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Weakly Borel-complete topological spaces

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Abstract. A Tychonoff space is *weakly Borel-complete* if each ultrafilter of Borel sets with the countable intersection property converges to some point in the space. This concept has been introduced by Z. Frolík in [4] under the name Baire–Borel–complete, with a different definition. The present paper studies such spaces, contrasting their properties with the Borel-complete and closed complete spaces discussed in [9] and the familiar realcompact spaces, and adds some new results on Borel-complete spaces. The primary difference in approach between [9] and the present work is the measure-theoretic language adopted here. For example, weak Borel-completeness is equivalent to each non-trivial 0-1 valued countably additive Forel measure having a non-empty support set (necessarily consisting of one point). Finally, we note that the present work has considerable overlap with the recent work of R. J. Gardner [6]; the details of this overlap are found at the end of section two.

Section 1. A space is Borel-complete (resp. closed complete) if each ultrafilter of Borel sets (resp. closed sets) with the countable intersection property is fixed at some point of the space; alternately Borel-completeness is equivalent to each σ-additive 0-1 Borel measure being a point mass measure. Therefore each Borel-complete space is weakly Borel-complete. For other background information the reader is referred to [9]. In particular, the Baire (resp. Borel) sets are the smallest σ-field which contains the zero sets of continuous real-valued mappings (resp. the closed sets).

THEOREM 1.1. *The following statements are equivalent.*

(i) *X is closed complete.*

(ii) *Each non-trivial regular σ-additive* 0-1 *Borel measure is a point mass measure.*

(iii) *For each closed ultrafilter $\mathscr{F}$ on X with $\bigcap\mathscr{F} = \emptyset$ there exists a σ-disjoint open refinement of $\{X-F\colon F\in\mathscr{F}\}$ and X has no closed discrete subspace of measurable power.*

To prove the above theorem, we will need the following lemma that was discovered during the writing of [9] (see 6.9–6.12 of [8]).

LEMMA. *Let $\mathscr{C}\subset\mathscr{P}(X)$ and let $\mathscr{F}$ be a $\mathscr{C}$-ultrafilter closed under countable intersections. Define*

$$\Sigma(\mathscr{F}) = \{S\subset X\colon S \text{ misses or contains some member of } \mathscr{F}\}.$$

Then the following statements hold:

(i) $\Sigma(\mathscr{F})$ *is a* σ*-field containing* $\mathscr{C}$.

(ii) $\mathscr{F}^* = \{A \subset X;\ A$ *contains some member of* $\mathscr{F}\}$ *is a* $\Sigma(\mathscr{F})$*-ultrafilter closed under countable intersections and* $\mathscr{F}^* \cap \mathscr{C} = \mathscr{F}$.

(iii) *If* $\mathscr{C} \subset \mathscr{B} \subset \Sigma(\mathscr{F})$ *and* $\mathscr{B}$ *is closed under finite intersections, then* $\mathscr{F}^* \cap \mathscr{B}$ *is a* $\mathscr{B}$*-ultrafilter with the countable intersection property.*

To prove Theorem 1.1, we first note that if μ is a regular σ-additive 0-1 Borel measure, then $\mathscr{F} = \{F\colon F$ is closed and $\mu(F) = 1\}$ is a closed ultrafilter with the countable intersection property; hence (i)$\to$(ii) in 1.1. To show (ii)$\to$(i), let $\mathscr{F}$ be a closed ultrafilter with the countable intersection property. Using the preceding lemma with $\mathscr{C}$ = closed sets and $\mathscr{B} = \mathrm{Borel}(X)$, we see that $\mathscr{F}^* \cap \mathscr{B}$ is a Borel ultrafilter with the countable intersection property and one easily checks that the 0-1 Borel measure μ associated with $\mathscr{F}^* \cap \mathscr{B}$ is regular, so from (ii) μ is concentrated at some point p and $\mathscr{F}$ is therefore fixed at p. The implication (i)$\to$(iii) is immediate, so we need only show (iii)$\to$(ii) to complete the proof. The argument used is a standard one. Let μ be a non-trivial 0-1 regular σ-additive Borel measure and set $\mathscr{F} = \{F\colon F$ is closed and $\mu(F) = 1\}$. $\mathscr{F}$ is free if μ is not concentrated at any point and under this assumption there exists a σ-disjoint open cover $\bigcup \mathscr{U}_i$ which refines $\{X-F\colon F \in \mathscr{F}\}$. For each i, define $B_i = \bigcup\{U \in \mathscr{U}_i\}$; then $\mu(\bigcup B_i) = 1$, so $\mu(B_j) = 1$ for some j. Define $\lambda\colon \mathscr{P}(\mathscr{U}_j) \to \{0, 1\}$ by $\lambda(\mathscr{S}) = \mu(\bigcup\{S \in \mathscr{S}\})$; then λ is an Ulam measure on $\mathscr{U}_j$, so choosing one point from each member of $\mathscr{U}_j$ gives a closed discrete subset of measurable power, which is a contradiction. Hence μ is concentrated at some point and condition (ii) is established.

