Universal maps of Cartesian products

by

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Abstract. Let $f\colon X\to I^n$ be a map of a compact space X such that $\dim[X-f^{-1}(\dot{I})]\leqslant n$. Then

THEOREM 1. *The following statements are equivalent*

(i) $f\times g\colon X\times Y\to I^n\times I^m$ *Is universal for every universal map* $g\colon Y\to I^m$ *of any compact space* Y *such that* $\dim[Y-g^{-1}(\dot{I}^m)]\leqslant m$, $m=0,1,2,\ldots$

(ii) $f\times g_{pk}\colon X\times M_{pk}\to I^n\times I^2$ *is universal for every prime* p *and positive integer* k.

Above $g_{pk}\colon M_{pk}\to I^2$ is the canonical map of Möbius band of order p^k.

THEOREM 2. *The following statements are equivalent*

(a) $f\times g_p\colon X\times M_p\to I^n\times I^2$ *is universal for every prime* p.

(b) *If the composition of* $f_1\colon[X, f^{-1}(\dot{I}^n)]\to(S^n,1)$ *and* $f_2\colon(S^n,1)\to(S^n,1)$ *is homotopic to* $f'\colon[X, f^{-1}(\dot{I}^n)]\to(S^n,1)$ *induced by* f, *then* f_2 *is a homotopy equivalence.*

Some algebraic results also given.

1. Introduction. A continuous map $f\colon X\to W$ is called *universal* if for every continuous map $g\colon X\to W$ there exists an $x\in X$ such that $f(x)=g(x)$ [2]. Universal maps give a common generalization of both the fixed point property and covering dimension theory; there is also a relation with stable cohomotopy groups [5]. These maps were studied by the first author in a sequence of papers, and also by S. Iliadis, S. Kwapień, and the second author [7, 8, 17], H. Shirmer renamed them "coincidence producing maps" in [13], and discussed them in subsequent papers [14, 15]. Of course there are numerous papers on generalized fixed point properties and on coincidence points, and these are often related to universal maps. However, some of these papers are just on universal maps and were written before the notion of a universal map was introduced; e.g. see C. N. Maxwell [9], J. Mioduszewski and M. Rochowski [10], Rosen [12] (in connection with [12] see C. Vora [18]).

In this paper we study universality of the cartesian product of universal maps into cubes. Let $f\colon X\to I^n$ and $g\colon Y\to I^m$ be universal maps. It is known that if X is an n-dimensional paracompact space, $m=1$, and Y is compact then $f\times g\colon X\times$ $\times Y\to I^{n+1}$ is universal (see [4]; a more complete result in this direction is in [5]). In general, $f\times g$ is not universal even if X is a finite polyhedron (in fact, if X is a cube), $Y=I$, $m=1$, and $g=\mathrm{id}_I$ [5]. But if X and Y are compact oriented manifolds (possibly with boundary) of dimensions n and m respectively, then $f\times g$ is universal [6]. The restriction that X and Y be manifolds of these dimensions cannot be removed. There are universal maps $f\colon X\to I^2$, where X is a 2-dimensional polyhedron or a 4-dimensional closed manifold such that $f\times f\colon X\times X\to I^4$ is not universal [6]; for a concrete example see [17].

We consider the following question. Given families F and Φ of universal maps into cubes, find a condition on $f\in F$ which is equivalent to the condition:

$(*)$ $f\times g$ is universal for every $g\in\Phi$.

A geometric way to solve such a problem is to give a "small" family $\Phi_0\subseteq\Phi$ of explicit maps so that condition $(*)$ is equivalent to the condition:

$(*_0)$ $f\times g$ is universal for every $g\in\Phi_0$.

Here we solve this problem when $F=\Phi$ is the family of all universal maps of finite-dimensional compact spaces into cubes of the same dimension. In a sense we get the strongest possible result (see Theorem 1′ and the remarks which follow). We conjecture the following. Let $F=\Phi$ be the family of all universal maps of finite-dimensional compact spaces into cubes of the same dimension.

Let $\Phi_0\subseteq\Phi$ be an arbitrary family such that conditions $(*)$ and $(*_0)$ are equivalent for every $f\in F$. Then there exists a family $\Phi_1\subseteq\Phi_0$ such that $\Phi_0\setminus\Phi_1$ is infinite and $(*)$ is equivalent to the condition:

$(*_1)$ $f\times g$ is universal for every $g\in\Phi_1$ (for every $f\in F$).

