Functions of generalized variation

by

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Abstract. A class of functions, Λ-BV, of generalized bounded variation is defined in terms of certain sequences, Λ. The following are proved: The intersection of Λ-BV, taken over all sequences Λ, is the class of functions of bounded variation. The union of Λ-BV, taken over all sequences Λ, is the class of functions having a right and left hand limit at every point.

In his investigations on the uniform convergence of Fourier series, D. Waterman introduced a class of functions which he calls Λ-BV. These functions share many of the properties of functions of bounded variation. We will show that this class of functions lies between the regulated functions and the functions of bounded variation. That is, we will prove that the union of the Λ-BV functions over all sequences Λ are the regulated functions and the intersection of the Λ-BV functions over all sequences Λ is the set of functions of bounded variation. We will show that these results cannot be improved by taking countable unions or intersections.

We now give the basic definitions.

Let f be a real function defined on an interval $[a, b]$. If $I = [\alpha, \beta] \subset [a, b]$ we write $f(I) = f(\beta) - f(\alpha)$. Let $\Lambda = \{\lambda_n\}$ be a non-increasing sequence of positive numbers so that $\lambda_n \to 0$ and $\sum \lambda_n = \infty$. A function f is said to be of *Λ-bounded variation* (Λ-BV) if for all sequences $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ we have $\sum \lambda_n |f(I_n)| < \infty$. It may be shown [5] that this is equivalent to assuming that there is a number $M < \infty$ such that for all sequences $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ we have $\sum \lambda_n |f(I_n)| \leq M$. We call the supremum of $\sum \lambda_n |f(I_n)|$ over all sequences $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ the Λ-variation of f.

THEOREM 1. *Λ-BV is a linear space.*

Proof. For f, g in Λ-BV and α, β real

$$\sum \lambda_n |(\alpha f + \beta g)(I_n)| = \sum \lambda_n |\alpha f(I_n) + \beta g(I_n)| \leq |\alpha| \sum \lambda_n |f(I_n)| + |\beta| \sum \lambda_n |g(I_n)|$$

and this implies that $\alpha f + \beta g$ is in Λ-BV.

THEOREM 2. *If $f \in$ BV, then $f \in \Lambda$-BV.*

Proof. Since the λ_n's are monotone we have

$$\sum \lambda_n |f(I_n)| \leqslant \sum \lambda_1 |f(I_n| = \lambda_1 \sum |f(I_n)|$$

from which the theorem follows.

Theorems 3 and 4, which are due to Waterman [5], are included here for the sake of completeness.

THEOREM 3. *If $f \in \Lambda$-BV, then f is bounded.*

Proof. If f is not bounded there is a sequence $\{p_n\}$ in $[a, b]$ so that $|f(p_n)| \to \infty$. There is also a point x in $[a, b]$ and a subsequence $\{q_n\}$ of $\{p_n\}$ so that $q_n \to x$. Furthermore, there is a monotone subsequence $\{r_n\}$ of $\{q_n\}$ and finally the sequence $\{r_n\}$ has a subsequence $\{s_n\}$ so that $|f(s_{n+1})| \geqslant 1 + |f(s_n)|$. If we let I_n be the interval determined by the points s_n and s_{n+1}, then the I_n's will be non-overlapping and $|f(I_n)| = |f(s_{n+1}) - f(s_n)| \geqslant 1$. Thus $\sum \lambda_n |f(I_n)| \geqslant \sum \lambda_n = \infty$ and so f is not in Λ-BV.

THEOREM 4. *If $f \in \Lambda$-BV, then f has a right- and left-hand limit at every point of $[a, b]$.*

