Classification problems in K-categories

by

Melvyn B. Nathanson * (Carbondale, Ill.)

Abstract. We describe a large class of classification problems in linear algebra, or, more generally, in any abelian category or K-category. These problems can be partially ordered, in a natural sense, by difficulty. We prove that there are simple problems which are of maximum difficulty, that is, problems whose solution would give the solution of all classification problems in the category.

§ 1. Introduction. Let K be a ring with unit. A K-*category* is an abelian category $\mathscr{C}$ such that $\mathrm{Hom}(U, V)$ is a K-module for all objects U, V of $\mathscr{C}$, and composition $\mathrm{Hom}(V, W)\times\mathrm{Hom}(U, V)\to\mathrm{Hom}(U, W)$ is K-bilinear for all U, V, W of $\mathscr{C}$. For example, the category of finite-dimensional vector spaces over a field K is a K-category. We shall describe a large class of classification problems in any K-category.

A graph Γ will consist of a finite nonempty set of vertices $\Gamma_{\mathrm{ver}} = \{1, 2, \ldots, m\}$ and a finite set of arrows $\Gamma_{\mathrm{arr}} = \{a_1, a_2, \ldots, a_n\}$. Each arrow is of the form $a = (h, t)$, where h and t are vertices. We say that the arrow a goes from the tail t to the head h. If $t = h$, then the arrow $a = (h, h)$ will sometimes be called a loop. The set Γ_{arr} may also contain multiple arrows: $a_i = a_j$ for $i \neq j$.

Let t and h be vertices of the graph Γ. A composite arrow from t to h is a finite sequence $c = (a_{i_p}, a_{i_{p-1}}, \ldots, a_{i_2}, a_{i_1})$ of arrows of Γ, where $a_{i_k} = (h_{i_k}, t_{i_k})$, such that $t_{i_1} = t$, $h_{i_p} = h$, and $h_{i_k} = t_{i_{k+1}}$ for $k = 1, 2, \ldots, p-1$. If $h = t$, we introduce the formal symbol e_h, and consider this also as a composite arrow from h to h.

If $c_1, c_2, \ldots, c_q$ are composite arrows in Γ from t to h, then a linear combination

$$\lambda = \alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_q c_q$$

with coefficients $\alpha_i \in K$ is called a *relation on* Γ.

Let $\mathscr{C}$ be a K-category. To every graph Γ we shall construct a new category $\mathscr{C}(\Gamma)$. The objects of $\mathscr{C}(\Gamma)$ are the following "representations of Γ in $\mathscr{C}$". To each vertex $i \in \Gamma_{\mathrm{ver}}$ we assign an object U_i of $\mathscr{C}$. To each arrow $a = (h, t) \in \Gamma_{\mathrm{arr}}$ we assign a mor-

* This paper was written in the Summer of 1975 at the Institute of Mathematics of the Hebrew University of Jerusalem and the Department of Pure Mathematics of the Weizmann Institute of Science, Rehovot, Israel. The author wishes to thank both institutions for their hospitality.

phism f_a from U_t to U_h. The sets $\boldsymbol{U} = \{U_i; f_a\}_{i\in\Gamma_{\text{ver}}, a\in\Gamma_{\text{arr}}}$ will be the objects of the category $\mathscr{C}(\Gamma)$.

Let $\boldsymbol{U} = \{U_i; f_a\}$ and $\boldsymbol{V} = \{V_i; g_a\}$ be two representations of Γ in $\mathscr{C}$. A morphism from $\boldsymbol{U}$ to $\boldsymbol{V}$ is a family $\varphi = \{\varphi_i\}_{i\in\Gamma_{\text{ver}}}$, where φ_i is a morphism from U_i to V_i in $\mathscr{C}$, such that $\varphi_h \cdot f_a = g_a \cdot \varphi_t$ for every arrow $a = (h, t)$ in Γ_{arr}.

Let $c = (a_{i_p}, a_{i_{p-1}}, \ldots, a_{i_1})$ be a composite arrow in Γ from t to h. If $\boldsymbol{U} = \{U_i; f_a\}$ is a representation of Γ in $\mathscr{C}$, then $f_c = f_{a_{i_p}} \cdot f_{a_{i_{p-1}}} \cdot \ldots \cdot f_{a_{i_1}}$ is a morphism in $\mathscr{C}$ from U_t to U_h. If $h = t$ and $c = e_h$, let $f_c = f_{e_h} = 1_{U_h}$, where 1_{U_h} is the identity morphism of U_h in $\mathscr{C}$. Let $\lambda = \sum \alpha_j c_j$ be a relation on Γ, that is, a K-linear combination of composite arrows c_j from t to h. Then $f_\lambda = \sum \alpha_j f_{c_j}$ is a morphism in $\mathscr{C}$ from U_t to U_h. (Here we use the fact that $\operatorname{Hom}(U_t, U_h)$ is a K-module.) If Λ is a set of relations on Γ, we let $\mathscr{C}(\Gamma, \Lambda)$ be the full subcategory of $\mathscr{C}(\Gamma)$ consisting of all representations $\boldsymbol{U} = \{U_i; f_a\}$ such that $f_\lambda = 0$ for all $\lambda \in \Lambda$. If $\boldsymbol{U} \in \mathscr{C}(\Gamma, \Lambda)$, then $\boldsymbol{U}$ is called a *representation of Γ in $\mathscr{C}$* which satisfies the relations Λ. Clearly, $\mathscr{C}(\Gamma) = \mathscr{C}(\Gamma, \emptyset)$.

