On absolutely $k$th continuous functions

by

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Abstract. In [3] and [4] Russell derives some properties of the functions of bounded $k$th variation (BV functions). Here we introduce the notion of AC$^k$ functions and obtain some relations with those of BV$^k$ functions involving $k$th Riemann $*$-derivative. We also refine the definitions of BV$^k$ and AC$^k$ functions to obtain the classes of BV$^k_*$, BV$^k_0$, AC$^k_*$ and AC$^k_0$ functions and then study various interrelations of these classes.

1. Preliminaries and definitions. A. M. Russell in [3] obtained the definition of functions of bounded $k$th variation (BV$^k$ functions). In the definition there were certain restrictions which he removed in [4], where he investigated in detail the properties of functions of bounded $k$th variation. Prior to [3], [4], he obtained in [2] the properties of functions of second variation. In this paper we introduce the notion of AC$^k$ functions and investigate their properties. Also from BV$^k$ functions we derive the notions of BV$^k_*$ and BV$^k_0$ functions and obtain their relations. In the sequel, we shall need the following definitions and results from [4].

Definition 1(a). Let $f$ be a real-valued function defined on $[a, b]$ and let $x_0, x_1, ..., x_k$ be $k+1$ distinct points, not necessarily in the linear order, belonging to $[a, b]$. Define the $k$-th divided difference of $f$ as

$$Q(f; x_0, x_1, ..., x_k) = \frac{1}{k!} \sum_{i=0}^{k} \left[ f(x_i) \prod_{s \neq i} (x_i - x_s) \right] .$$

Definition 2. Let $x, x_1, ..., x_k$ be $k+1$ distinct points in $[a, b]$. Suppose that $h_i = x_i - x$ when $i = 1, 2, ..., k$ and that

$$0 < |h_1| < |h_2| < ... < |h_k| .$$

Then define the $k$-th Riemann $*$-derivative by

$$D^k_f(x) = k! \lim_{h_0, h_1, ..., h_k \to 0} \lim_{h_0 \to 0} \lim_{h_1 \to 0} ... \lim_{h_k \to 0} Q(f; x, x_1, ..., x_k) ,$$

if the iterated limit exists. The right and the left Riemann $*$-derivative are defined in the obvious way.
This definition has certain connections with the kth Riemann derivative as discussed in [1].

We shall call a subdivision of \([a, b]\) at \(x_0, x_1, ..., x_n\) a \(\pi\)-subdivision of \([a, b]\) when \(a=x_0<x_1<...<x_n=b\) and denote it by \(\pi(x_0, x_1, ..., x_n)\).

**Definition 3.** The total k-th variation of \(f\) on \([a, b]\) is defined by

\[ V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q_k(f; x_i, x_{i+1}, ..., x_{i+k})| \]

If \(V_k(f; a, b)<\infty\), we say that \(f\) is of bounded k-th variation on \([a, b]\) and write \(f \in BV_k([a, b])\). The summations over which the sup is taken are called approximating sums for \(V_k(f; a, b)\).

**Lemma 1.** \(Q_k(f; x_0, x_1, ..., x_k) = 0\) for all choices of \(x_0, x_1, ..., x_k\) if \(f\) is a polynomial of degree \(k-1\) at most.

**Lemma 2.** \(Q_k(f; x_0, x_1, ..., x_k)\) is independent of the order in which the points \(x_0, x_1, ..., x_k\) appear.

**Lemma 3.** \(Q_k(f; x_0, x_1, ..., x_k)\) is of degree \(k-1\) at most.

**Theorem 3.** The addition of extra points of subdivision to an existing subdivision does not decrease the approximating sums for \(V_k(f; a, b)\).

**Theorem 4.** If \(f \in BV_k([a, b])\) and \(f\) has a \((k-1)\)-th Riemann *-derivative at \(c\), where \(a<c<b\), then \(f \in BV_k([a, b])\).

**Lemma 1.** If \(g \in AC_k([a, b])\), then \(g\) possesses the \((k-1)\)-th Riemann *-derivative in \([a, b]\).

**Proof.** Let \(a<c<b\) and \(\varepsilon>0\) be arbitrary. There exists a \(\delta(\varepsilon)>0\) such that the condition of the definition of \(AC_k([a, b])\) functions is satisfied with \(\varepsilon\) replaced by \(\delta(\varepsilon)(k-1)\). We choose points \(x_{p+k-1} < x_{p+k-2} < ... < x_{p+1} < x_p < x_{p+1} < ... < x_{p+k-1} < \delta\).