Proof. It is sufficient to consider left-hand limits only. Suppose there is a point x in $(a, b]$ at which f does not have a left-hand limit. If $L = \overline{\lim_{t \to x-}} f(t)$ and $l = \underline{\lim_{t \to x-}} f(t)$, then $L > l$. Set $\delta = \frac{1}{3}(L - l)$. There are sequences $\{P_n\}$ and $\{p_n\}$ so that $P_1 < P_2 < \ldots \to x$, $f(P_n) \to L$, $f(P_n) \geqslant L - \delta$ and $p_1 < p_2 < \ldots \to x$, $f(p_n) \to l$, $f(p_n) \leqslant l + \delta$. We choose subsequences $\{Q_n\}$ of $\{P_n\}$ and $\{q_n\}$ of $\{p_n\}$ such that $q_1 < Q_1 < q_2 < Q_2 < \ldots$ If $I_n = [q_n, Q_n]$ then the intervals I_n are disjoint and $|f(I_n)| \geqslant (L - \delta) - (l + \delta) = \delta$. Thus $\sum \lambda_n |f(I_n)| \geqslant \sum \lambda_n \delta = \infty$ and so f is not in Λ-BV.

As a partial converse to Theorem 2 we have the following:

THEOREM 5. *If f is in Λ-BV for every sequence Λ, then $f \in$ BV.*

Proof. Since $f \in \Lambda$-BV for at least one choice of Λ, Theorem 3 says that f is bounded. Thus there are numbers m, M so that $m \leqslant f \leqslant M$. Defining $F = (f - m)/(M - m)$ we have $0 \leqslant F \leqslant 1$ and thus $|F(I)| \leqslant 1$ for every subinterval I of $[a, b]$. By Theorem 1, F is in Λ-BV for every Λ and the theorem will follow if we show that F is in BV.

If F is not BV, there is a point x in $[a, b]$ such that F is not of bounded variation on any neighborhood of x ([2], p. 328). Let $\{d_n\}$ be a sequence of positive numbers so that $\sum d_n = \infty$. There is a finite partition P_1 of $[a, b]$ so that

$$\sum_{I \in P_1} |F(I)| \geqslant d_1 + 2 .$$

The point x is either an interior point of one interval in P_1 or an endpoint of at most two intervals in P_1. If we remove this one, or possibly two, intervals from P_1 and call the remaining collection of intervals Q_1 then, since $|F(I)| \leqslant 1$, we will have

$$\sum_{I \in Q_1} |F(I) \geqslant d_1 .$$

If Q_1 has q_1 intervals we write $Q_1 = \{I_k^1 | \ k = 1, 2, \ldots, q_1\}$ and writing $\lambda(k)$ instead of λ_k, we define

$$\lambda(1) = \lambda(2) = \ldots = \lambda(q_1) = 1 .$$

We then have

$$\sum_1^{q_1} \lambda(k) |F(I_k^1)| \geqslant d_1 .$$

The first step in our induction process is now complete.

Assuming that n steps in our induction process have been completed, we proceed to the next step as follows. The one, or possibly two, intervals that were removed from P_n to form Q_n, form a neighborhood U_n of x. Since F is not of bounded variation on U_n, there is a finite partition P_{n+1} of U_n so that

$$\sum_{I \in P_{n+1}} |F(I)| \geqslant (n+1) d_{n+1} + 2 .$$

The point x is either an interior point of one interval in P_{n+1} or an endpoint of at most two intervals in P_{n+1}. If we remove this one, or possibly two, intervals from P_{n+1} and call the remaining collection of intervals Q_{n+1}, then since $|F(I) \leqslant 1$ we will have

$$\sum_{I \in Q_{n+1}} |F(I)| \geqslant (n+1) d_{n+1} .$$

If Q_{n+1} has q_{n+1} intervals we write $Q_{n+1} = \{I_k^{n+1} | \ k = 1, 2, \ldots, q_{n+1}\}$ and define

$$\lambda(r_n + 1) = \lambda(r_n + 2) = \ldots = \lambda(r_n + q_{n+1} = \frac{1}{n+1}$$

where $r_n = \sum_0^n q_k$ and $q_0 = 0$. We then have

$$\sum_{k=1}^{q_{n+1}} \lambda(r_n + k) |F(I_k^{n+1})| \geqslant d_{n+1} .$$

We observe that because the intervals of Q_{n+1} are within U_n that all the intervals of $Q_1 \cup Q_2 \cup \ldots \cup Q_{n+1}$ are pairwise non-overlapping. It follows that

$$\sum_{i=1}^{n+1} \sum_{k=1}^{q_i} \lambda(r_{i-1} + k) |F(I_k^i)| \geqslant \sum_{i=1}^{n+1} d_i .$$