Let $\mathscr{C}$ be a K-category, and let Λ be a set of relations on the graph Γ. Then $\mathscr{C}(\Gamma, \Lambda)$ is again a K-category [11]. In particular, if $\boldsymbol{U} = \{U_i; f_a\}$ and $\boldsymbol{V} = \{V_i; g_a\}$ are representations in $\mathscr{C}(\Gamma, \Lambda)$, then their direct sum is $\boldsymbol{U} \oplus \boldsymbol{V} = \{U_i \oplus V_i; f_a \oplus g_a\}$. A nonzero representation in $\mathscr{C}(\Gamma, \Lambda)$ is called *indecomposable* if it is not isomorphic to the direct sum of two nonzero representations. In the special case when $\mathscr{C}$ is the category of finite-dimensional vector spaces over a field K, then the Krull–Schmidt theorem holds in every category $\mathscr{C}(\Gamma, \Lambda)$: Every representation in $\mathscr{C}(\Gamma, \Lambda)$ can be written uniquely (up to order) as the direct sum of indecomposable representations.

To every graph Γ and every set Λ of relations on Γ there is the following classification problem in the K-category $\mathscr{C}$: Describe the isomorphism classes of representations of Γ in $\mathscr{C}$ which satisfy the relations Λ, and, in particular, describe the indecomposable representations. These problems have been intensively studied in the case when $\mathscr{C}$ is the category of finite-dimensional vector spaces over a field K [1–10, 12].

However, there is an abundance of graphs and of relations on the graphs, and a corresponding abundance of classification problems in any K-category. It is useful to be able to compare the difficulty of two classification problems, and to reduce one problem to another problem. This can be done in the following way.

Let Γ_1 and Γ_2 be graphs, and let Λ_1 and Λ_2 be sets of relations on Γ_1 and Γ_2, respectively. Then the classification of representations of Γ_1 in $\mathscr{C}$ which satisfy the relations Λ_1 can be reduced to the classification of representations of Γ_2 in $\mathscr{C}$ which satisfy the relations Λ_2 if there exists a functor $F: \mathscr{C}(\Gamma_1, \Lambda_1) \to \mathscr{C}(\Gamma_2, \Lambda_2)$ such that for any $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}(\Gamma_1, \Lambda_1)$

(1) $\boldsymbol{U} \cong \boldsymbol{V}$ in $\mathscr{C}(\Gamma_1, \Lambda_1)$ if and only if $F(\boldsymbol{U}) \cong F(\boldsymbol{V})$ in $\mathscr{C}(\Gamma_2, \Lambda_2)$ and

(2) $\boldsymbol{U}$ is indecomposable in $\mathscr{C}(\Gamma_1, \Lambda_1)$ if and only if $F(\boldsymbol{U})$ is indecomposable in $\mathscr{C}(\Gamma_2, \Lambda_2)$.

In this case we also say that the classification problem $\mathscr{C}(\Gamma_2, \Lambda_2)$ is more difficult than the classification problem $\mathscr{C}(\Gamma_1, \Lambda_1)$, and we write $\mathscr{C}(\Gamma_1, \Lambda_1) \leqslant \mathscr{C}(\Gamma_2, \Lambda_2)$.

The object of this paper is to prove that there exist classification problems which are of *maximum* difficulty, and that such problems can be "simple". More precisely, let Γ^* be the graph consisting of one vertex $\Gamma^*_{\text{ver}} = \{1\}$ and two loops $\Gamma^*_{\text{arr}} = \{a_1, a_2\}$, where $a_1 = a_2 = (1,1)$. Let Λ^* consist of the following five relations:

$$\Lambda^* = \{a_1 a_2 - a_2 a_1, a_1^3, a_1^2 a_2, a_1 a_2^2, a_2^3\}.$$

We shall prove that $\mathscr{C}(\Gamma^*, \Lambda^*)$ is a problem of maximum difficulty, that is,

$$\mathscr{C}(\Gamma, \Lambda) \leqslant \mathscr{C}(\Gamma^*, \Lambda^*)$$

for every K-category $\mathscr{C}$ and every graph Γ with relations Λ. This means that the classification of a pair of commuting endomorphisms which satisfy nilpotency conditions of index 3 would solve all classification problems in $\mathscr{C}$. The proof of this result depends on a ring-theoretic lemma which uses an idea of Gel'fand–Ponomarev [9].

§ 2. Large sets of endomorphisms. In this section we prove that any classification problem can be reduced to the simultaneous classification of n endomorphisms for sufficiently large n. Let $\Gamma^{(m,t)}$ be the graph consisting of m vertices $\Gamma_{\text{ver}} = \{1, 2, \ldots, m\}$ and $m^2 t$ arrows $\Gamma_{\text{arr}} = \{a_{ij}^{(k)}\}$, where $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, t$, and $a_{ij}^{(k)} = (i, j)$. That is, there are exactly t arrows from j to i for every pair (i, j) of vertices. In particular, $\Gamma^{(1,n)}$ is the graph consisting of one vertex and n loops. A representation of $\Gamma^{(1,n)}$ in $\mathscr{C}$ is of the form $\boldsymbol{U} = \{U; r_1, r_2, \ldots, r_n\}$, where U is an object of $\mathscr{C}$ and $r_1, r_2, \ldots, r_n$ are endomorphisms of U. If $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}(\Gamma^{(1,n)})$, then $\boldsymbol{U} \cong \boldsymbol{V}$ if and only if there exists an isomorphism $\varphi: U \to V$ in $\mathscr{C}$ such that $s_i = \varphi r_i \varphi^{-1}$ for $i = 1, 2, \ldots, n$.