2. Statement of main result. In order to formulate the principal result we need to define the following special maps. Consider the annulus $S^1\times I$. For a given prime p and natural number k we identify every p^k points of the circle $S^1\times\{0\}$ which divide it into p^k equal arcs. Let m_{pk} denote the 2-dimensional polyhedron obtained from $S^1\times I$ with these identifications (M_{pk} is a "Möbius band of order p^k"). Define a map $g_{pk}\colon M_{pk}\to I^2$ as follows: g_{pk} maps the image of $S^1\times\{0\}$ in M_{pk} to the centre of I^2, g_{pk} maps the image of $S^1\times\{1\}$ in M_{pk} onto the boundary $\dot I^2$ of I^2 by the map $z\to z^{p^{k-1}}$ (where we identify $S^1\times\{1\}$ and $\dot I^2$ with $S^1\subset C$), and g_{pk} maps the image of a radial line segment from $S^1\times\{0\}$ to $S^1\times\{1\}$ in M_{pk} to a radial line segment from the centre of I^2 to the boundary $\dot I^2$ of I^2.

We are now ready to state the main theorem (compact spaces are assumed to be Hausdorff).

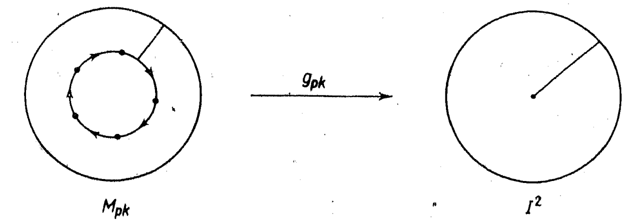
THEOREM 1. *For a given map $f\colon X\to I^n$ of a compact space X into the n-dimensional cube I^n, where the $\dim(X-f^{-1}(\dot I^n))\leqslant n$, the following statements are equivalent:*

(i) *$f\times g\colon X\times Y\to I^n\times I^m$ is universal for every universal map $g\colon Y\to I^m$ of a compact space Y, where the $\dim(Y-g^{-1}(\dot I^m))\leqslant m$, $m=0,1,2,\ldots$*

(ii) *$f\times g_{pk}\colon X\times M_{pk}\to I^n\times I^2$ is universal for every prime p and natural number k.*

The proof of this theorem is preceded by a theorem in which we characterize a class of right indecomposable maps (see Section 4) in terms of cartesian products.

3. Preliminaries. In this paragraph we will collect auxiliary definitions and known results needed in the sequel. We use Čech cohomology with integral coefficients.



PROPOSITION 1 ([3]). *A map $h\colon Z\to I^k$ is universal if and only if there does not exist an extension H of the map $h|h^{-1}(\dot I^k)\colon h^{-1}(\dot I^k)\to I^k$ to all of Z.*

Let $h\colon Z\to I^k$, and set $D=h^{-1}(\dot I^k)$. Let $\bar s_{k-1}$ be a generator of $H^{k-1}(\dot I^k)$ and let $s_k=\delta\bar s_{k-1}$, where $\delta\colon H^{k-1}(\dot I^k)\to H^k(I^k,\dot I^k)$.

PROPOSITION 2. *Let Z be a compact space with $\dim(Z-D)\leqslant k$. Then $h^*(s_k)=0$ in $H^k(Z,D)$ if and only if h is not universal.*

Proof. This follows from Proposition 1 and the Hopf extension theorem (e.g., see Spanier [16] and [11], Appendix by Kodama).

Let s_n and s_m be generators of $H^n(I^n,\dot I^n)$ and $H^m(I^m,\dot I^m)$, respectively. Identifying I^{n+m} with $I^n\times I^m$, it follows that $s_n\times s_m$ is a generator of $H^{n+m}(I^{n+m},\dot I^{n+m})$. If $f\colon(X,A)\to(I^n,\dot I^n)$ and $g\colon(Y,B)\to(I^m,\dot I^m)$, then

$$(f\times g)^*(s_n\times s_m)=f^*(s_n)\times g^*(s_m)\in H^{n+m}((X,A)\times(Y,B)).$$

Note that if $A=f^{-1}(\dot I^n)$ and $B=g^{-1}(\dot I^m)$, then $(X,A)\times(Y,B)=(X\times Y,C)$, where $C=(f\times g)^{-1}(\dot I^{n+m})$.