In this way, we construct a sequence of numbers $\{\lambda(k)\}$ and a sequence $\{I_k^n | \ k = 1, 2, \ldots, q_n;\ n = 1, 2, \ldots\}$ of non-overlapping subintervals of $[a, b]$ so that $\lambda(k)$ decreases to zero, $\sum \lambda(k) = \infty$ and

$$\sum_{i=1}^{\infty} \sum_{k=1}^{q_i} \lambda(r_{i-1} + k) |F(I_k^i)| = \infty .$$

Thus F is not in Λ-BV for this particular sequence of λ's and our proof by contradiction is complete.

To show that Theorem 5 is best possible we need some preliminary results.

LEMMA 1. *Let $\{a_n\}$ be a sequence of positive numbers tending to zero. Then there exists a decreasing sequence $\{b_n\}$ of positive numbers tending to zero such that $\sum b_n = \infty$ and $\sum a_n b_n < \infty$.*

Proof. Setting $N_0 = 0$, we choose a positive integer N_1 such that $a_n \leqslant \frac{1}{2}$ for $n \geqslant N_1$ and put $b_n = (N_1 - N_0)^{-1}$ for $N_0 < n \leqslant N_1$. Assume that integers $N_0, N_1, \ldots, N_k$ have been chosen so that

$$N_0 < N_1 < \ldots < N_k,$$

(1) $$a_n \leqslant 1/2^k \quad \text{for} \quad n \geqslant N_k$$

and

$$b_n > 0 \quad \text{for} \quad N_0 < n \leqslant N_k.$$

We then choose an integer N_{k+1} so that

(2) $$N_{k+1} \geqslant N_k + k,$$

(3) $$N_{k+1} - N_k > b_{N_k}^{-1}$$

and

$$a_n \leqslant 1/2^{k+1} \quad \text{for} \quad n \geqslant N_{k+1}$$

and then put

(4) $$b_n = (N_{k+1} - N_k)^{-1} \quad \text{for} \quad N_k < n \leqslant N_{k+1}.$$

For $N_k < n \leqslant N_{k+1}$, we have by (3) and (4) $b_n = (N_{k+1} - N_k)^{-1} < b_{N_k}$ and by (2) $b_n \leqslant 1/k$ thus showing that the b_n's decrease to zero. By (4) we have

$$\sum_{N_k+1}^{N_{k+1}} b_n = 1$$

and thus $\sum b_n = \infty$. Finally by (1) and (4)

$$\sum_{N_k+1}^{N_{k+1}} a_n b_n \leqslant \sum_{N_k+1}^{N_{k+1}} \frac{1}{2^k} b_n = \frac{1}{2^k} \sum_{N_k+1}^{N_{k+1}} b_n = \frac{1}{2^k}$$

and thus $\sum a_n b_n < \infty$.

We now extend Lemma 1 to a countable number of sequences.

LEMMA 2. *For each positive integer n let $\{a(n,k)\}$ be a sequence of positive numbers tending to zero such that*

(5) $$a(n,k) \leqslant a(n+1,k) \quad (n, k = 1, 2, \ldots).$$

Then there exists a decreasing sequence, $\{B(k)\}$, of positive numbers tending to zero such that $\sum B(k) = \infty$ and $\sum_k a(n,k) B(k) < \infty$ $(n = 1, 2, \ldots)$.

Proof. Let $\{c_n\}$, $\{d_n\}$ and $\{\varepsilon_n\}$ be sequences of positive numbers such that $\sum c_n < \infty$, $\sum d_n = \infty$ and ε_n decreases to zero. For each positive integer n we apply Lemma 1 to the sequence $\{a(n,k)\}$ to obtain a decreasing sequence $\{b(n,k)\}$ of positive numbers tending to zero such that $\sum_k b(n,k) = \infty$ and $\sum_k a(n,k) b(n,k) < \infty$.