THEOREM 1. *Let Γ be a graph with m vertices and at most t arrows from j to i for every pair (i, j) of vertices, where $t \geqslant 1$. Let Λ be a set of relations on Γ. Let $n = mt + m(t-1) + 1$. Then*

$$\mathscr{C}(\Gamma, \Lambda) \leqslant \mathscr{C}(\Gamma^{(1,n)})$$

for every K-category $\mathscr{C}$.

Proof. If Γ is a graph and Λ_1 and Λ_2 are sets of relations on Γ such that $\Lambda_2 \subseteq \Lambda_1$, then $\mathscr{C}(\Gamma, \Lambda_1)$ is a full subcategory of $\mathscr{C}(\Gamma, \Lambda_2)$, and it suffices to consider only the inclusion functor $\mathscr{C}(\Gamma, \Lambda_1) \to \mathscr{C}(\Gamma, \Lambda_2)$ in order to show that $\mathscr{C}(\Gamma, \Lambda_1) \leqslant \mathscr{C}(\Gamma, \Lambda_2)$. In particular, with $\Lambda_2 = \emptyset$ and $\Lambda_1 = \Lambda$, we have

(1) $$\mathscr{C}(\Gamma, \Lambda) \leqslant \mathscr{C}(\Gamma).$$

Similarly, if Γ_1 is a subgraph of Γ_2, then there is a natural injection of $\mathscr{C}(\Gamma_1)$ into $\mathscr{C}(\Gamma_2)$, since every representation of Γ_1 in $\mathscr{C}$ can be extended to a representation of Γ_2 in $\mathscr{C}$ by assigning to each additional vertex of Γ_2 the zero object of $\mathscr{C}$ and to each additional arrow of Γ_2 the zero morphism. In particular, if the graph Γ has m vertices and at most $t \geqslant 1$ arrows between any two vertices, then Γ can be considered a subgraph of $\Gamma^{(m,t)}$. Therefore,

(2) $$\mathscr{C}(\Gamma) \leqslant \mathscr{C}(\Gamma^{(m,t)}).$$

The last step is to show that

(3) $$\mathscr{C}(\Gamma^{(m,t)}) \leqslant \mathscr{C}(\Gamma^{(1,n)})$$

where $n = mt+m(t-1)+1$.

Let $U = \{U_i; f_{ij}^{(k)}\}$ be in $\mathscr{C}(\Gamma^{(m,t)})$, where $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, t$, and $f_{ij}^{(k)}\colon U_j \to U_i$ is the morphism corresponding to the arrow $a_{ij}^{(k)} = (i, j)$ of $\Gamma^{(m,t)}$. Let $U^{\#} = \bigoplus_{p=1}^{mt} U_p^{\#}$, where $U_{i+m(k-1)}^{\#} = U_i$ for $k = 1, 2, \ldots, t$ and $i = 1, 2, \ldots, m$. Then $U_p^{\#} = U_{p+m}^{\#}$ for $p = 1, 2, \ldots, m(t-1)$. Every endomorphism of $U^{\#}$ is defined by an $(mt)\times(mt)$ matrix whose (r, s)-th component is a morphism from $U_s^{\#}$ to $U_r^{\#}$. In particular, for $p = 1, 2, \ldots, mt$, let $\delta^p = (\delta_{rs}^p)$ be the endomorphism of $U^{\#}$ defined by

$$\delta_{rs}^p = \begin{cases} 1_{U_p^{\#}} & \text{if } r = s = p, \\ 0 & \text{otherwise} \end{cases}$$

where $1_{U_p^{\#}}$ is the identity morphism of $U_p^{\#}$ in $\mathscr{C}$. For $q = 1, 2, \ldots, m(t-1)$, let $\varepsilon^q = (\varepsilon_{rs}^q)$ be the endomorphism of $U^{\#}$ defined by

$$\varepsilon_{rs}^q = \begin{cases} 1_{U_q^{\#}} & \text{if } r = q \text{ and } s = q+m, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, let $f^{\#} = (f_{rs}^{\#})$ be the endomorphism of $U^{\#}$ defined by

$$f_{rs}^{\#} = \begin{cases} f_{ij}^{(k)} & \text{if } r = i+m(k-1) \text{ and } s = j+m(k-1), \\ 0 & \text{otherwise}. \end{cases}$$

Then $U^{\#} = \{U^{\#}; \delta^p, \varepsilon^q, f^{\#}\}$ for $p = 1, 2, \ldots, mt$ and $q = 1, 2, \ldots, m(t-1)$ is a representation of $\Gamma^{(1,n)}$ in $\mathscr{C}$. We construct the functor $F\colon \mathscr{C}(\Gamma^{(m,t)}) \to \mathscr{C}(\Gamma^{(1,n)}$ in $\mathscr{C}$. We construct the functor $F\colon \mathscr{C}(\Gamma^{(m,t)}) \to \mathscr{C}(\Gamma^{(1,n)})$ by setting $F(U) = U^{\#}$.