Since the canonical homomorphism $H^n(X,A)\otimes H^m(Y,B)\to H^{n+m}((X,A)\times(Y,B))$, under which $u\otimes v\to u\times v$, is a monomorphism (see [12], Appendix by Kodama), we have:

PROPOSITION 3. *Suppose $f\colon X\to I^n$ and $g\colon Y\to I^m$ are maps of compact spaces X and Y with $\dim(X-A)\leqslant n$ and $\dim(Y-B)\leqslant m$, respectively, where $A=f^{-1}(\dot I^n)$ and $B=g^{-1}(\dot I^m)$. Then $f^*(s_n)\otimes g^*(s_m)=0$ in $H^n(X,A)\otimes H^m(Y,B)$ if and only if $f\times g\colon X\times Y\to I^n\times I^m$ is not universal.*

We recall that a subgroup H of an abelian group F is said to be *pure* if the equation $nx=h\in H$, where n is an integer, is solvable in H whenever it is solvable in the whole group F.

PROPOSITION 4. *If H is a pure subgroup of F and F/H is finitely generated, then H is a direct summand of F* (e.g., see Fuchs [1], Corollary 28.3).

4. Related results. We make the following definition:

DEFINITION. A map (of pairs) $f\colon(X,A)\to(W,E)$ is said to be *homotopically right indecomposable* if for all maps $f_1\colon(X,A)\to(W,E)$ and $f_2\colon(W,E)\to(W,E)$ such that $f_2\circ f_1\simeq f$, it follows that f_2 is a homotopy self-equivalence.

The notion of right indecomposability is naturally defined for arbitrary categories.

If $f: (X, A)\to(I^n, \dot{I}^n)$ is a given map, let $f': (X, A)\to(S^n, 1)$ be the composition $q\circ f$, where $q: (I^n, \dot{I}^n)\to(I^n/\dot{I}^n, *)\approx(S^n, 1)$. Let $M_p = M_{p1}$ and $g_p = g_{p1}$.

THEOREM 2. *Let $f: (X, A)\to(I^n, \dot{I}^n)$ be a given map of a compact pair (X, A), where $A = f^{-1}(\dot{I}^n)$ and $\dim(X-A)\leqslant n$. Then the following statements are equivalent:*

(a) *$f\times g_p: X\times M_p\to I^n\times I^2$ is universal for every prime p.*

(b) *$f': (X, A)\to(S^n, 1)$ is homotopically right indecomposable.*

The proof of Theorem 2 will follow from the following sequence of lemmas.

LEMMA 1. *Let F be an abelian group, and let $a\in F$. Then the following statements are equivalent:*

(a′) *$\alpha\otimes\beta_p\neq 0$ in $F\otimes C_p$ for all primes p, where β_p is a generator of the (cyclic) group C_p of order p.*

(b′) *α is not divisible by any integer other than ± 1.*

Proof. That (a′) implies (b′) is immediate. Suppose that $\alpha\otimes\beta_p = 0$ in $F\otimes C_p$ for some prime p. Since the tensor product is continuous with respect to the direct limit, we can assume that F is finitely generated. Let F_f+F_t refer to a fixed decomposition of F into the direct sum of a free group F_f and a torsion group F_t. Let $\alpha = \alpha_f+\alpha_t$, where $\alpha_f\in F_f$ and $\alpha_t\in F_t$. Then $\alpha\otimes\beta_p = \alpha_f\otimes\beta_p+\alpha_t\otimes\beta_p = 0$, and hence $\alpha_f\otimes\beta_p = 0$ and $\alpha_t\otimes\beta_p = 0$.

Since F_f is isomorphic to the direct sum of infinite cyclic groups, we will write $\alpha_f = \sum_{i=1}^{r}\alpha_i e_i$ (where $\{e_1, \ldots, e_r\}$ forms the standard basis for $\bigoplus_{i=1}^{r} C$). Then

$$0 = \alpha_f\otimes\beta_p = \sum_{i=1}^{r}\alpha_i(e_i\otimes\beta_p),$$

and since $\{e_1\otimes\beta_p, \ldots, e_r\otimes\beta_p\}$ forms a basis for $\bigoplus_{i=1}^{r} C/p$, it follows that $p|\alpha_i$ for $i = 1, \ldots, r$ and hence $p|\alpha_f$.

