p, \\ (1 - \lambda)p & \text{for } x = p. \end{cases}$$

As in the proof of Lemma 3.2, it can be shown that $h$ is an embedding of $\text{CNS}$ into $\text{CNS}$. By Lemma 3.2, there is an embedding $H$ of $\text{CNS}$ into $\text{CNS}$ such that $H(x) \in [px]$ for each $x \in \text{CNS}$, $H(A) = h(A)$, and $H/\text{Bd}N$ is the identity. It follows that $H(A) = S$. We choose $N$ to be $H^{-1}(S)$. It is straightforward to show that $N$ is the required neighborhood of $A$.

**Lemma 3.4.** Let $A$ be a compact, properly starlike w.r.t. $p$ set in a locally compact, SC-WR metric space $(X, d)$ and let $U$ be an open set containing $A$. Then there is an ideal starlike w.r.t. $p$ neighborhood $N$ of $A$ such that $\text{CNS}$ is compact, $\text{CNS} \subset U$, and no nondegenerate segment from $p$ in $A$ has its terminal point in $\text{Bd}N$.

**Lemma 3.5.** Let $A$ be a compact, properly starlike w.r.t. $p$ set in a locally compact, SC-WR-C metric space $(X, d)$, let $U$ be an open set containing $A$, and let $s > 0$. Then there is an ideal starlike w.r.t. $p$ neighborhood $N$ of $A$ such that $\text{CNS}$ is compact, $N \subset U$, and $\text{Ed}_p N \subset (p, s)$.

Proof. Let $S$ denote the collection of segments in $A$ from $p$ which cannot be extended in $A$. Let $s \in S$ and suppose the conclusion is false for $s$ as a properly starlike w.r.t. $p$ set. Let $\{N_i(s)\}$ be a nested neighborhood base of ideal starlike w.r.t. $p$ sets for $s$ such that $\text{CNS}(s)$ is compact and $s \cap \text{Ed}_p N_i(s) = \emptyset$ for each $i$ (Lemma 3.4). We choose $x_i \in \text{Ed}_p N_i(s) - \text{N}(p, s)$, for each $i$. Then $\{x_i\}$ is contained in the compact set $\text{Ed}_p N_i(s) - \text{N}(p, s)$.

We may assume $x_i \to x$, where $x \in \text{Ed}_p N_i(s) - \text{N}(p, s)$. But $x \in (\bigcap \text{CNS}(s)) - \text{N}(p, s)$, this implies $x \in s - p$. A contraction. Thus for $x \in S$, we have a neighborhood $N(x)$ of $x$ such that $N(x) \subset U$, $\text{CNS}$ is compact, and $N \cap \text{Ed}_p N_i(s) = \emptyset$ for each $i$. Since $A$ is covered by $\{N(x) : x \in S\}$, we may choose an ideal starlike w.r.t. $p$ neighborhood $N$ of $A$ such that $\text{CNS}$ is compact and $N \subset \text{CNS}$ (Lemma 3.5). Since $\text{Ed}_p N \subset \text{Ed}_p N_i(s)$, we have $\text{Ed}_p N \subset (p, s)$.

**Lemma 3.6.** Let $(X, d)$ be a locally compact, SC-WR metric space and let $U$ be an open set in $X$ containing $p$ such that $\text{CNS} = \text{compact and } \text{Ed}_p U = \emptyset$. Then $p$ is starlike w.r.t. $p$ if and only if $U = \bigcup V_x$ where each $V_x$ is ideal starlike w.r.t. $p$ neighborhood and $\text{CNS} = V_{x+1}$ for each $n$.