THEOREM 1.2. *The following statements are equivalent.*

(i) X *is weakly Borel-complete.*

(ii) X *is closed complete and each* 0-1 σ*-additive Borel measure is* $\uparrow$*-additive.*

(iii) *Each* 0-1 σ*-additive Borel measure is* $\uparrow$*-additive.*

(iv) *Each* 0-1 σ*-additive Borel measure is additive.*

A Borel measure μ is $\uparrow$-additive if the empty intersection of a family of zero sets $\{Z_s\}$ indexed over a directed set implies that $\mu(Z_s) \to 0$; the measure is additive if the above condition is satisfied for arbitrary closed sets Z_s.

The proof that (i)$\to$(iv) is left to the reader. To establish that (iii)$\to$(ii), let $\mathscr{F}$ be a closed ultrafilter with the countable intersection property with $\mathscr{Z} = \{F_s\colon s \in D\}$ the family of zero sets which belong to $\mathscr{F}$ and make D into a directed set by defining $s < t$ when $F_t \subset F_s$. If $\bigcap F_s = \emptyset$, (iii) implies that for some t, $\mu(F_s) = 0$ for $s > t$, where μ is the Borel measure associated with $\mathscr{F}$ used in the proof of (ii)$\to$(i) of 1.1. But from the definition of μ it follows that $F_t \notin \mathscr{F}$, which is a contradiction. Hence there exists a point p in $\bigcap F_s$. One easily shows that p belongs to $\bigcap \mathscr{F}$, so $\mathscr{F}$ is fixed and X is closed complete.

To show (ii)$\to$(i), let μ be a 0-1 σ-additive Borel measure. By assumption μ is $\uparrow$-additive, so by ([12], comments following 2.5), $\mu|_{\mathrm{Baire}(X)}$ has an extension to a regular 0-1 σ-additive Borel measure ν. Since X is closed complete, it follows from 1.1 that ν is concentrated at some point p. If $p \in U$, U open, choose a cozero set C such that $p \in C \subset U$; then $\nu(C) = 1 = \mu(C)$, so $\mu(U) = 1$, which shows that μ is supported at p and therefore that X is weakly Borel-complete.

We comment that portions of the preceding results are analogues of results which are known for realcompact spaces. For example, the analogue of 1,1(iii) is: X is realcompact if and only if for each free ultrafilter of zero sets $\mathscr{F}$, there exists a σ-discrete cozero refinement of $\{X-F\colon F \in \mathscr{F}\}$, and the analogue of 1.2(iii) is ([12], 3.2): X is realcompact if and only if each 0-1 additive Baire measure is $\uparrow$-additive.

THEOREM 1.3. X *is Borel-complete if and only if* X *and* $X-(p)$ *(for each* p *in* X*) are weakly Borel-complete.*

Since Borel-completeness is a hereditary property ([9]), the condition is clearly necessary. To show the sufficiency of the condition, let μ be a 0-1 σ-additive Borel measure. Since X is weakly Borel-complete, μ has a support point p. Assume that $\mu(p) = 0$. Since $S = X-(p)$ is weakly Borel-complete, $\mu_0 = \mu|_S$ has a support point x in S. Choose disjoint open sets U and V containing p and x respectively; then

$$\mu_0(U-(p)) = \mu(U) = 1 \quad \text{and} \quad \mu_0(V-(p)) = \mu(V) = 1\,,$$

which is a contradiction. Hence μ is concentrated at the point p, which completes the proof.