Now decompose F_t as the direct sum $C_{pi_1}+\ldots+C_{pi_s}+L$, where no element of L has order divisible by p. Then writing

$$\alpha_t = \sum_{j=1}^{s}\alpha_{pi_j}f_j+\lambda,$$

where f_j is a generator of C_{pi_j}, we obtain

$$0 = \alpha_t\otimes\beta_p = \sum_{j=1}^{s}\alpha_{pi_j}(f_j\otimes\beta_p)+\lambda\otimes\beta_p = \sum_{j=1}^{s}\alpha_{pi_j}(f_j\otimes\beta_p).$$

Since $f_j\otimes\beta_p$ is a generator for $C_{pi_j}\otimes C_p\simeq C_p$, $p|\alpha_{pi_j}$ for $j = 1, \ldots, s$. Since $p|\lambda$, $p|\alpha_t$.

Thus $p|\alpha_f$ and $p|\alpha_t$, implying $p|\alpha$, contradicting the assumption that no integer $\neq\pm 1$ divides α. Hence (b′) implies (a′).

In Lemmas 2 and 3 let $F = H^n(X, A)$ and $\alpha = f^*(s_n)\in F$.

LEMMA 2. *The following statements are equivalent:*

(a) *$f\times g_p: X\times M_p\to I^n\times I^2$ is universal for every prime p.*

(a′) *$\alpha\otimes\beta_p\neq 0$ in $F\otimes C_p$ for all primes p, where β_p is a generator of the (cyclic) group C_p of order p.*

Proof. This follows from Proposition 3 and the easily checked fact that $H^2(M_p, B)$ is cyclic of order p and is generated by $g^*(s_2)$, where $B = g_p^{-1}(\dot{I}^2)$.

LEMMA 3. *The following statements are equivalent:*

(b) *$f': (X, A)\to(S^n, 1)$ is homotopically right indecomposable.*

(b′) *α is not divisible by any integer $k\neq\pm 1$.*

Proof. Suppose α is divisible by $k\neq\pm 1$; then $\alpha = k\alpha_k$ for some $\alpha_k\in F$. By the Hopf classification theorem (e.g., see Spanier [4]), there is a map $h_k: (X, A)\to(S^n,1)$ such that $h_k^*(\bar{s}_n) = \alpha_k$ and hence $f^*(\bar{s}_n) = kh_k^*(s_n)$. Let $d_k: (S^n, 1)\to(S^n, 1)$ be the map induced by the map $z\to z^k$ from $S^1\subset C$ to itself. Then $d_k^*(\bar{s}_n) = k\bar{s}_n$, and hence $f^*(s_n) = (d_k\circ h_k)^*(\bar{s}_n)$. Since the collapsing map $q: (I^n, \dot{I}^n)\to(S^n, 1)$ induces an isomorphism $q^*: H^n(S^n, 1)\to H^n(I^n, \dot{I}^n)$ where $q^*(\bar{s}_n) = s_n$, we again apply the Hopf classification theorem to show that $f'\simeq d_k\circ h_k$. Since $k\neq\pm 1$, d_k is not a homotopy equivalence and hence f' is not homotopically right indecomposable. Thus (b) implies (b′).

Suppose f' is not homotopically right indecomposable, i.e., there are maps $f_1: (X, A)\to(S^n, 1)$ and $f_2: (S^n, 1)\to(S^n, 1)$ such that $f_2\circ f_1\simeq f'$, but f_2 is not a homotopy equivalence. Since $[S^n, 1; S^n, 1] = Z$ and by the Hopf classification theorem, $f_2^*(\bar{s}_n) = k\bar{s}_n$ for some integer k. Since f_2 is not a homotopy self-equivalence, $k\neq\pm 1$. Hence for some integer $k\neq\pm 1$, there exists a map $f_1: (X, A)\to(S^n, 1)$ such that $(f_1)^*(\bar{s}_n) = (f_2\circ f_1)(\bar{s}_n) = kf_1^*(\bar{s}_n)$. Since $q^*(\bar{s}_n) = s_n$, $\alpha = f^*(s_n) = (f')^*(\bar{s}_n)$ and $\alpha = kf_1^*(\bar{s}_n)$, i.e., $k|\alpha$ and hence (b′) implies (b).