We will choose two sequences of integers $\{r_n\}$ and $\{s_n\}$ and set $t_n = \sum_1^n (s_k - r_k + 1)$ and $t_0 = 0$. We let $r_1 = 1$ and choose s_1 so that $s_1 \geqslant r_1$, $b(1, s_1) \leqslant \varepsilon_1$, $\sum_{k=r_1}^{s_1} b(1,k) \geqslant d_1$ and $\sum_{k=1+t_1}^{\infty} a(2,k) b(2,k) \leqslant c_2$.

Assuming $r_1, r_2, \ldots, r_n, s_1, s_2, \ldots, s_n$ have been chosen we proceed as follows. We choose r_{n+1} so that

(6) $$r_{n+1} > 1 + s_n$$

and

(7) $$b(n+1, r_{n+1}) \leqslant b(n, s_n).$$

Then we choose s_{n+1} so that

$$s_{n+1} \geqslant r_{n+1},$$

(8) $$b(n+1, s_{n+1}) \leqslant \varepsilon_{n+1}.$$

(9) $$\sum_{k=r_{n+1}}^{s_{n+1}} b(n+1, k) \geqslant d_{n+1}$$

and

(10) $$\sum_{k=1+t_{n+1}}^{\infty} a(n+2, k) b(n+2, k) \leqslant c_{n+2}.$$

It follows from (6) that $t_n \leqslant s_n$ and hence

(11) $$r_{n+1} \geqslant 1 + t_n.$$

We now choose the sequence $\{B(k)\}$ of our lemma to be the following sequence of $b(n,k)$'s:

$$b(1,1), b(1,2), \ldots, b(1, s_1),$$
$$b(2, r_2), b(2, r_2+1), \ldots, b(2, s_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b(n, r_n), b(n, r_n+1), \ldots, b(n, s_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Because $b(n,k)$ decreases to zero, for a fixed n, (7) and (8) show that the sequence $\{B(k)\}$ decreases to zero. The divergence of $\sum B(k)$ follows from (9). Finally, we need to prove the convergence of $\sum_k a(l+1,k)B(k)$ for $l=0,1,2,\ldots$ Since

$$\sum_k a(l+1,k)B(k)=\sum_{n=0}^{\infty}\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)$$

we use (5), (11) and then (10) to obtain, for $n>l$,

$$\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)$$
$$\leqslant \sum_{k=0}^{s_{n+1}-r_{n+1}} a(n+1,t_n+k+1)b(n+1,r_{n+1}+k)$$
$$\leqslant \sum_{k=0}^{s_{n+1}-r_{n+1}} a(n+1,t_n+k+1)b(n+1,t_n+k+1)$$
$$=\sum_{k=1+t_n}^{t_{n+1}} a(n+1,k)b(n+1,k)\leqslant c_{n+1}.$$

Thus

$$\sum_{n=l}^{\infty}\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)\leqslant \sum_{n=l}^{\infty} c_{n+1}$$

and therefore $\sum_k a(l+1,k)B(k)<\infty$ for $l=0,1,2,\ldots$

LEMMA 3. *Let $\{a_n\}$ be a decreasing sequence of positive numbers. If $\{b_n\}$ is a sequence of positive numbers tending to zero and $\{B_n\}$ is the sequence $\{b_n\}$ arranged in decreasing order, then $\sum a_k b_k\leqslant\sum a_k B_k$.*

Proof. Fix an integer n and consider the set $\{b_1,b_2,\ldots,b_n\}$. Let $b_1'\geqslant b_2'\geqslant\ldots\geqslant b_n'$ be this set arranged in decreasing order. By [1], Th. 368, p. 261, $\sum_1^n a_i b_i\leqslant\sum_1^n a_i b_i'$. But, it is clear that $b_i'\leqslant B_i$ for $i=1,2,\ldots,n$ and so

$$\sum_1^n a_i b_i\leqslant\sum_1^n a_i b_i'\leqslant\sum_1^n a_i B_i.$$

THEOREM 6. *Let $\Lambda_1=\{\lambda_k^1\}$ and $\Lambda_2=\{\lambda_k^2\}$ be two decreasing sequences of positive numbers tending to zero such that $\sum_k\lambda_k^1=\sum_k\lambda_k^2=\infty$. If $\Lambda_3=\{\lambda_k^3\}$, where $\lambda_k^3=\lambda_k^1+\lambda_k^2$, then Λ_3-BV $=\Lambda_1$-BV $\cap\,\Lambda_2$-BV.*