Proof. Sufficiency is straightforward. As for necessity, let $z$ be a segment from $p$ to $\text{Bd} U$ and let $p$ be considered the first point of $z$ where $h(s)$ is the first point on $s$ where $h$ hits $\text{Bd} U$. Then $U$ is starlike w.r.t. $p$ implies $U = \bigcup\{[px(s)] - x(s)\}$. Now let $s_1 > 0$ such that $e_1 < \frac{d(p, \text{Bd} U)}{2}$. It can be shown that there is $\delta_1 > 0$ such that $y \not\in \text{N}(\text{Bd} U, e_1) \implies [px(s)] \cap \text{N}(\text{Bd} U, e_1) = \emptyset$. Fix a segment $s$ with respect to the linear ordering on $[px(s)]$, let $y_1(s) = \sup\{y \in [px(s)] : y \not\in [yx(s)]\}$ such that $d(y', \text{Bd} U) > s_1 \implies \text{CNS} = \bigcup \{[yx(s)] \cap \text{Bd} U \}$, then $A_1 \subset \text{Bd} U$. Since $A_1$ is starlike w.r.t. $p$ and $\text{CNS}$ is compact, $A_1$ is starlike w.r.t. $p$ by Lemma 3.3. We obtain an ideal starlike w.r.t. $p$ neighborhood $N_x$ such that $\text{CNS} = N_x \subset \text{Bd} N_x$. It follows that $\text{d}(\text{Bd} V_x, \text{Bd} U) < s_1$. Let $s_2 > 0$ such that $s_3 < \frac{d(\text{Bd} V_x, \text{Bd} U)}{2}$. As above, we obtain an ideal starlike w.r.t. $p$ neighborhood $N_y$ such that $\text{CNS} = N_y \subset \text{Bd} N_y$, and $\text{d}(\text{Bd} V_y, \text{Bd} U) < s_2$. Necessity now follows by induction.

**4. Radially-shrinkable and radially-pointlike decompositions.** All spaces in this section are locally compact SC-WR metric spaces. We show that radially-shrinkable decompositions are radially-pointlike (Theorem 4.3); this result is an important cog of Section 5. We also establish two results for radially-shrinkable decompositions previously established for shrinkable decompositions (Theorems 7 and 10 of [9]).

**Theorem 4.1.** Let $X$ be an open set in $(X, d)$ containing a compact, starlike w.r.t. $p$ set $A$ and let $f$ be an embedding of $\text{CNS}$ into $\text{CIU}$ such that $f(x) \in [px]$ for each $x \in \text{CIU}$. Then (1) if $Y$ is a starlike w.r.t. $p$ neighborhood of $A$ such that $\text{CNS}$ is compact and $Y \subset U$, then $f(\text{CNS}) \subset Y$, and (2) if $f(\text{Bd} U) = \text{Bd} Y$, then $f(\text{CNS}) \subset Y$. (3) There is a homeomorphic $F$ of $\text{CNS}$ onto $\text{CIU}$ such that $f(x) = [px]$ for each $x \in \text{CIU}$, $f(A = f, d)$ and $F(\text{Bd} F)$ is the identity.

Proof. (1) follows from the fact that $\text{CNS}$ is starlike w.r.t. $p$. (2) follows from (1), Lemma 3.3, and Lemma 3.2.

**Theorem 4.2.** Let $G$ be a 0-dimensional decompositions of $(X, d)$ such that $\text{H}(G) = H(G')$. If $G$ is radially shrinkable in $(X, d)$, then $G'$ is radially shrinkable in $(X, d)$.

Proof. Some details are the same as in Theorem 7 of [9]; we sketch the differences. Let $H(G) = \{q_x \in \emptyset\}$ and let $\{h_x\}, \{q_x\}$, and $\{p_x\}$ be the collections of maps, compact, starlike sets, and points, respectively, given us by the radially-shrinkability of $G$. We claim that $\{h_x : q_x \in H(G')\}, \{q_x : q_x \in H(G')\}$, and $\{p_x : p_x \in H(G')\}$ are the required collections for $G'$. Let $s > 0$ and let $U$ be an open set containing $\bigcup H(G')$. Then $\{U - X \cup G'(s)\}$ is an open cover of $H(G)$ and is refined by $s$, a disjoint collection of open sets ([9], Theorem 1). Let $U'$ be the union of all components of $U$ which intersect $\bigcup G'(s)$. Then $U'$ is an open subset of $U$ (Proposition 2.1(2)). Since $G$ is radially shrinkable, there is a homeomorphism $h$ of $X$ onto $X$ such that $h(U - X \cup G'(s))$ is the identity, diam $h_g < \epsilon$ for each $g_x \in H(G')$, and for each $q_x \in G'(s)$ there are $V_x$ and $f_x$ such that $h_x, q_x, p_x, U, h_y, V_y$, and $f_a$ satisfy the remaining radial-shrinkability conditions at $g_x$ for $G$. Define

$$h(x) = \begin{cases} e & \text{for } x \in U \cup U', \\ h(x) & \text{for } x \in U' \end{cases}$$

then $h$ is a homeomorphism of $X$ onto $X$ such that $H(U - X \cup G'(s))$ is the identity, and diam $h_g < \epsilon$ for each $g_x \in H(G')$. Let $g_x \in H(G')$. Choose a starlike w.r.t. $p_x$ neighborhood $W_x$ such that $q_x \subset W_x \subset h_x(V_x \cup U')$ and $U W_x$ is compact (Lemma 3.3). Letting $F_x = f_x(\text{CNS})$, it is easy to verify using Lemma 4.1(1) that $h_x, q_x, p_x, U, H, h_x, (W_x),$ and $F_x$ satisfy the remaining radial-shrinkability conditions at $g_x$ for $G'$. 