Proof of Theorem 2. By Lemma 1, (a′) is equivalent to (b′). Hence by Lemmas 2 and 3, statements (a) and (b) are equivalent.

5. Algebraic formulation. In this paragraph we formulate and prove an algebraic substitute of Theorem 1, with topological spaces replaced by abelian groups.

THEOREM 3. *Let F be a given abelian group, and let $\alpha\in F$. Then the following statements are equivalent:*

(i′) *$\alpha\otimes\beta\neq 0$ in $F\otimes G$ for any non-zero element of an arbitrary abelian group G.*

(ii′) *$\alpha\otimes\beta_{pk}\neq 0$ in $F\otimes C_{pk}$ for all primes p and natural numbers k, where β_{pk} is an element of order p in the cyclic group C_{pk} of order p^k.*

(iii′) *The subgroup generated by α is a direct summand of every finitely generated subgroup of F containing α.*

Proof. Clearly (i′) implies (ii′). To show that (ii′) implies (iii′) and that (iii′) implies (i′), we can assume that F is finitely generated.

To show that (ii′) implies (iii′), we first note that by Lemma 1 α is not divisible by any integer $\neq\pm 1$, and thus $|\alpha| = \infty$. Let $H = \langle\alpha\rangle$; we will show that H is a pure subgroup of F. This is for some integers r and s there is an element $\gamma \in F$ such that $r\alpha = s\gamma$, then there exists an element $\delta \in H$ such that $r\alpha = s\delta$.

Suppose first that $(r, s) = 1$; then there exist integers a and b such that $ar+bs = 1$, and thus $\alpha = ar\alpha+bs\alpha = as\gamma+bs\alpha$, i.e., $\alpha = s(a\gamma+b\alpha)$. Hence $s|\alpha$ and $s = \pm 1$. But then $r\alpha = \pm\gamma \in H$, and we take $\delta = \pm r\alpha$.

Now suppose $(r, s) = m \neq 1$; then $r = mr_0$ and $s = ms_0$, where $(r_0, s_0) = 1$. Then $r\alpha = s\gamma$ implies $m(r_0\alpha - s_0\gamma) = 0$. Let F_f+F_t refer to a fixed decomposition of F into the direct sum of a free group F_f and a torsion group F_t. Let $\alpha = \alpha_f+\alpha_t$ and $\gamma = \gamma_f+\gamma_t$, where α_f and γ_f and α_t and $\gamma_t \in F_t$. Note that $\alpha_f \neq 0$ since $|\alpha| = \infty$. Then $m(r_0\alpha_f - s_0\gamma_f) = 0$ and thus $r_0\alpha_f - s_0\gamma_f = 0$, i.e., $r_0\alpha_f = s_0\gamma_f$.

We now show that α_f is not divisible by any integer other than ± 1. Suppose α_f is divisible by the prime p, i.e., $\alpha_f = p\alpha_f'$. Then

$$\alpha\otimes\beta_{pk} = p\alpha_f'\otimes\beta_{pk}+\alpha_t\otimes\beta_{pk} = \alpha_t\otimes\beta_{pk},$$

since $|\beta_{pk}| = p$. F_t can be decomposed as the direct sum $C_{pi_1}+\ldots+C_{pi_n}+L$, where no element of L has order divisible by p. Then writing $\alpha_t = \sum_{j=1}^{n}\alpha_{pi_j}+\lambda$ we have

$$\alpha_t\otimes\beta_{pk} = \sum_{j=1}^{n}(\alpha_{pi_j}\otimes\beta_{pk})+\lambda\otimes\beta_{pk} = \sum_{j=1}^{n}(\alpha_{pi_j}\otimes\beta_{pk}).$$

Choosing $k>\max\{i_1, \ldots, i_n\}$, $\alpha_{pi_j}\otimes\beta_{pk} = 0$ for $j = 1, \ldots, n$. Thus for this choice of k, $\alpha\otimes\beta_{pk} = \alpha_t\otimes\beta_{pk} = 0$, contradicting the assumptions of statement (ii′). Thus α_f is indeed not divisible by any integer other than ± 1.