Let $V = \{V_i; g_{ij}^{(k)}\}$ be another object in $\mathscr{C}(\Gamma^{(m,t)})$, and let $V^{\#} = F(V) = \{V^{\#}; \delta^p, \varepsilon^q, g^{\#}\} \in \mathscr{C}(\Gamma^{(1,n)})$. Let $\varphi = (\varphi_i)_{i=1,\ldots,m}$ be a morphism in $\mathscr{C}(\Gamma^{(m,t)})$ from U to V. Then $\varphi_i\colon U_i \to V_i$ is a morphism in $\mathscr{C}$ such that $\varphi_i f_{ij}^{(k)} = g_{ij}^{(k)} \varphi_j$ for $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, t$. We define $F(\varphi)\colon U^{\#} \to V^{\#}$ in $\mathscr{C}(\Gamma^{(1,n)})$ by

$$(F(\varphi))_{rs} = \begin{cases} \varphi_i & \text{if } r = s = i+m(k-1) \text{ for some} \\ & \quad i = 1, 2, \ldots, m \text{ and } k = 1, 2, \ldots, t, \\ 0 & \text{otherwise}. \end{cases}$$

It follows directly by matrix multiplication that

$$\begin{aligned} F(\varphi)\delta^p &= \delta^p F(\varphi), \\ F(\varphi)\varepsilon^q &= \varepsilon^q F(\varphi), \\ F(\varphi) f^{\#} &= g^{\#} F(\varphi) \end{aligned}$$

and so $F(\varphi)$ is a morphism from $F(U)$ to $F(V)$ in the category $\mathscr{C}(\Gamma^{(1,n)})$.

Now suppose that $U, V \in \mathscr{C}(\Gamma^{(m,t)})$, and that $\psi\colon F(U) \to F(V)$ is a morphism in $\mathscr{C}(\Gamma^{(1,n)})$. Then $\psi\colon U^{\#} \to V^{\#}$ is a morphism in $\mathscr{C}$ such that

$$\begin{aligned} \psi\delta^p &= \delta^p\psi, \\ \psi\varepsilon^q &= \varepsilon^q\psi, \\ \psi f^{\#} &= g^{\#}\psi. \end{aligned}$$

Moreover, ψ has the matrix representation $\psi = (\psi_{rs})$, where $\psi_{rs}\colon U_s^{\#} \to V_r^{\#}$. By multiplying the matrices for ψ and δ^p we see that the condition $\psi\delta^p = \delta^q\psi$ implies that $\psi_{rs} = 0$ for $r \neq s$, and so the matrix for ψ is diagonal. By multiplying the matrices for ψ and ε^q, we see that the condition $\psi\varepsilon^q = \varepsilon^q\psi$ implies that $\psi_{qq} = \psi_{q+m,q+m}$ for $q = 1, 2, \ldots, m(t-1)$. Therefore, there exist morphisms $\varphi_i\colon U_i \to V_i$ for $i = 1, 2, \ldots, m$ such that

$$\psi_{pq} = \begin{cases} \varphi_i & \text{if } p = q = i+m(k-1), \\ 0 & \text{otherwise}. \end{cases}$$

Finally, the condition $\psi f^{\#} = g^{\#}\psi$ implies that

$$\varphi_i f_{ij}^{(k)} = g_{ij}^{(k)} \varphi_j$$

for $i, j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, t$, and so $\varphi = (\varphi_i)_{i=1,\ldots,m}$ is a morphism from U to V in $\mathscr{C}(\Gamma^{(m,t)})$ such that $F(\varphi) = \psi$. Therefore, F imbeds $\mathscr{C}(\Gamma^{(m,t)})$ as a full subcategory of $\mathscr{C}(\Gamma^{(1,n)})$. In particular, $\psi\colon F(U) \to F(V)$ is an isomorphism in $\mathscr{C}(\Gamma^{(1,n)})$ if and only if $\psi = F(\varphi)$, where $\varphi\colon U \to V$ is an isomorphism in $\mathscr{C}(\Gamma^{(m,t)})$. Moreover, in any abelian category, and, in particular, in the categories $\mathscr{C}(\Gamma^{(m,t)})$ and $\mathscr{C}(\Gamma^{(1,n)})$, decomposability is equivalent to a resolution of the identity morphism into a sum of orthogonal idempotents. If follows that a representation $U \in \mathscr{C}(\Gamma^{(m,t)})$ is indecomposable if and only it $F(U) \in \mathscr{C}(\Gamma^{(1,n)})$ is indecomposable. This proves that $\mathscr{C}(\Gamma^{(m,t)}) \leqslant \mathscr{C}(\Gamma^{(1,n)})$. Theorem 1 now follows from (1), (2), and (3).

§ 3. A ring-theoretic lemma. Let $\mathscr{C}$ be an additive category, and let $U, V \in \mathscr{C}$. Then $R = \mathrm{Hom}(U, U)$ and $S = \mathrm{Hom}(V, V)$ are rings, and $H = \mathrm{Hom}(U, V)$ is an $S-R$ bimodule. Let $U^{\#} = \bigoplus_{j=1}^{2n} U$ and let $V^{\#} = \bigoplus_{i=1}^{2n} V$. Then $\mathrm{Hom}(U^{\#}, U^{\#}) = M_{2n}(R)$, $\mathrm{Hom}(V^{\#}, V^{\#}) = M_{2n}(S)$, and $\mathrm{Hom}(U^{\#}, V^{\#}) = M_{2n}(H)$, where $M_{2n}(X)$ denotes the set of $2n \times 2n$ matrices with elements in X. Moreover, $M_{2n}(H)$ is an $M_{2n}(S) - M_{2n}(R)$ bimodule. This indicates the application of the following technical lemma.