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Theorem 4.2. Let G be a 0-dimensional decomposition of $(X,d)$. Then G is radially-shrinkable in $(X,d)$ if and only if G is radially-shrinkable in $(X,d)$ at each element of $H(G)$.

Proof. The proof of Theorem 10 of [9] may be modified to obtain this proof in virtually the same way the proof of Theorem 7 of [9] is modified to obtain the proof of Theorem 4.1 (see [14]).

Theorem 4.3. Let G be a 0-dimensional radially-shrinkable decomposition of $(X,d)$. Then G is radially-pointlike in $(X,d)$.

Proof. Let $g \in H(G)$ and let $h, q, p$ be such that $h$ is a space homeomorphism taking $g$ onto $q$ and $q$ is a compact, starlike w.r.t. p subset, given us by the radially-shrinkability of G. Let U be an open set containing g. We must construct V and H satisfying the radially-pointlike conditions for q in order to conclude G is radially pointlike. The rest of the proof is divided into several parts.

(i) For each $s > 0$, there is an open set O containing g such that for each open subset $W$ of O containing $g$, there are homeomorphisms $H_s$ and $F_s$ such that $H_s$ is a homeomorphism of $X$ onto $X$, $H_s((X-h^{-1})(U))$ is the identity, $diam H_s(g)<e$, $F_s$ is a homeomorphism of $h(Cl W)$ onto $h(Cl W)$, $F_s(Bd h(Cl W))$ is the identity, $F_s(h(x)) \in [ph(x)]$ for each $x \in Cl W$, and $H_s(Cl W) = h^{-1}\ast F_s + h$.

Let $G_1$ be the decomposition of $X$ such that $H(G_1) = (g)$. Then by Theorem 4.1, $G_1$ is a radially-shrinkable decomposition. Thus we have a homeomorphism $h_1$ from $X$ onto $X$ such that $h_1((X-h^{-1})(U))$ is the identity, $diam h_1(g)<e$, and there are O and f such that $h_1$, g, f, $h_1^{-1}(U)$, $h_1$, O, and f satisfy the remaining radially-shrinkability conditions at $g$ for $G_1$. Let $W$ be any open set containing g such that $W \subset O$. By Lemma 4.1 there is a homeomorphism $F_1$ taking $h(Cl W)$ onto $h(Cl W)$ such that $F_1(h(x)) \in [ph(x)]$ for each $x \in Cl W$, $F_1(f) = f$, and $F_1(Bd h(Cl W))$ is the identity. Define

$$H_1(x) = \begin{cases} x & \text{for } x \in X - Cl W, \\ h^{-1}(F_1(h(x))) & \text{for } x \in Cl W. \end{cases}$$

It follows that $H_1$ is a homeomorphism of $X$ onto $X$ and $H_1((X-h^{-1})(U))$ is the identity. It also follows that $diam H_1(g) = diam h_1(g)$; hence $diam H_1(g)<e$.

(ii) Construction of V of H.

Let $\{N_k\}$ be a neighborhood base for g such that $N_k \subset Cl N_{k+1} \subset N_{k+2} \subset \cdots$, $Cl N_k$ is compact, $h(Cl N_k) \subset U$, and $h(N_k)$ is starlike w.r.t. p (Lemma 3.3). Choose $M_s$ to be $N_k$ and set $V = h(N_k)$. By the uniform continuity of $h$ on $Cl M_s$, there is a sequence of positive numbers $\{\delta_s\}$ such that $d(x,y) < 2\delta_s$ implies $d(h(x),h(y)) < 1/n$ for each $x,y \in Cl M_s$. For each positive integer n, let $O_n$ be the open set containing g given us by (i) for $\delta_s$ and h. Assuming $M_s \in \{N_k\}$ has been chosen, choose $M_{s+1} \in \{N_k\}$ such that $Cl M_{s+1} \subset M_s \cup O_n$. Then by (i) we have collections of homeomorphisms $\{H_s\}$ and $\{F_s\}$ such that for each $n$ the following hold: $H_s$ is a homeomorphism of $X$ onto $X$, $H_s((X-h^{-1})(U))$ is the identity, $diam H_s(g)<\delta_s$, $F_s$ is a homeomorphism of $h(Cl M_s)$ onto $h(Cl M_s)$, $F_s(Bd h(Cl M_s))$ is the identity, $F_s(h(x)) \in [ph(x)]$ for each $x \in Cl M_s$, and $H_s(Cl M_s) = h^{-1}\ast F_s + h$.