Proof. If $\{I_k\}$ is a sequence of non-overlapping subintervals of $[a,b]$ and f is a function defined on $[a,b]$, then the theorem follows from the equality

$$\sum_k\lambda_k^3|f(I_k)|=\sum_k\lambda_k^1|f(I_k)+\sum_k\lambda_k^2|f(I_k)|.$$

We now show that Theorem 5 is best possible in the following sense.

THEOREM 7. *For each positive integer n let $\Lambda_n=\{a(n,k)\}$ be a decreasing sequence of positive numbers tending to zero such that $\sum_k a(n,k)=\infty$. Then there is a function in $\bigcap_1^\infty\Lambda_n$-BV which is not of bounded variation.*

Proof. For each positive integer n let Λ^n be the sequence $\{A(n,k)\}$ where $A(n,k)=\sum_{i=1}^n a(i,k)$. By theorem 6 we have

$$\Lambda^n\text{-BV}=\Lambda_1\text{-BV}\cap\Lambda_2\text{-BV}\cap\ldots\cap\Lambda_n\text{-BV}.$$

By applying Lemma 2 to the sequences $A(n,k)$ we obtain a decreasing sequence, $\{B(k)\}$, of positive numbers tending to zero such that $\sum B(k)=\infty$ and

$$\sum_k A(n,k)B(k)<\infty$$

for $n=1,2,\ldots$

We define a function f on the interval $[a,b]$ as follows. Let $\{c_k\}$ be a sequence of points in $[a,b]$ such that $a=c_0<c_1<c_2<\ldots<b$ and $\lim c_n=b$. Setting $B(0)=0$, let $f(c_n)=\sum_0^n(-1)^{k+1}B(k)$, $f(b)=\sum_0^\infty(-1)^{k+1}B(k)$ and extend f linearly to the remainder of $[a,b]$. Then we see that f is a continuous function whose total variation equals $\sum B(k)=\infty$. Thus f is not of bounded variation.

If I_k is the interval $[c_{k-1},c_k]$, using Lemma 3 we see that the Λ^n variation of f equals

$$\sum_k A(n,k)|f(I_k)|=\sum_k A(n,k)B(k)<\infty.$$

Thus f is in Λ^n-BV$\subseteq\Lambda_n$-BV and our theorem follows.

THEOREM 8. *If φ is a monotone function mapping $[a,b]$ into $[c,d]$ and f is Λ-BV on $[c,d]$, then $f\circ\varphi$ is Λ-BV on $[a,b]$.*

Proof. Let $I_n=[p_n,q_n]$ be a sequence of non-overlapping subintervals of $[a,b]$. Let J_n be the interval determined by the points $\varphi(p_n)$ and $\varphi(q_n)$. Then because φ is monotone $\varphi(I_n)\subseteq J_n\subseteq[c,d]$ and the intervals J_n are non-overlapping. Because $f\in\Lambda$-BV on $[c,d]$

$$\sum\lambda_n|f\circ\varphi(I_n)|=\Sigma\lambda_n|f\circ\varphi(q_n)-f\circ\varphi(p_n)|$$
$$=\Sigma\lambda_n|f[\varphi(q_n)]-f[\varphi(p_n)]|=\Sigma\lambda_n|f(J_n)|<\infty$$

and thus $f;\varphi$ is in Λ-BV on $[a,b]$.

THEOREM 9. *If f is a continuous function, then $f\in\Lambda$-BV for some sequence Λ.*

Proof. For $\delta>0$ define $\omega(\delta)$, the modulus of continuity of f, by

$$\omega(\delta)=\sup\{|f(t)-f(t')|:\ t,t'\in[a,b],\ |t-t'|\leqslant\delta\}.$$

Clearly, $\omega(\delta)$ is increasing and $\omega(\delta)\to 0$ as $\delta\to 0$ because of the uniform continuity of f on $[a,b]$.