Choose a positive integer \(i\) such that \(p-k+1 < i < p-1\) and consider the elementary system consisting of a single interval

\[ I(x_{i+k}, ..., x_{i+k-1}): (x_i, x_{i+k}) \]

Using Lemma 4 of [4], we get

\[ |Q_k(g; x_{i+k}, ..., x_{i+k-1}) - Q_k(g; x_i, ..., x_{i+k-1})| < \delta \]

Since \(i\) may assume \(k-1\) values in \(p-k+1 < i < p-1\), we see that the above inequality is true for any one of these values of \(i\), viz., for \(i = p-k+1, ..., p-1\).

It may be noted, however, that these \(k-1\) intervals taken together do not form an elementary system, because the intervals are overlapping.

Combining now the \(k-1\) inequalities, we obtain

\[ |Q_k(g; x_i, ..., x_{i+k-1}) - Q_k(g; x_i, ..., x_{i+k-1})| < \delta \]

for \(i = p-k+1, ..., p-1\) and \(j = p-k+1, ..., p-1\).

Hence \(g(x)\) possesses the \((k-1)\)-th Riemann *-derivative at \(c\).

**Note.** With suitable modifications it may be shown that \(D^*_k g(a)\) and \(D^*_k g(b)\) exist.

**Theorem 5.** If \(g \in AC_k([a, b])\), then \(g \in BV_k([a, b])\).

**Proof.** There exists a \(\delta(\varepsilon) > 0\) such that for any elementary system

\[ I(x_{i+k}, ..., x_{i+k-1}): (x_i, x_{i+k}), \quad i = 1, 2, ..., n, \quad \text{in \([a, b]\)} \]

with \(\sum_{i=1}^{n} (x_{i+k} - x_{i+k}) < \delta\) the relation

\[ \sum_{i=1}^{n} (x_{i+k} - x_{i+k})|Q_k(g; x_{i+k}, ..., x_{i+k-1})| < \delta \]

is satisfied.

The interval \([a, b]\) is broken up into a finite number of sub-intervals \([c_0, c_1], [c_1, c_2], ..., [c_{n-1}, c_n]\) \((a = c_0 < c_1 < ... < c_n = b)\) such that \(c_{i+1} - c_i < \delta\) for each \(i = 0, 1, ..., n-1\).

We keep \(\varepsilon\) fixed temporarily and consider any \(\pi(x_0, x_1, ..., x_n)\) subdivision of \([c_0, c_{n+1}]\).

The sets of intervals \((x_i, x_{i+k}), \quad i \in A_r = \{ r, r+k, 2k+r, ..., \leq n \}\) and \(r = 0, 1, 2, ..., k-1\) form \(k\) elementary systems

\[ I(x_{i+k}, ..., x_{i+k-1}): (x_i, x_{i+k}), \quad i \in A_r \text{ and } r = 0, 1, 2, ..., k-1. \]
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(i) If \( y \notin [x_{i-k}, x_{i+k}] \) for some \( i \), \( 0 \leq k < m \), then \( y \) may or may not coincide with any \( x_{i-j} \), \( -k \leq j \leq k \). Since the relation (2) is satisfied for all choices of \( x_{i-j} \) in \( [x_{i-k}, x_{i+k}] \), we may take \( y \) coincident with a suitable \( x_{i-j} \).

(ii) If \( x_{i-k} < y < x_{i+k} \), then we can easily introduce a new interval and a set of points satisfying (2) and (4) and \( y \) coinciding with a suitable new point.

Thus in any case we may suppose that \( y \), \( 0 \leq k \leq m \), coincides with \( x_{i-j} \) for some \( i \) and some \( j \) for \( 0 \leq j \leq n \), \( -k \leq j \leq k \).

Let \( J \) be any positive integer such that \( 1 \leq j \leq k \). Consider the elementary system

\[ I_k = \{(x_{i-1}, x_{i+1}), \ldots, x_{j-k+1} \} : (x_{i-1}, x_{i+j-k-1}) \] for \( i = 1, 2, \ldots, n \)

where \( x_{i-1} = x_{i-k} \) and \( x_{i+j-k} = x_{i+k} \).

As \( J \) ranges from 1 to \( k \), we obtain \( k \) numbers of elementary systems

\[ I_k = \{(x_{i-1}, x_{i+1}), \ldots, x_{j-k+1} \} : (x_{i-1}, x_{i+j-k-1}) \] for \( i = 1, 2, \ldots, n \)

where \( x_{i-1} = x_{i-k} \) and \( x_{i+j-k} = x_{i+k} \).