For each $n$ define $h_n(x) = F_n(\cdots(F_n(\cdots(F_n(h(x)))\cdots) for $x \in Cl (h(M_{s+n})-h(Cl M_{s+n}))$. Define

$$H^n(x) = \begin{cases} x & \text{for } x \in X - h(M_s), \\ h_n(x) & \text{for } x \in Cl h(M_{s+n}). \end{cases}$$

and

$$H(X) = \{H^n(x) for x \in X - \{h(M_s)-h(Cl M_{s+n})\}, h_n(x) for x \in Cl h(M_{s+n}).$$

Clearly H is well defined on $X - q$.

(iii) $x_n \to q$ implies $H(x_n) \to p$.

Let $y_n = h^{-1}(x_n)$, let $s > 0$, and let N be a neighborhood of q such that $h^{-1}(N) \subset U$. Since $H_N$ is uniformly continuous on $Cl M_s$, there is $\zeta > 0$ such that $\zeta < diam g$ and $d(x,y) < \zeta$ implies $d(H_N(x),H_N(y)) < \delta_N(x,y) = \zeta$ for $x,y \in Cl M_s$. Let J be so large that $m \geq J$ implies $\{x_n \in Cl M_{s+n}\}$ and $d(y_n, q) < diam g$. Then $m \geq J$ implies $d(H_N(y_n),H_N(q)) < 2\delta_N$, which implies $d(F_N(x_n), F_N(q)) = d(h(h_N(y_n)), h(h_N(y))) < 1/N$.

So $d(F_N(x_n), p) < 2\zeta < 2/N < e$. Now suppose $x \in h(M_s)-h(Cl M_{s+n})$, where $m \geq J$ and $n \geq N$. Since $H_N(x_n) = F_n(\cdots(F_n(x_n))\cdots), it follows that $d(H_N(x),p) < d(F_N(\cdots(F_N(x_n))\cdots),p) < d(F_N(x_n), p) < 2\zeta < e$.

It follows that $x_n \to q$ implies $H(x_n) \to p$.

(iv) H is continuous on $X - q$.

Since $\{h(M_s)\}$ is a neighborhood base for q, then $\{h(M_s)-h(Cl M_{s+n})\}$ is locally null collection of disjoint, open sets. Since each of $H^n$ and $h_n$ is continuous on its domain, it follows from Theorem 2 of [9], that H is continuous on $X - q$.

(v) $H(x) \in [px]$ for $x \in X - q$.

If $x \in X - V$, then $H(x) = x$. If $x \in V$, assume x is in some $h(M_s)-h(Cl M_{s+n})$.

Then $H(x) = F_n(\cdots(F_n(x))) \in [pF_n(\cdots(F_n(x)))\cdots] = [pF_n(x)] \subset [px]$ by applying Lemma 3.1 inductively to $\{F_1, \cdots, F_n\}$.

(vi) $H(X - q) = V - p$ and $H(Cl V - q) = Cl V - p$.

Since H is the identity on $X - q$, assume H shows $H(X - q) = V - p$. Let $x \in X - q$ and assume $x \in h(M_s)-h(Cl M_{s+n})$. For each $k$, $F_k(\cdots) \in q$ if and only if $x = p$. Since $x \in V - q$, then none of $F_k(\cdots)$, $\cdots, F_1(\cdots) equals p$ by induction, $H(x) \neq p$. Thus $H(V - q) \subset V - p$. Now let y $\in V - p$ and suppose $y \in [py] - x$ where we assume $H(x) = x$. From (v) we have $H^{-1}(y) \subset [py]$. We suppose there is no preimage of y on $[py]$, then the continuity of $H$, $H(([py]) = ([py])$.