Let $I_n = [a_n, b_n]$ be a sequence of non-overlapping subintervals of $[a, b]$. For each positive integer m define

$$E_m = \left\{I_k\colon \omega\left(\frac{b-a}{m}\right) \geqslant |f(I_k)| > \omega\left(\frac{b-a}{m+1}\right)\right\}.$$

If

$$|I_k| \leqslant \frac{b-a}{m+1},$$

then

$$|f(I_k)| = |f(b_k)-f(a_k)| \leqslant \omega(|b_k-a_k|) \leqslant \omega\left(\frac{b-a}{m+1}\right).$$

Thus $I_k \in E_m$ only if

$$|I_k| > \frac{b-a}{m+1}.$$

Since the intervals I_k are non-overlapping and contained in $[a, b]$, it follows that E_m contains at most m intervals. Also if $I_p \in E_r$ and $I_q \in E_{r+s}$, then

$$|f(I_q)| \leqslant \omega\left(\frac{b-a}{r+s}\right) \leqslant \omega\left(\frac{b-a}{r+1}\right) < |f(I_p)|.$$

Thus, by considering those intervals in E_1, then those in E_2, etc., the intervals may be relabeled J_k so that

$$(12) \qquad |f(J_1)| \geqslant |f(J_2)| \geqslant \ldots \geqslant |f(J_n)| \geqslant \ldots \to 0.$$

Now we want to show that

$$|f(J_n)| \leqslant \omega\left(\frac{b-a}{n}\right).$$

Indeed, if m was an integer for which

$$|f(J_m)| > \omega\left(\frac{b-a}{m}\right),$$

then

$$|f(J_1)| \geqslant |f(J_2)| \geqslant \ldots \geqslant |f(J_m)| > \omega\left(\frac{b-a}{m}\right).$$

This implies that

$$|J_k| > \frac{b-a}{m} \quad (k = 1, 2, \ldots, m)$$

which is impossible since the intervals J_k $(k = 1, 2, \ldots, m)$ are non-overlapping and contained in $[a, b]$. Thus the sequence in (12) is term by term less than or equal to the sequence

$$(13) \qquad \omega\left(\frac{b-a}{1}\right), \ \omega\left(\frac{b-a}{2}\right), \ldots, \omega\left(\frac{b-a}{n}\right), \ldots$$

Because sequence (13) decreases to zero, we may apply Lemma 1 to obtain a decreasing sequence, $\{\lambda_n\}$, of positive numbers tending to zero such that

$$\sum \lambda_n = \infty \quad \text{and} \quad \Sigma \lambda_n \omega\left(\frac{b-a}{n}\right) < \infty.$$

Applying Lemma 3 to the sequences $\{\lambda_n\}$ and $\{|f(I_n)|\}$ we obtain

$$\sum \lambda_n |f(I_n)| \leqslant \sum \lambda_n |f(J_n)| \leqslant \sum \lambda_n \omega\left(\frac{b-a}{n}\right) < \infty.$$

Since the λ_n's are a fixed sequence depending only on $\omega(\delta)$ and not on the I_k's, Theorem 9 follows.

In what follows we need to make use of a theorem due to Sierpiński [4].

THEOREM. *A function has a right- and left-hand limit at each point if and only if it is the composition of a continuous function with a monotone function.*

Sierpiński proved his theorem for functions defined on the entire line. His theorem is easily extended to functions defined on a finite interval. For, if F, defined on $[a, b]$, has a right- and left-hand limit at every point, extend F to a function $\bar{F}$ defined on $(-\infty, \infty)$ by setting $\bar{F}(x) = F(a)$ for $x<a$ and $\bar{F}(x) = F(b)$ for $x>b$. Then by Sierpiński's theorem, there exists a continuous function $\bar{f}$ and a monotone function $\bar{\varphi}$ such that $\bar{F} = \bar{f} \circ \bar{\varphi}$. If we let φ be the restriction of $\bar{\varphi}$ to the interval $[a, b]$ and f the restriction of $\bar{f}$ to the smallest closed interval containing the range of φ, then φ will be monotone, f will be continuous and $F = f \circ \varphi$.