Now, as before, it follows that $s_0 = \pm 1$; thus $s = \pm m$. Now let $\delta = \pm r_0\alpha$; then $r\alpha = m(r_0\alpha) = \pm s(r_0\alpha) = s(\pm r_0\alpha) = s\delta$, where $\delta \in H$.

Thus we have shown that if F is finitely generated, $H = \langle\alpha\rangle$ is a pure subgroup of F. Clearly F/H is finitely generated; hence by Proposition 4, H is a direct summand of F. Hence (ii′) implies (iii′).

Finally, it is easy to see that (iii′) implies (i′). Indeed, if $\alpha\otimes\beta = 0$ in $F\otimes G$, where β and G are as in (i′), $\alpha\otimes\beta = 0$ in $F_0\otimes G$ for some finitely generated subgroup F_0 of F. Since $H = \langle\alpha\rangle$ is a direct summand of F_0, $\alpha\otimes\beta = 0$ in $H\otimes G$. But $H\otimes G \cong G$ by the isomorphism $\alpha\otimes\beta\to\beta$. Thus $\beta = 0$ in G, a contradiction.

6. Proof of Theorem 1. We will use the previously established notation; in particular, $f\colon X\to I^n$ is a map of a compact space X with $\dim(X-A)\leqslant n$, where $A = f^{-1}(\dot{I}^n)$. We set $F = H^n(X, A)$ and $\alpha = f^*(s_2) \in F$.

LEMMA 4. *The following statements are equivalent*:

(i) $f\times g\colon X\times Y\to I^n\times I^m$ *is universal for every universal map* $g\colon Y\to I^m$ *of a compact space* Y *with* $\dim(Y-B)\leqslant m$, *where* $B = g^{-1}(\dot{I}^m)$, $m = 0, 1, \ldots$

(i′) $\alpha\otimes\beta \neq 0$ *in* $F\otimes G$ *for any non-zero element* β *of an arbitrary abelian group* G.

Proof. By Propositions 2 and 3, (i′) implies (i). To show that (i) implies (i′), we first recall that (i′) is equivalent to (ii′) of Theorem 2. Suppose $\alpha\otimes\beta = 0$ in $F\otimes G$; then $\alpha\otimes\beta_{pk} = 0$ in $F\otimes C_{pk}$ for some prime p and integer k, where β_{pk} is an element of order p in the cyclic group C_{pk} of order p^k. Consider the compact pair (M_{pk}, B) where M_{pk} is the two-dimensional polyhedron described in paragraph 0, and B is the image of $S^1\times\{1\}$ in M_{pk} $(B\approx S^1)$. Then it is easily shown that $H^2(M_{pk}, B) = C_{pk}$, and if $\beta_{pk} = qe_{pk}$, where e_{pk} is a generator of C_{pk}, then $(g_{pq})^*(s_2) = \beta_{pk}$ $(q \equiv 0 \bmod (p^{k-1}))$. Thus if we set $Y = M_{pk}$ and $g = g_{pq}$, then $g\colon Y\to I^2$ is a universal map of a compact space Y with $\dim(Y-B) = 2$. Since $\alpha\times\beta_{pk} = 0$ in $H^n(X, A)\times\times H^m(Y, B)$, $f\times g_{pq}$ is not universal by Proposition 4. It follows that (ii′), and hence (i′), implies (i).

LEMMA 5. *Given a prime p and a natural number k the following statements are equivalent*:

(ii$_{pk}$) $f\times g_{pk}\colon X\times M_{pk}\to I^n\times I^2$ *is universal.*

(ii′$_{pk}$) $\alpha\times\beta_{pk} \neq 0$ *in* $F\otimes C_{pk}$, *where* β_{pk} *is of order p in the cyclic group* C_{pk} *of order* p^k.

Proof. This follows from Proposition 3 and the fact that $H^2(M_{pk}, B) = C_{pk}$ and $g^*_{pk}(s_2) \in H^2(M_{pk}, B)$ is of order p.

Proof of Theorem 1. By Lemmas 4 and 5, (i) is equivalent to (i′) and (ii) is equivalent to (ii′). By Theorem 3, (i′) and (ii′) are equivalent, and Theorem 1 follows.