From (ii) we have $d(x,y) < s$ implies $d(H(x),p) < e$. This contradicts the fact that $d(x,y) < s$. Thus $H(V - q) = V - p$. 


(vii) $H^{-1}$ is continuous on $X - p$.

It is straightforward to show (using (v), the properties of the $F_i$'s, and Lemma 3.1) that $H$ is one-to-one. It is also straightforward to show that for each $n$, $F_n$ takes open sets of $h(M_n)$ onto open sets of $h(M_n)$. Hence it follows from (iii) that \( \{h(M_n) - h(M_{n+1})\} \) is a locally null collection of disjoint, open sets. We now apply Theorem 2 of [9], to obtain the continuity of $H^{-1}$ on $X - p$.

5. Properly starlike-equivalent decompositions. In this section $(X, d)$ is a locally compact, SC-WR-CE metric space. We show that locally null, properly starlike-equivalent decompositions of $(X, d)$ are radially-shrinkable. We then show that for locally null decompositions of $(X, d)$, some of the properties studied in this paper are equivalent. These two results include Theorem A as stated in Section 1. We must first establish a result (Lemma 5.1) in which we construct a preliminary shrinking which moves along segments and moves any edge points; if a map moves along segments and moves edge points, it cannot be an onto map and hence cannot be a shrinking. The reader might find it helpful to refer to the space $(X(2), d_2)$ of the examples of Section 2 while working the proof of Lemma 5.1; he may also wish to consult [14].

**Lemma 5.1.** Let $G$ be a monotone, locally null decomposition of $(X, d)$, let $g \in H(G)$ be a compact, properly starlike w.r.t. $p$ set, let $W$ be an open set containing $g$, and let $\varepsilon > 0$. Then there is an open set $U$, an open set $M$, and a homeomorphism $h$ from $X$ onto $X$ satisfying

1. $g = M = CLM \subset U \subset W$;
2. $U \cap h(\{\emptyset\}) = \emptyset$;
3. $M$ is ideally starlike w.r.t. $p$;
4. $h(x) \in [px]$ for each $x \in CLM$;
5. $h(X - M)$ is the identity; and
6. $(diam h') < \varepsilon$ for each $g' \in H(G(U))$.

**Proof.** The proof is given in three parts.

(i) Construction of the "controls" and the open sets $U$ and $M$.

Because of Proposition 2.2 we may choose a neighborhood base $\{U_n\}$ for $g$ such that $Bu_n \cap (\{H(G)\} - \emptyset) < U_n \subset W$ for each $n$. Let $G'$ be the decomposition of $X$ such that $H(G') = H(G) - \{g\}$. Because of Proposition 2.2 we may choose a neighborhood base $\{V_n\}$ for $p$ such that $Bu_n \cap (\{H(G')\} - \emptyset) < V_n$ for each $n$. There is a nested neighborhood base $\{N_n\}$ for $g$ such that $CLM_n$ is compact, each $N_n$ is ideally starlike w.r.t. $p$, and no nondegenerate segment from $p$ in $g$ has its terminal point on the edge w.r.t. $p$ of any $N_n$ (Lemma 3.4). Choose $F_n \subset V_n$ such that $CLM_n \cap N_n(p, 1/2)$ and $\delta_j > 0$ such that if $a, b \in Bu_n \cap Ed_n, a$ and $d(a, b) < \delta_j$, then for every ideally starlike w.r.t $p$ neighborhood $N$ with $N \subset N_n$, $d([pa] \cap (Bu_n \cup Ed_n), [pb] \cap (Bu_n \cup Ed_n)) < 1/2\varepsilon$. Let $\delta_j > 0$ such that if $a, b \in Bu_n \cap Ed_n, a$ and $d(a, b) < \delta_j$, then $d(a, b) < \delta_j$. Let $\delta_j = d(Bu_n, g)$ and let $\delta_j > 0$ such that $\delta_j < \Omega$ and $\delta_j < \Omega$. Since $G$ is locally null and $g$ is compact, we may choose $U \in \{U_n\}$ such that $U \cap N_n$ and if $g' \in H(G) - \{g\}$, then $g' < U$ only if $diam g' < \delta_j$. Let $R_i = N(i, i\delta_j), i = 1, 2, 3, ..., \infty$ We now choose $M_1, ..., M_k$ members of $\{N_n\}$ such that