THEOREM 10. *If F has a right- and left-hand limit at every point of* $[a, b]$, *then* $F \in \Lambda$-BV *on* $[a, b]$ *for some sequence* Λ.

Proof. By the discussion following Sierpiński's theorem there exists a monotone function φ defined on $[a, b]$ and a continuous function f defined on the smallest closed interval, say $[c, d]$, containing the range of φ such that $F = f \circ \varphi$. Let ψ be the linear function mapping the interval $[c, d]$ onto the interval $[a, b]$. Then $\psi \circ \varphi$ is a monotone function mapping $[a, b]$ into $[a, b]$ and $f \circ \psi^{-1}$ is a continuous function defined on $[a, b]$. By Theorem 9, $f \circ \psi^{-1}$ is in Λ-BV on $[a, b]$ for some sequence Λ and by Theorem 8 $F = (f \circ \psi^{-1}) \circ (\psi \circ \varphi)$ is in Λ-BV on $[a, b]$.

THEOREM 11. *If g is continuous and F is in* Λ-BV *on* $[a, b]$, *then $g \circ F$ is in* Λ'-BV *on* $[a, b]$ *for some sequence* Λ'.

Proof. By Theorem 4 F has a right- and left-hand limit at every point of $[a, b]$. By the proof of Theorem 10 we know that $F = h \circ \theta$ where h is a continuous function defined on $[a,b]$ and θ is a monotone function mapping $[a,b]$ into $[a,b]$. Since $g\circ F=(g\circ h)\circ\theta$, and $g\circ h$ is continuous, Theorems 8 and 9 imply that $g\circ F$ is in Λ'-BV on $[a,b]$ for some sequence Λ'.

It should be noted that in Theorems 11 the sequences Λ and Λ' are not necessarily the same. If they were, then by choosing F to be a BV function we would have $F\in\Lambda$-BV for every sequence Λ by Theorem 2 and then also $g\circ F\in\Lambda$-BV for every sequence Λ. Then by Theorem 4, $g\circ F$ would be BV. We would have thus proved the following: a continuous function composed with a BV function is again BV, which is false.

To show that Theorem 10 is best possible we need another result, which is similar to Lemma 2.

LEMMA 4. *For each positive integer n let $\{a(n,k)\}$ be a decreasing sequence of positive numbers tending to zero such that $\sum_k a(n,k)=\infty$. Then there exists a decreasing sequence $\{B(k)\}$ of positive numbers tending to zero such that $\sum B(k)=\infty$ and $\sum_k a(n,k)B(k)=\infty$ $(n=1,2,\ldots)$.*

Proof. Let $\{d_n\}$ and $\{\varepsilon_n\}$ be two sequences of positive numbers such that $\sum d_n=\infty$ and ε_n decreases to zero. For positive integers n,k let

$$w(n,k)=\sum_{i=1}^{k}a(n,i).$$

Then, for n fixed

$$w(n,k)\to\infty \quad \text{as} \quad k\to\infty,$$

$$\sum_k \frac{a(n,k)}{w(n,k)}=\infty$$

by the Abel–Dini theorem [3] p. 290, and also

$$\sum_k \frac{1}{w(n,k)}=\infty.$$

For positive integers n,k let

$$b(n,k)=\sum_{i=1}^{n}\frac{1}{w(i,k)}.$$

Then, for n fixed

$$b(n,k)\to 0 \quad \text{as} \quad k\to\infty,$$

$$b(n,k)\geqslant\frac{1}{w(i,k)} \quad (i=1,2,\ldots,n),$$

$$\sum_k b(n,k)=\infty$$

and

$$\sum_k a(i,k)b(n,k)\geqslant\sum_k\frac{a(i,k)}{w(i,k)}=\infty \quad (i=1,2,\ldots,n).$$

We will choose two sequences of integers $\{r_n\}$ and $\{s_n\}$ and set $t_n=\sum_1^n(s_k-r_k+1)$ and $t_0=0$. We let $r_1=1$ and choose s_1 so that

$$s_1\geqslant r_1,\quad b(1,s_1)\leqslant\varepsilon_1,$$

$$\sum_{k=r_1}^{s_1}b(1,k)\geqslant d_1 \quad\text{and}\quad \sum_{k=r_1}^{s_1}a(1,k)b(1,k)\geqslant d_1.$$