Section 2. Using any one of the characterizations of the weakly Borel-complete property, one may show that the class of weakly Borel-complete spaces is closed under the formation of products, non-measurable sums, perfect pre-images, and subspaces which are members of $\varrho(\mathscr{C})$ (the smallest family containing the closed sets $\mathscr{C}$ closed under countable unions and countable intersections). Thus in particular the weakly Borel-complete spaces form an epi-reflective subcategory of Tychonoff spaces and the corresponding reflection $\nu_0 X$ of a space X may be compared with the realcompactification νX. We note that $\nu_0 X$ may be viewed as a subspace of νX which contains X (since each realcompact space is weakly Borel-complete by 2.1 below).

It would be interesting to have a description of the points in $\nu X - X$ which correspond to points of $\nu_0 X$, analogous to the well known description of the points of νX, where νX is viewed as a subspace of the Stone–Čech compactification βX: $\nu X = \{p \in \beta X\colon$ each member of $\varrho(\mathscr{U}_p)$ intersects $X\}$, where $\mathscr{U}_p$ is the family of open sets in βX containing p. One possible description of the points in $\nu_0 X$ is the following. Let $\mathscr{U}_p^0$ be the family of open sets in βX which meet each member of $\varrho(\mathscr{U}_p)$. Then one may show that X weakly Borel-complete implies the existence of a member of $\varrho(\mathscr{U}_p^0)$ which missis X for each $\varrho \in \beta X - X$. We have been unable to establish the converse of this statement.

THEOREM 2.1. *Each almost realcompact (and hence each realcompact space)* *is weakly Borel-complete. If X has no closed discrete subspace of measurable power, then each of the following types of spaces are weakly Borel-complete*: (i) *metacompact*, (ii) *subparacompact*, *and* (iii) *screenable*.

The proof of parts (ii) and (iii) are routine based on the proof of the implication (iii)→(i) of 1.1. Now assume that X is almost realcompact (see [3]) and let μ be a 0-1 σ-additive Borel measure. Let $\mathscr{F} = \{U: U \text{ is open and } \mu(U) = 1\}$ generate an open ultrafilter $\mathscr{F} \subset \overline{\mathscr{F}}$. If there exists $\{F_i\} \subset \overline{\mathscr{F}}$ with $\bigcap F_i = \emptyset$, then $\mu(X - \overline{F}_j) = 1$ for some j, so $X - \overline{F}_j \in \mathscr{F}$. But $F_j \cap (X - \overline{F}_j) = \emptyset$, which is a contradiction. Hence $\overline{\mathscr{F}}$ has what is called in [3] the C.C.I.P. property. Since X is almost realcompact, there exists $p \in \bigcap\{\overline{F}: F \in \mathscr{F}\}$ and one shows without difficulty that μ is supported at p.

Now suppose that X is metacompact and let $\mathscr{F}$ be a Borel ultrafilter. Assuming that $\mathscr{F}$ does not converge, we will show that $\mathscr{F}$ does not have the countable intersection property (¹). Let $\mathscr{F}_0 = \{F \in \mathscr{F}: F \text{ is closed}\}$. Since $\mathscr{F}$ does not converge, $\mathscr{F}_0$ is free; let $\mathscr{U}$ be a point finite open cover that refines $\{X - F: F \in \mathscr{F}_0\}$. Using Zorn's lemma one may find a subspace H maximal with respect to the property that $|H \cap \mathscr{U}| \leqslant 1$ for each $U \in \mathscr{U}$. It follows that H is a closed discrete subspace and $\{\mathrm{St}(x, \mathscr{U}): x \in H\}$ covers X. Then $\mathscr{U}_0 = \{U \in \mathscr{U}: U \cap H \neq \emptyset\}$ is a point finite cover of X, which guarantees that $|\mathscr{U}_0| = |H|$.