1. $R_i$ contains $N_i$;
2. $g \in M_1 \subset M_2 \subset \ldots \subset M_k \subset CLM_k \subset \{AU\}$ $\cap \{N_n\}$;
3. if $g' \in H(G) - \{g\}$, and $g' \cap M_k \neq \emptyset$, then $g' \cap M_k = \emptyset, i \neq k$ implies $g' \subset M_i$, and in any case diam $g' < \delta_j$;
4. for each $i, g \cap Ed_i M_i = \emptyset$ and $Ed_i M_i \subset R_i$ (Lemma 3.5); and
5. if $a \in \{Bu_n \cup Ed_n\} - R_i$ and $i, j \in \{1, ..., k\}$, then $d([pa] \cap (Bu_n \cup Ed_n), [pb] \cap (Bu_n \cup Ed_n)) < \delta_j$.

We choose $M$ to be $M_k$.

(ii) Construction of the shrinking $h$ satisfying conditions (4) and (5) of the conclusion.

Let $x \in CLM_k - p$ and let $s$ be the segment from $p$ to a point in $Bu_n \cup Ed_n$, so that $[px] = s$. For $i \in \{1, ..., k\}$, let

$$m(x) = \begin{cases} s \cap \{Bu_n \cup Ed_n\} & \text{if } i \in \{1, ..., k\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that each of $m_i(x)$ and $r_i(x)$ is continuous on $CLM_k - p$. We now define a map $H$ from $\{\{Bu_n \cup Ed_n\} - CLM\}_{i=1, ..., k}$ into $CLM_k$ by

$$H(m_i) = m_i \text{ for } d(p, m_i) < \varepsilon, r_i(m_i),$$

where $m \in \{Bu_n \cup Ed_n\}$ and $i \in \{1, ..., k\}$. It follows from properties (4) and (5) of $\{M_1, ..., M_k\}$ that $H(\{Bu_n \cup Ed_n\})$ is the identity. It can be shown that $H$ is continuous on its domain. We now define $h$ from $CLM_k$ into $CLM_k$ by

$$h(x) = \begin{cases} \frac{d(m(x), x) + d(x, m_x)}{d(m(x), x) + d(m(x), x)} & \text{for } x \in CLM_k - CLM_k, \\ d(m(x), x) + \frac{d(m(x), x)}{d(m(x), x)} & \text{for } x \in CLM_k - p, \\ d(m(x), x) + \frac{d(m(x), x)}{d(m(x), x)} & \text{for } x = p. \end{cases}$$

It follows that $h(x) \in [px]$ for each $x \in CLM_k$ and that $h$ is continuous on $CLM_k - p$ and hence, by Lemma 3.1, on $CLM_k$. It can be shown that $h$ is one-to-one on $CLM_k$ by the definition and continuity of $h$ and have $h([m(x)]_k) = [m(x)]_k$. It can now be shown that $H(\{Bu_n \cup Ed_n\}) = CLM_k$. 

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We now define $h$ from $X$ onto $Y$ by
\[ h(x) = \begin{cases} \varphi(x) & x \in f^{-1}_s(U_s) \text{ and } n \geq 1, \\ x & x \in X \setminus \bigcup f^{-1}_s(U_s). \end{cases} \]
It can be shown that $h$ and $h^{-1}$ satisfy the conditions of Theorem 2 of [9] and thus $h$ is a homeomorphism. It is not difficult to show that $h^{-1}$, $\{q_x\}, \{p_x\}$, $U$, $h$, $\{f^{-1}_s(V_x)\}$, and $\{f_x\}$ satisfy the conditions of radial-shrinkability for $G$.

**Proposition 5.1.** Let $G$ be a decomposition of a locally compact, SC metric space $(Y, e)$, where $G$ need not be u.c. If $G$ is radial-pointlike in $(Y, e)$, then $G$ is properly starlike-equivalent in $(Y, e)$.

**Proof.** The proof follows by contradiction.

**Theorem 5.2 (Theorem A).** Let $G$ be a locally null decomposition of $(X, d)$. Then the following are equivalent:

1. $G$ is properly starlike-equivalent in $(X, d)$;
2. $G$ is radially shrinkable in $(X, d)$;
3. $G$ is radially shrinkable in $(X, d)$ at each element of $H(G)$; and
4. $G$ is radially-pointlike in $(X, d)$.

If any of the above hold, $X/G \cong X$.