Assuming $r_1,r_2,\ldots,r_n,s_1,s_2,\ldots,s_n$ have been chosen we proceed as follows. We choose r_{n+1} so that

(14) $$r_{n+1}\geqslant 1+s_n$$

and

(15) $$b(n+1,r_{n+1})\leqslant b(n,s_n).$$

Then we choose s_{n+1} so that

$$s_{n+1}\geqslant r_{n+1},$$

(16) $$b(n+1,s_{n+1})\leqslant\varepsilon_{n+1},$$

(17) $$\sum_{k=r_{n+1}}^{s_{n+1}}b(n+1,k)\geqslant d_{n+1}$$

and

(18) $$\sum_{k=r_{n+1}}^{s_{n+1}}a(i,k)b(n+1,k)\geqslant d_{n+1} \quad (i=1,2,\ldots,n+1).$$

It follows from (14) that $t_n\leqslant s_n$ and hence

(19) $$r_{n+1}\geqslant 1+t_n.$$

We now choose the sequence $\{B(k)\}$ of our lemma to be the following sequence of $b(n,k)$'s:

$$b(1,1),b(1,2),\ldots,b(1,s_1),$$
$$b(2,r_2),b(2,r_2+1),\ldots,b(2,s_2),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$b(n,r_n),b(n,r_{n+1}),\ldots,b(n,s_n),$$
$$\ldots\ldots\ldots\ldots\ldots$$

Because $b(n,k)$ decreases to zero, for a fixed n, (15) and (16) show that the sequence $\{B(k)\}$ decreases to zero. The divergence of $\sum B(k)$ follows from (17). Finally, we need to prove the divergence of $\sum_k a(l+1,k)B(k)$ for $l=0,1,2,\ldots$ Since

$$\sum_k a(l+1,k)B(k) = \sum_{n=0}^{\infty}\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)$$

we use (19) and then (18) to obtain, for $n\geqslant l$,

$$\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)$$
$$\geqslant \sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,r_{n+1}+k)b(n+1,r_{n+1}+k)$$
$$= \sum_{k=r_{n+1}}^{s_{n+1}} a(l+1,k)b(n+1,k)\geqslant d_{n+1}.$$

Thus

$$\sum_{n=l}^{\infty}\sum_{k=0}^{s_{n+1}-r_{n+1}} a(l+1,t_n+k+1)b(n+1,r_{n+1}+k)\geqslant \sum_{n=l}^{\infty} d_{n+1}$$

and so $\sum_k a(l+1,k)B(k)=\infty$ for $l=0,1,2,\ldots$

We now show that Theorem 10 is best possible in the following sense.

THEOREM 12. *For each positive integer n let $\Lambda_n=\{a(n,k)\}$ be a decreasing sequence of positive numbers tending to zero such that $\sum_k a(n,k)=\infty$. Then there is a continuous function which is not in $\bigcup_1^{\infty}\Lambda_n$-BV.*

Proof. Using Lemma 4 there is a decreasing sequence $\{B(k)\}$ of positive numbers tending to zero such that $\sum B(k)=\infty$ and $\sum_k a(n,k)B(k)=\infty$ for $n=1,2,\ldots$

We define a function f on the interval $[a,b]$ as follows. Let $\{c_k\}$ be a sequence of points in $[a,b]$ such that $a=c_0<c_1<c_2<\ldots<b$ and $\lim_n c_n=b$. Setting $B(0)=0$, let

$$f(c_n)=\sum_0^n(-1)^{k+1}B(k),\qquad f(b)=\sum_0^{\infty}(-1)^{k+1}B(k)$$

and extend f linearly to the remainder of $[a,b]$. Then we see that f is a continuous function. If I_k is the interval $[c_{k-1},c_k]$, using Lemma 3 we see that the Λ_n variation of f equals $\sum_k a(n,k)|f(I_k)| = \sum_k a(n,k)B(k)=\infty$. Thus f is not in Λ_n-BV and our theorem follows.

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