For each $F \in \mathscr{F}_0$, define $F^{\#} = \{U \in \mathscr{U}_0: F \cap U \neq \emptyset\}$. Since $\mathscr{U}_0$ is a cover, each $F^{\#} \neq \emptyset$, so $\bigcap_{i=1}^{n} (F_i^{\#}) \supset (\bigcap_{i=1}^{n} F_i)^{\#}$, $\{F_i\} \subset \mathscr{F}_0$, shows that the family $\{F^{\#}: F \in \mathscr{F}_0\}$ generates an ultrafilter $\mathscr{F}^{\#}$ on the set $\mathscr{U}_0$. $\mathscr{F}^{\#}$ is free since $\mathscr{U}_0$ refines $\{X - F: F \in \mathscr{F}_0\}$. The measurability restriction on $|H|$ shows that $\mathscr{F}^{\#}$ does not have the countable intersection property, so there exists a decreasing sequence $\{\mathscr{E}_i\}$ of members of $\mathscr{F}^{\#}$ such that $\bigcap \mathscr{E}_i = \emptyset$. Now $G_i = X - \bigcup\{U \in \mathscr{E}_i\} \notin \mathscr{F}_0$, for each i (otherwise one obtains $G_i^{\#} \cap \mathscr{E}_i = \emptyset$, which is a contradiction), so there exists $F_i \in \mathscr{F}$ such that $G_i \cap F_i = \emptyset$. We will show that $\bigcap F_i = \emptyset$ by showing that $\bigcap(X - G_i) = \emptyset$, which will complete the proof of the result. If $x \notin G_i$, for each i, then $x \in U_i$ for some $U_i \in \mathscr{E}_i$; thus $\mathscr{U}$ point finite implies that $U_i = U_j$ for $i \geqslant j$ and hence that $U_i \in \bigcap \mathscr{E}_i$, which is a contradiction.

COROLLARY 2.2. *Each metacompact or subparacompact space with points G_δ sets is Borel-complete provided it has no closed discrete subspace of measurable power.*

COROLLARY 2.3. *Each metacompact normal space is realcompact provided it has no closed discrete subspace of measurable power.*

Corollary 2.2 follows from 1.3 and the fact that F_δ sets inherit the weak Borel-complete property. Corollary 2.3 follows from 2.1(i) since metacompact normal spaces are countably paracompact and [9] closed complete normal spaces with this property are realcompact. We note that 2.3 was first proved in [14] and reproved in ([17], Corollary 2).

For the convenience of the reader we summarize the overlap between [6] and the

(¹) In fairness, the authors note that the idea for the following proof is based on the imaginative proof of ([17], implication (ii) → (i) of the theorem).

present work. Theorem 1.1(i) and (ii) is essentially Theorem 3.5 in [6]. Although [6] does not define weakly Borel-complete spaces, property B in [6] is the real-valued (as opposed to 0-1 valued) analogue of this concept, while the τ-additive concept used here is the weakly τ-additive concept in [6] (for regular spaces). Our Theorem 2.1(i)–(iii) may be deduced from Theorem 3.9 and the remark after 4.2 in [6]; in fact the proof shows that weakly θ-refinable (resp. θ-refinable) and no discrete (resp. closed discrete) subspace of measurable power implies closed complete (resp. weakly Borel-complete). (The method used in the proof of our 2.1 may also be modified to obtain these results). As a final note, we comment that the weakly Borel-complete concept and results 1.3, 2.1, and 2.2 were announced by the authors in Notices Amer. Math. Soc. August, 1972.

Section 3. In the following section we collect some results involving our filter properties which may be obtained for locally compact spaces and for spaces with special countably generated σ-fields.