We remark that it is easy to see that (ii′$_{p,k+1}$) implies (ii′$_{pk}$) for $k = 1, 2, \ldots$ Thus we can formulate Theorem 1 in a somewhat sharper way.

THEOREM 1′. *For a given map* $f\colon X\to I^n$ *of a compact space X into the n-dimensional cube* I^n, *where* $\dim(X-f^{-1}(\dot{I}^n))\leqslant n$, *the following statements are equivalent*:

(i) $f\times g\colon X\times Y\to I^n\times I^m$ *is universal for every universal map* $g\colon Y\to I^m$ *of a compact space* Y, *where the* $\dim(Y-g^{-1}(\dot{I}^m))\leqslant m$, $m = 0, 1, 2, \ldots$

(ii$_0$) *There exists a family* $\Phi_0\subseteq\{g_{pk}\colon p$ *is a prime*, $k = 1, 2, \ldots\}$ *such that for every prime p, the map g_{pk} belongs to Φ_0 for infinitely many k, and $f\times g$ is universal for every* $g \in \Phi_0$.

For example, let us identify I^2 with the unit disc in the complex plane, and let $f_{pk}\colon I^2\to I^2$ be given by $f_{pk}(z) = z^{p^k}$. Then $f^*_{pk}(s_2)\times g_{qr} = 0$ if and only if $q = p$ and $r>k$. This means that $f_{pk}\times g_{qr}$ is universal if and only if $p \neq q$ or $r\leqslant k$. In particular $f_{pk}\times g_{p,k+1}$ is not universal. This shows that Theorem 1′ is the sharpest possible in the following sense. If $\Psi\subseteq\{g_{pk}\colon p$ is a prime, $k = 1, 2, \ldots\}$ is a family such that condition (i) of Theorem 1′ is equivalent to the condition "$f\times g$ is universal for every $g \in \Psi$", then Ψ is one of the families Φ_0 specified by condition (ii$_0$) of Theorem 1′.

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Weakly Borel-complete topological spaces

by

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Abstract. A Tychonoff space is *weakly Borel-complete* if each ultrafilter of Borel sets with the countable intersection property converges to some point in the space. This concept has been introduced by Z. Frolík in [4] under the name Baire–Borel–complete, with a different definition. The present paper studies such spaces, contrasting their properties with the Borel-complete and closed complete spaces discussed in [9] and the familiar realcompact spaces, and adds some new results on Borel-complete spaces. The primary difference in approach between [9] and the present work is the measure-theoretic language adopted here. For example, weak Borel-completeness is equivalent to each non-trivial 0-1 valued countably additive Forel measure having a non-empty support set (necessarily consisting of one point). Finally, we note that the present work has considerable overlap with the recent work of R. J. Gardner [6]; the details of this overlap are found at the end of section two.

Section 1. A space is Borel-complete (resp. closed complete) if each ultrafilter of Borel sets (resp. closed sets) with the countable intersection property is fixed at some point of the space; alternately Borel-completeness is equivalent to each σ-additive 0-1 Borel measure being a point mass measure. Therefore each Borel-complete space is weakly Borel-complete. For other background information the reader is referred to [9]. In particular, the Baire (resp. Borel) sets are the smallest σ-field which contains the zero sets of continuous real-valued mappings (resp. the closed sets).

THEOREM 1.1. *The following statements are equivalent.*

(i) *X is closed complete.*

(ii) *Each non-trivial regular σ-additive 0-1 Borel measure is a point mass measure.*

(iii) *For each closed ultrafilter $\mathscr{F}$ on X with $\bigcap\mathscr{F} = \emptyset$ there exists a σ-disjoint open refinement of $\{X-F\colon F\in\mathscr{F}\}$ and X has no closed discrete subspace of measurable power.*

To prove the above theorem, we will need the following lemma that was discovered during the writing of [9] (see 6.9–6.12 of [8]).

LEMMA. *Let $\mathscr{C}\subset\mathscr{P}(X)$ and let $\mathscr{F}$ be a $\mathscr{C}$-ultrafilter closed under countable intersections. Define*

$$\Sigma(\mathscr{F}) = \{S\subset X\colon S \text{ misses or contains some member of } \mathscr{F}\}.$$