LEMMA. *Let R be a ring with 1, and let $r = (r_1, r_2, \ldots, r_n)$ be an n-tuple of elements of R. Let $\alpha(r) = (\alpha_{ij}(r))$ and $\beta(r) = (\beta_{ij}(r))$ be the following matrices in $M_{2n}(R)$:*

$$\alpha_{ij}(r) = \begin{cases} r_{j-2} & \text{if } (i,j) = (1,j) \text{ for } j = 3, 4, \ldots, n+1, \\ 1 & \text{if } (i,j) = (1,2) \text{ or if } (i,j) = (i, n+i) \\ & \quad \text{for } i = 2, 3, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_{ij}(r) = \begin{cases} r_{j-1} & \text{if} \quad (i,j) = (1,j) \text{ for } j = 2, 3, \ldots, n+1, \\ 1 & \text{if} \quad (i,j) = (i, n+i-1) \text{ for } i = 3, 4, \ldots, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

I. $\alpha(r)\beta(r) = \beta(r)\alpha(r)$,

II. $\alpha(r)^3 = \alpha(r)^2\beta(r) = \alpha(r)\beta(r)^2 = \beta(r)^3 = 0$.

Let S be a ring with 1, let $s = (s_1, s_2, \ldots, s_n)$ be an n-tuple of elements of S, and let $\alpha(s)$ and $\beta(s)$ be the corresponding matrices in $M_{2n}(S)$. Let H be an $S-R$ bimodule. If the matrix $\Phi = (\varphi_{ij})$ in $M_{2n}(H)$ satisfies

III. $\Phi\alpha(r) = \alpha(s)\Phi$,

IV. $\Phi\beta(r) = \beta(s)\Phi$

then

V. *There exists $\varphi \in H$ such that $\varphi_{ii} = \varphi$ for $i = 1, 2, \ldots, 2n$,*

VI. *$\varphi r_j = s_j\varphi$ for $j = 1, 2, \ldots, n$,*

VII. *$\varphi_{ij} = 0$ for $j < i$.*

Proof. By multiplying the matrices $\alpha(r)$ and $\beta(r)$, we see immediately that $\alpha(r)\beta(r) = \beta(r)\alpha(r)$ and $\alpha(r)^3 = \alpha(r)^2\beta(r) = \alpha(r)\beta(r)^2 = \beta(r)^3 = 0$. This proves I and II.

Let $\Phi = (\varphi_{ij}) \in M_{2n}(H)$ satisfy conditions III and IV. The (i,j)-th components of $\Phi\alpha(r)$ and $\alpha(s)\Phi$ can be computed explicitly:

$$\big(\Phi\alpha(r)\big)_{ij} = \begin{cases} 0 & \text{if} \quad j = 1, \\ \varphi_{i1} & \text{if} \quad j = 2, \\ \varphi_{i1} r_{j-2} & \text{if} \quad j = 3, 4, \ldots, n+1, \\ \varphi_{i,j-n} & \text{if} \quad j = n+2, \ldots, 2n, \end{cases}$$

$$\big(\alpha(s)\Phi\big)_{ij} = \begin{cases} \varphi_{2j} + \sum_{k=1}^{n-1} s_k \varphi_{k+2,j} & \text{if} \quad i = 1, \\ \varphi_{n+i,j} & \text{if} \quad i = 2, 3, \ldots, n, \\ 0 & \text{if} \quad i = n+1, n+2, \ldots, 2n. \end{cases}$$

Similarly, the (i,j)-th components of $\Phi\beta(r)$ and $\beta(s)\Phi$ can be explicitly computed.

$$\big(\Phi\beta(r)\big)_{ij} = \begin{cases} 0 & \text{if} \quad j = 1, \\ \varphi_{i1} r_{j-1} & \text{if} \quad j = 2, 3, \ldots, n+1, \\ \varphi_{i,j-n+1} & \text{if} \quad j = n+2, n+3, \ldots, 2n, \end{cases}$$

$$\big(\beta(s)\Phi\big)_{ij} = \begin{cases} \sum_{k=1}^{n} s_k \varphi_{k+1,j} & \text{if} \quad i = 1, \\ \varphi_{n+i-1,j} & \text{if} \quad i = 3, 4, \ldots, n+1, \\ 0 & \text{if} \quad i = 2 \text{ or } i = n+2, n+3, \ldots, 2n. \end{cases}$$

The proofs of statements V, VI, VII proceed by direct computation in nine steps.

(1) For $i = n+1, n+2, \ldots, 2n$ and $j = 2, 3, \ldots, n$, we have

$$\varphi_{ij} = \big(\Phi\alpha(r)\big)_{i,j+n} = \big(\alpha(s)\Phi\big)_{i,j+n} = 0.$$

(2) For $i = n+1, n+2, \ldots, 2n$ we have

$$\varphi_{i1} = \big(\Phi\alpha(r)\big)_{i2} = \big(\alpha(s)\Phi\big)_{i2} = 0.$$

(3) It follows from (1) that for $i = 2, 3, \ldots, n$

$$\varphi_{i1} = \big(\Phi\alpha(r)\big)_{i2} = \big(\alpha(s)\Phi\big)_{i2} = \varphi_{n+i,2} = 0.$$