**Proof.** The circle of implications follows from the theorems of Sections 4 and 5.

**Corollary 5.1.** Each compact starlike subset of $E^s$ is radially-pointlike.

Other consequences of Theorem 5.2 are given in Examples 1 and 2 of Section 1.

6. **Star-0-dimensional decompositions.** In this section $(X, d)$ is a locally compact, SC-WR-CM metric space. We recall from Section 1 that Price has shown [13] that a decomposition $G$ of $E^s$ yields $E^s$ for each $g \in H(G)$, there is a collection of $n$-cells $B_k$ such that $(\text{Int } B_k)$ is a neighborhood base for $g$ and $\text{Bd } B_k \cap \{H(G)\} = \emptyset$ for each $k$. Such a decomposition is star-0-dimensional, but star-0-dimensional decompositions may not satisfy Price's conditions because there are open $n$-cells which are starlike but whose closures are not $n$-cells. In this section we prove Theorem B after first proving Lemma 6.1, a result analogous to Lemma 5.1.

**Lemma 6.1.** Let $G$ be a monotone decomposition of an open starlike w.r.t. $p$ set $U$ in $(X, d)$ such that $G$ is compact and $E_d U = \emptyset$. Then for each $x > 0$, there is an ideal starlike w.r.t. $p$ neighborhood $V$ such that $GV \subset U$, and there is a homeomorphism $h$ from $G U$ onto $G \subset U$ satisfying these conditions: $h(x) \in [p_x]$ for each $x \in G X$, $h(U - V)$ is the identity, and $d(h(x), e) < \epsilon$ for each $x \in G(U)$.

**Proof.** Let $\alpha > 0$. From Lemma 3.3 we may obtain an ideal starlike w.r.t. $p$ neighborhood of $N$ containing $G X$ such that $C X \subset U$ is compact and $E_d N = \emptyset$.

By Lemma 3.6, $G X = \bigcup V_x$ where each $V_x$ is ideally starlike w.r.t. $p$, $C X \subset V_x$, for each $x$, and each $E_d V_x = \emptyset$. Let $\delta > 0$ such that if $a, b \in B_d X$ and $d(a, b) < \delta$, then $d([p_x]) \cap B_d M, [p_b] \cap B_d M, < \frac{\delta}{2}$ for each ideal starlike w.r.t. $p$ neighbor-
hool. $M \subseteq N$. Let $\delta > 0$ such that if $a, b \in B d N$, $c \in [p a] - V_1$ and $d \in [p b] - V_1$, and $d(c, d) < \delta$, then $d(a, b) < \delta_1$. Let $\delta > 0$ such that $\delta < \min \{\delta_1, \delta_2, \frac{1}{2} \varepsilon\}$. It follows that there is an integer $J$ such that if $a \in B d N$ and $n, m > J$, then

$$d([p a] \cap B d V_n, [p a] \cap B d V_m) < \delta.$$ 

Let $V_n \in \{V_r\}$ such that $n_r > J$ and if diam $g \geq \delta$ (where $g \in H(G)$), then $g \subseteq CL V_n$. Let $V_n \in \{V_r\}$ such that $n_r > J$ and if $g \cap CL V_n \neq \emptyset$, then $g \subseteq V_n$. We continue this process inductively until a $V_n$ has been chosen and $k$ is so large that $\text{Cl} U = N(p, kR) \cap N$. We choose $V$ to be $V_n$. Let $R_i = N(p, iR) \cap N$ for $i \in \{1, \ldots, k\}$. Each $R_i$ is ideally starlike w.r.t. $p$. Let $x \in \text{Cl} U - p$. Then $x \in [p a]$ where $a \in B d N$. For $i \in \{1, \ldots, k\}$, let $r_i(x) = [p a] \cap B d R_i$ and $v_i(x) = [p a] \cap B d V_i$. The procedure is now completely analogous to that of Lemma 5.1.

**Theorem 6.1 (Theorem B).** Let $G$ be a star-0-dimensional decomposition of $(X, d)$. Then $G$ is shrinkable. Hence $X/G \cong X$.