THEOREM 3.1. *Each locally compact closed complete space is weakly Borel-complete; hence each locally compact space X with the property that $X - \{p\}$ is closed complete for all $p \in X$, is Borel complete.*

Let μ be a 0-1 σ-additive Borel measure. For each open set G, define $\mu_*(G) = \sup\{\mu(K): K \subset G, K \text{ compact}\}$ and for each Borel set H, define $\mu^*(H) = \inf\{\mu_*(G): H \subset G, G \text{ open}\}$. By a standard argument involving local compactness ([10], 51), μ^* is a regular 0-1 σ-additive Borel measure, so by 1,1(ii) μ^* is concentrated at some point p. If U is an open set containing p, then $\mu^*(U) = 1 = \mu_*(U) \leqslant \mu(U)$, so μ is supported at p; hence the space is weakly Borel-complete.

THEOREM 3.2. (CH) *Each compact Borel-complete topological group is metrizable; each locally compact abelian Borel-complete topological group is metrizable.*

Assume that G is a compact Borel-complete group. From ([11], 25.35) G is dyadic (that is, the continuous image of a product of two point spaces). From [2], a dyadic space is metrizable (assuming (CH)) if it contains no copy of βN. Since βN is not Borel-complete ([9]), it follows that G is metrizable. Now suppose that G is a locally compact abelian Borel-complete group. From ([11], 24.30), G is isomorphic to $R^n \times G_0$, where G_0 is a locally compact group containing an open compact subgroup G_1; hence $G_0 = \sum a_s G_1$ (topological sum of cosets) since G_1 is clopen in G_0. Since G_1 is Borel-complete, the result above shows that G_1 is metrizable and hence that G_1 is metrizable (as a sum of metrizable spaces); thus G is metrizable and the proof of 3.2 is complete.

In view of 3.2, it should be noted that there exists a nonmetrizable Borel-complete topological group. Example 4.22 of [11] is a countable (and hence Borel-complete) topological group G in which each point is a G_δ set, but G is not metrizable. Therefore $\mathrm{Baire}(G) = \mathrm{Borel}(G) = \mathscr{P}(G)$ is a countably generated σ-field. It is interesting to note in fact that the assumption of a special countably generated σ-field associated with X may imply that X is Borel-complete, as the following result shows.

Theorem 3.3. *If either Baire(X) or Borel(X) is a countably generated σ-field, then X is Borel-complete.*

If Baire(X) is countably generated, by a result of Marczewski ([13]), there exists a one-to-one mapping $X\xrightarrow{f}M$ onto a metric space such that f^{-1} (Borel(M)) = Baire(X). Since Baire(X) is countably generated, M has a most $2^{\aleph_0}$ points, so by ([7], 15.24) M is realcompact. Since a space is realcompact exactly when each Baire ultrafilter with the countable intersection property is fixed (which may be proved using the lemma in 1.1), it follows that X is realcompact. Furthermore, since each point of M is a zero set, each point of X is a Baire set and hence is a G_δ set ([10], Theorem D of Section 51), so X is hereditarily realcompact ([7], 8.15) and hence Borel-complete by 1.3 and 2.1. If Borel(X) is countably generated, the result from [10] mentioned above may be similarly used to show that X is Borel-complete.

In conclusion, we comment that 3.2 and 3.3, combined, answer for topological groups the following question mentioned in [5]: if X is compact and Borel(X) is a countably generated σ-field, must X be metrizable?

The following examples consider some of the ideas that have been discussed in the paper.

Example 1. The Dowker space constructed in [15] is a closed complete normal space which is not weakly Borel-complete; this has been shown in [16]. The authors have no example of a Borel-complete normal space that is not realcompact; since each closed complete normal countably paracompact space is realcompact ([9]), such an example would necessarily be a Dowker space.

Example 2. Example 3.7 of [9] is a locally compact metacompact space that is not almost realcompact.

Example 3. The space constructed in Problem 51 of [7] is a pseudocompact Borel-complete space that is not realcompact.

Example 4. A weakly Borel-complete extremally disconnected space need not be almost realcompact. For example, the projective resolution of any weakly Borel-complete space that is not almost realcompact is such a space, since the perfect image of an almost realcompact space is almost realcompact and the perfect preimage of a weakly Borel-complete space is weakly Borel-complete.

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