(4) For $i = 2, 3, \ldots, n$ and $j = 2, 3, \ldots, n$ we have the recurrence relations

$$\begin{aligned} \varphi_{ij} &= \big(\Phi\alpha(r)\big)_{i,n+j} = \big(\alpha(s)\Phi\big)_{i,n+j} \\ &= \varphi_{n+i,n+j} = \big(\beta(s)\Phi\big)_{i+1,n+j} \\ &= \big(\Phi\beta(r)\big)_{i+1,n+j} = \varphi_{i+1,j+1}. \end{aligned}$$

(5) In particular, it follows from (4) that

$$\varphi = \varphi_{2,2} = \varphi_{3,3} = \ldots = \varphi_{n,n} = \varphi_{n+1,n+1} = \ldots = \varphi_{2n,2n}$$

for some $\varphi \in H$. Also, since $\varphi_{n+i,j} = 0$ for $j = 1, 2, \ldots, n$, we have

$$\varphi_{ij} = \varphi_{n+i,n+j} = 0$$

for $i = 3, 4, \ldots, n$ and $j = 2, 3, \ldots, i-1$.

(6) It follows from (3) that

$$\begin{aligned} \varphi_{1,1} &= \big(\Phi\alpha(r)\big)_{1,2} = \big(\alpha(s)\Phi\big)_{1,2} \\ &= \varphi_{2,2} + \sum_{k=1}^{n-1} s_k \varphi_{k+2,2} \\ &= \varphi_{2,2} = \varphi. \end{aligned}$$

Therefore, $\varphi_{ii} = \varphi$ for all i. This proves V.

(7) For $i = 2, 3, \ldots, n$ we have

$$\varphi_{n+i,n+1} = \big(\alpha(s)\Phi\big)_{i,n+1} = \big(\Phi\alpha(r)\big)_{i,n+1} = \varphi_{i1} r_{n-1} = 0.$$

It follows from (1), (2), (3), (5), (7) that $\varphi_{ij} = 0$ for $j < i$. This proves VII.

(8) For $j = 3, 4, n+1$ we have

$$\varphi_{2j} = \big(\Phi\beta(r)\big)_{2,j+n-1} = \big(\beta(s)\Phi\big)_{2,j+n-1} = 0.$$

It follows from the recurrence relation (4) that $\varphi_{ij} = 0$ for $i = 2, 3, \ldots, n$ and $j = i+1, i+2, \ldots, n+1$.

(9) Finally, we have for $j = 1, 2, \ldots, n$

$$\varphi r_j = \varphi_{1,1} r_j = (\Phi\beta(r))_{i,j+1}$$
$$= (\beta(s)\Phi)_{1,j+1} = \sum_{k=1}^{n} s_k \varphi_{k+1,j+1}$$
$$= s_j \varphi_{j+1,j+1} = s_j \varphi .$$

This proves VI.

Remark. It would be of interest to construct matrices $\alpha(r)$ and $\beta(r)$ in $M_p(R)$ for some p, not necessarily $p = 2n$, which would satisfy I, V, VI, VII, and, instead of II, the stronger condition

II′. $\alpha(r)^3 = \alpha(r)^2\beta(r) = \beta(r)^2 = 0$.

§ 4. Problems of maximum difficulty. In this section we reduce the classification of n endomorphisms to the classification of a pair of commuting nilpotent endomorphisms.

THEOREM 2. *Let Γ^* be the graph consisting of one vertex $\Gamma^*_{\text{ver}} = \{1\}$ and two loops $\Gamma^*_{\text{arr}} = \{a_1, a_2\}$, where $a_1 = a_2 = (1, 1)$. Let Λ^* consist of the following five relations on Γ^*:*

$$\Lambda^* = \{a_1a_2 - a_2a_1, a_1^3, a_1^2, a_2, a_1, a_2^2, a_2^3\}.$$

Then

$$\mathscr{C}(\Gamma^{(1,n)}) \leqslant \mathscr{C}(\Gamma^*, \Lambda^*)$$

for every K-category $\mathscr{C}$ and for all n.

Proof. Let $U = (U; r_1, r_2, \ldots, r_n) \in \mathscr{C}(\Gamma^{(1,n)})$, where $U \in \mathscr{C}$ and $r_1, r_2, \ldots, r_n \in \operatorname{Hom}(U, U) = R$. Let $r = (r_1, r_2, \ldots, r_n)$, and let $\alpha(r), \beta(r) \in M_{2n}(R)$ be the matrices constructed in the lemma. Let $U^{\#} = \bigoplus_{j=1}^{2n} U$. By the lemma, the matrices $\alpha(r)$ and $\beta(r)$ are endomorphisms of $U^{\#}$ which satisfy the relations Λ^*. Therefore,

$$F(U) = (U^{\#}; \alpha(r), \beta(r)) \in \mathscr{C}(\Gamma^*, \Lambda^*).$$

Let $V = (V; s_1, s_2, \ldots, s_n) \in \mathscr{C}(\Gamma^{(1,n)})$, where $V \in \mathscr{C}$ and $s_1, s_2, \ldots, s_n \in \operatorname{Hom}(V, V) = S$. Setting $s = (s_1, s_2, \ldots, s_n)$ and $V^{\#} = \bigoplus_{i=1}^{2n} V$, we have

$$F(V) = (V^{\#}; \alpha(s), \beta(s)) \in \mathscr{C}(\Gamma^*, \Lambda^*).$$

Let $\varphi: U \to V$ be a morphism in $\mathscr{C}(\Gamma^{(1,n)})$. Then $\varphi \in \operatorname{Hom}(U, V) = H$, and $\varphi r_i = s_i\varphi$ for $i = 1, 2, \ldots, n$. Let $F(\varphi): U^{\#} \to V^{\#}$ be the matrix in $M_{2n}(H)$ defined by

$$(F(\varphi))_{ij} = \begin{cases} \varphi & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then

$$F(\varphi)\alpha(r) = \alpha(s)F(\varphi),$$
$$F(\varphi)\beta(r) = \beta(s)F(\varphi).$$

Therefore, $F(\varphi)$: $F(U) \to F(V)$ is a morphism in $\mathscr{C}(\Gamma^*, \Lambda^*)$. We have thus constructed a functor F: $\mathscr{C}(\Gamma^{(1,n)}) \to \mathscr{C}(\Gamma^*, \Lambda^*)$.