**Proof.** Let $g \in H(G)$, and let $W$ be an open set about $g$ such that $C(W)$ is compact and $B d W \cap \bigcup H(G) = \emptyset$. Let $\varepsilon > 0$ and let $U$ be an open set containing $\bigcup H(G(W))$. Let $G_\varepsilon(g) = \{g' \in H(G(W)) \setminus \text{diam}(g') < \varepsilon\}$. Then $\bigcup G_\varepsilon(g)$ is compact. For each $g' \in G_\varepsilon(g)$, let $O(g')$ be an open set containing $g'$ such that $O(g') \subseteq U$, $B d O(g') \cap \bigcup H(G) = \emptyset$, and $\text{Cl}(O(g'))$ is homeomorphic to an open, starlike set with compact closure and empty edge w.r.t. $p$. Using Lemma 6.1, we proceed exactly as in Lemma 1.2 of [13] to obtain a homeomorphism $h$ of $X$ onto $X$ such that $h((X - U)$ is the identity and $\text{diam}(g') < \varepsilon$ for each $g' \in H(G(W))$. Hence $O(W)$ is shrinkable, i.e., $G$ is shrinkable at $g$. By Theorem 10 of [9], $G$ is shrinkable and by Theorem 4 of [9], $X/G \cong X$.

**Corollary 6.1.** Let $G$ be a decomposition of $E^3$ such that $H(G)$ is countable.

1. The following are equivalent:
   (i) $G$ is star-0-dimensional;
   (ii) $G$ is shrinkable;
   (iii) $E^3/G \cong E^3$; and
   (iv) $G$ satisfies Prices condition.

2. If $G$ is starlike, then $G$ is star-0-dimensional.

**Proof.** (1) follows from Theorem 6.1 and Theorem 1.4 of [13]; (2) follows from (1) and Theorem 2 of [4].

Other consequences of Theorem 6.1 are given in Example 3 of Section 1.

**References**


[5] — *A decomposition of $E^3$ into points and tame arcs such that the decomposition space is topologically different from $E^3$, Ann. of Math. 65 (1957), pp. 484-500.


Remarks of the elementary theories of formal
and convergent power series

by

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Abstract. In § 1 an example is given of two fields $F_1$, $F_2$ of characteristic 0 such that $F_1 = F_2$ but $F_1[[x_1, x_2]] \not\cong F_2[[x_1, x_2]]$. In § 2 it is shown that $(C(x, y), C(x), C[[x, y]], C[[x]])$, where $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$.

In [3] and [4] Ax and Kochen and Eršov showed among other things that the ring of convergent power series $C[[x]]$, over the complex numbers $C$, is an elementary subring of the ring of formal power series $C[[x]]$ over $C$. This means that the same first order statements (in the language of valued rings) with constants from $C[[x]]$, are true in both rings. (This is denoted $C[[x]] \cong C[[x]]$.) Also they showed that if fields $F_1$ and $F_2$ of characteristic 0 are elementarily equivalent, denoted $F_1 \equiv F_2$ (i.e. the same first order statements in the language of fields are true of $F_1$ and $F_2$) then $F_1[[x]] = F_2[[x]]$ as valued rings (i.e. the same first order statements, in the language of valued rings, are true about $F_1[[x]]$ and $F_2[[x]]$). It is natural to ask whether these results extend to power series rings in several variables. In Section 1, we show that one can have fields $F_1 \equiv F_2$ but $F_1[[x_1, x_2]] \not\cong F_2[[x_1, x_2]]$. In Section 2 we show that a slightly stronger statement than $C(x_1, ..., x_d) \cong C[[x_1, ..., x_d]]$ is false (*). These remarks contradict some results claimed in [7].

Section 1. Eršov [4] showed that for any field $F$ and for $n \geq 2$, $F[[x_1, ..., x_n]]$ is undecidable. We shall give a slightly different proof of this for the case that $F$ has characteristic zero and use this proof to show that we can have $F_1 \equiv F_2$ of characteristic 0 but $F_1[[x_1, ..., x_n]] \not\cong F_2[[x_1, ..., x_n]]$ ($n \geq 2)$ as rings. Let $F$ be a field of characteristic zero.

For the sake of clarity, we begin by showing that $F = F[[x_1, x_2]]$ is undecidable as an $F$ algebra with $x_1$ and $x_2$ picked out, i.e., that $F$ as a ring under the operations of addition and multiplication, with constants for $x_1$ and $x_2$, and with an additional predicate which picks out a particular lifting of the residue field $F$

(*) (Added in proof) Some of the results of this paper and some extensions have been discovered independently by F. Delon, Résultats d'indécidabilité dans les anneaux de séries formelles (to appear).