Let Φ: $F(U) \to F(V)$ be any morphism in $\mathscr{C}(\Gamma^*, \Lambda^*)$. Then $\Phi = (\varphi_{ij}) \in M_{2n}(H)$ satisfies III and IV of the lemma. Therefore, $\varphi_{ij} = 0$ for $j < i$, and $\varphi_{11} = \varphi_{22} = \ldots = \varphi_{2n,2n} = \varphi \in H$, where φ: $U \to V$ is a morphism in $\mathscr{C}(\Gamma^{(1,n)})$. If Φ: $F(U) \to F(V)$ is an isomorphism, then there exists Ψ: $F(V) \to F(U)$ such that $\Psi\Phi = 1_{F(U)}$ and $\Phi\Psi = 1_{F(V)}$. Again by the lemma we see that $\Psi = (\psi_{ij})$, where $\psi_{ij} = 0$ for $j < i$, and $\psi_{11} = \psi_{22} = \ldots = \psi_{2n,2n} = \psi$ is a morphism from V to U in $\mathscr{C}(\Gamma^{(1,n)})$. Multiplying the matrices for Φ and Ψ, we obtain

$$(\Phi\Psi)_{ii} = \varphi\psi$$

for $i = 1, 2, \ldots, 2n$. But $\Phi = 1_{F(V)}$ implies that $(\Phi\Psi)_{ii} = 1_V$. Therefore, $\varphi\psi = 1_V$. Similarly, $\psi\varphi = 1_U$, and so $\varphi = \psi^{-1}$ is an isomorphism in $\mathscr{C}(\Gamma^{(1,n)})$. Therefore, if $F(U) \cong F(V)$ in $\mathscr{C}(\Gamma^*, \Lambda^*)$, then $U \cong V$ in $\mathscr{C}(\Gamma^{(1,n)})$.

Similarly, if $U \in \mathscr{C}(\Gamma^{(1,n)})$ and if $F(U)$ is decomposable in $\mathscr{C}(\Gamma^*, \Lambda^*)$, then the identity morphism $1_{F(V)}$ can be decomposed into a finite sum of nonzero orthogonal idempotents in $\mathscr{C}(\Gamma^*, \Lambda^*)$. Applying the lemma to this decomposition, we see that the identity 1_U is also a finite sum of nonzero orthogonal idempotents in $\mathscr{C}(\Gamma^{(1,n)})$. and so U is decomposable in $\mathscr{C}(\Gamma^{(1,n)})$. On the other hand, the functor F is additive, and if U is decomposable in $\mathscr{C}(\Gamma^{(1,n)})$, then $F(U)$ is decomposable in $\mathscr{C}(\Gamma^*, \Lambda^*)$. This proves the theorem.

Combining Theorems 1 and 2 we obtain

THEOREM 3. *Let Γ be a graph and let Λ be a set of relations on Γ. Then*

$$\mathscr{C}(\Gamma, \Lambda) \leqslant \mathscr{C}(\Gamma^*, \Lambda^*)$$

for every K-category $\mathscr{C}$.

If $\mathscr{C}$ is the category of finite-dimensional vector spaces over a field K, then Theorem 3 can be restated in the following form. Let $K[x, y]$ be the algebra of polynomials in two commuting variables x and y, and let $I = \langle x^3, x^2y, xy^2, y^3 \rangle$ be the ideal of $K[x, y]$ generated by the monomials of degree 3. Then $A = K[x, y]/I$ is a six-dimensional K-algebra. An object in the category $\mathscr{C}(\Gamma^*, \Lambda^*)$ is simply a finite-dimensional A-module, and the classification of such A-modules is a problem of maximum difficulty in linear algebra.

The classification of a pair of commuting endomorphisms which satisfy nilpotency conditions of index 3 is not the unique classification problem of maximum difficulty. Any problem which includes the classification of a pair of endomorphisms is also of maximum difficulty. Here is an example.

THEOREM 4. *Let Γ^{**} be the graph consisting of one vertex $\Gamma^{*}_{\mathrm{ver}} = \{1\}$ and three loops $\Gamma^{**}_{\mathrm{arr}} = \{a_1, a_2, a_3\}$, where $a_1 = a_2 = a_3 = (1, 1)$. Let Λ^{**} consist of the following nine relations on Γ^{**}:*

$$\Lambda^* = \{a_1^2, a_2^2, a_3^2, a_1a_2, a_2a_1, a_2a_3, a_3a_2, a_1a_3, a_3a_1\}.$$

Let Γ be a graph and let Λ be a set of relations on Γ. Then

$$\mathscr{C}(\Gamma, \Lambda) \leqslant \mathscr{C}(\Gamma^{**}, \Lambda^{**})$$

for every K-category $\mathscr{C}$.

Proof. By Theorem 3, it suffices to prove that $\mathscr{C}(\Gamma^*) \leqslant \mathscr{C}(\Gamma^{**}, \Lambda^{**})$. Let $U = \{U; r_1, r_2\} \in \mathscr{C}(\Gamma^*)$, where $U \in \mathscr{C}$ and $r_1, r_2 \in \mathrm{Hom}(U, U) = R$. Let $U^{\#} = U \oplus U$. Let $\mathrm{Hom}(U^{\#}, U^{\#}) = M_2(R)$. If $r \in R$, define $\gamma(r) \in M_2(R)$ by

$$\gamma(r) = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}.$$

Then the morphisms $\gamma(1_U), \gamma(r_1), \gamma(r_2) \in M_2(R) = \mathrm{Hom}(U^{\#}, U^{\#})$ satisfy the relations Λ^{**}, and so

$$F(U) = \{U^{\#}; \gamma(1_U), \gamma(r_1), \gamma(r_2)\}$$

belongs to $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$.

Let $V = \{V; s_1, s_2\} \in \mathscr{C}(\Gamma^*)$, and let

$$F(V) = \{V^{\#}; \gamma(1_V), \gamma(s_1), \gamma(s_2)\} \in \mathscr{C}(\Gamma^{**}, \Lambda^{**})$$

If φ: $U \to V$ is a morphism in $\mathscr{C}(\Gamma^*)$, then

$$F(\varphi) = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}$$

is a morphism $F(\varphi)$: $F(U) \to F(V)$ in $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$. Thus, F: $\mathscr{C}(\Gamma^*) \to \mathscr{C}(\Gamma^{**}, \Lambda^{**})$ is a functor.

Let $U, V \in \mathscr{C}(\Gamma^*)$, and let Φ: $F(U) \to (V)$ be a morphism in $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$. Then $\Phi = (\varphi_{ij}) \in M_2(H)$, where $H = \mathrm{Hom}(U, V)$. Then

$$\Phi\gamma(1_U) = \gamma(1_V)\Phi$$

implies that $\varphi_{21} = 0$ and that $\varphi_{11} = \varphi_{22} = \varphi \in H$. Since $\varphi\gamma(r_i) = \gamma(s_i)\Phi$ for $i = 1, 2$, we see that $\varphi r_i = s_i \varphi$ for $i = 1, 2$ and so φ: $U \to V$ is a morphism in $\mathscr{C}(\Gamma^*)$. It follows that if Φ: $F(U) \to F(V)$ is an isomorphism in $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$, then φ: $U \to V$ is an isomorphism in $\mathscr{C}(\Gamma^*)$, and so $U \cong V$ in $\mathscr{C}(\Gamma^*)$ if and only if $F(U) \cong F(V)$ in $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$. Similarly, U is decomposable in $\mathscr{C}(\Gamma^*)$ if and only if $F(U)$ is decomposable in $\mathscr{C}(\Gamma^{**}, \Lambda^{**})$. This proves the theorem.

Let $B = K[x, y, z]/J$, where J is the ideal of $K[x, y, z]$ generated by the monomials of degree 2. Then B is a K-algebra of dimension 4, and the classification of finite-dimensional B-modules is another problem of maximum difficulty in linear algebra.

References

[1] I. N. Bernstein, I. M. Gel'fand, and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspekhi Mat. Nauk 28 (2) (1973), pp. 19–33; Russian Math. Surveys 28 (2) (1973), pp. 17–32.

[2] S. Brenner, *Quivers with commutativity conditions and some phenomenology of forms*, Lect. Notes in Math. 488 (1975), pp. 29–53.

[3] V. Dlab and C. M. Ringel, *On algebras of finite representation type*, J. Algebra 33 (1975), pp. 306P394.

[4] P. Gabriel, *Unzerlegbare Darstellungen*, I, Manuscripta Math. 6 (1972), pp. 71–103.

[5] — *Indecomposable representations* II, Istit. Naz. Atta Mat., Symp. Math. 11 (1973), pp. 81–104.

[6] — *Problèmes actuels de théorie des représentations*, Enseignement Math. 20 (1974), pp. 323–332.

[7] — *Représentations Indécomposables*, Séminaire Bourbaki 1973/74, no. 444, pp. 1–27.

[8] I. M. Gel'fand, *The cohomology of infinite-dimensional Lie algebras*, Actes Congrès Internat. Math. Nice 1970, vol. 1 (1970), pp. 95–111.

[9] — and V. A. Ponomarev, *Remarks on the classification of a pair of commuting linear transformations in a finite-dimensional space*, Funkcional. Anal. i Prilozen. 3 (4) (1969), pp. 81–82; Functional Anal. Appl. 3 (1969), pp. 325–326.

[10] — *Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space*, Colloq. Math. Soc. Janos Bolyai 5, Hilbert space operators, Tihany, Hungary, 1970, pp. 163–237.

[11] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohuku Math. J. 9 (1957), pp. 119–221.

[12] L. A. Nazarova, *Representations of quivers of infinite type*, Izv. Akad. Nauk SSR Ser. Mat. 37 (1973), pp. 752–791; Math. USSR Izv. 7 (1973), pp. 749–792.

SOUTHERN ILLINOIS UNIVERSITY
DEPARTMENT OF MATHEMATICS
Carbondale, Illinois 62901

Accepté par la Rédaction le 28. 2. 1977