From (3), it follows that

\[ \sum_{i=1}^{m} Q_{i,j} \leq \sum_{j=1}^{k} \sum_{i=1}^{m} Q_{i,j} < \varepsilon / 2k \]

for \( j = 1, 2, \ldots, k \) and so

\[ \sum_{j=1}^{k} \sum_{i=1}^{m} Q_{i,j} < \varepsilon / 2. \]

On the other hand, by using (3),

\[ Q_{i,j} \leq \frac{\varepsilon (x_{i+k} - x_{i-k})}{2(k+1)(b-a)^2} \]

for \( i = 0, 1, \ldots, n \) and \( j = -k, \ldots, 0 \).

This gives

\[ \sum_{i=0}^{n} Q_{i,j} < \frac{\varepsilon (x_{i+k} - x_{i-k})}{2(k+1)(b-a)^2}, \quad i = 0, 1, \ldots, n. \]

Consequently

\[ \sum_{i=0}^{n} \sum_{j=-k}^{k} Q_{i,j} < \varepsilon / 2. \]

Since the addition of extra points does not decrease the approximating sum for \( V \) (Theorem 3, [4]), we have

\[ \sum_{i=0}^{n} \sum_{j=-k}^{k} Q_{i,j} < \sum_{i=1}^{m} \sum_{j=1}^{k} Q_{i,j} + \sum_{i=1}^{m} \sum_{j=-k}^{k} Q_{i,j} < \varepsilon / 2 + \varepsilon / 2 = \varepsilon, \]

by (5) and (6).
Since \( \pi \) is any subdivision of \([a, b]\) and \( \varepsilon > 0 \) is arbitrary, it follows that

\[
V = V_1[g; a, b] = 0.
\]

**Proof of the theorem.** We consider a sequence of points \( a \leq x_1 < x_2 < \ldots < x_k = b \) of \([a, b]\).

For any collection of \(2k\) points \( x_i \), \( i = 0, 1, \ldots, 2k-1\) with \( 0 \leq x_0 < x_1 < \ldots < x_{2k-1} \leq b \),

\[
Q_{a,b}(x_0, x_1, \ldots, x_{2k-1}) = \left| \sum_{i=0}^{k-1} (x_{2i} - x_{2i+1})Q_i(g; x_{2i+1}, x_{2i+2}) \right|
\]

and write

\[
\alpha = \sum_{i=1}^{n} (x_{i,k} - x_{i,0})Q_i(g; x_{i,0}, x_{i,1}, \ldots, x_{i,k}).
\]

where \( m \angle = \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) \).

**Definition 2.** The function \( g(x) \) is said to be absolutely \( k\)-th continuous from above on \([a, b]\) if for an arbitrary \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any elementary system \( I(x_{1,0}, \ldots, x_{k-1}) \), \( n > 1 \), \( \ldots, n \in [a, b] \) with \( m \angle < \delta \) the relation \( \alpha \angle < \varepsilon \) is satisfied. If it is said to be absolutely \( k\)-th continuous from below if the relation \( \alpha \angle > -\varepsilon \) holds when \( m \angle > \delta \). If \( g(x) \) is absolutely \( k\)-th continuous from above (or from below) on \([a, b]\), we write \( g \in AC_{+}^k [a, b] \) (or \( g \in AC_{-}^k [a, b] \)).

**Definition 3.** The least upper bound and the greatest lower bound of the aggregate \( \{\alpha\} \) of sums of all possible elementary systems \( I \) in \([a, b]\) are called, respectively, the positive and the negative \( k\)-th variation of \( g(x) \) in \([a, b]\) and are designated by \( V_{+}^k[g; a, b] \) and \( V_{-}^k[g; a, b] \).

Henceforth we shall assume that \((k-1)\)-th divided differences of \( g \) in \([a, b]\) are bounded in absolute value by \( \lambda \).

**Lemma 3.** \( V_{+}^k[g; a, b] \geq 0 \) and \( V_{-}^k[g; a, b] \leq 0 \).

The proof is omitted.

**Definition 4.** If \( V_{+}^k[g; a, b] < +\infty \), we say that \( g \in BV_{+}^k[a, b] \) and if \( V_{-}^k[g; a, b] < -\infty \), then \( g \in BV_{-}^k[a, b] \).

**Lemma 4.** Let \( x_{0,1}, x_{1,1}, \ldots, x_{0,k} \), \( x_{1,k} \), \( x_{2,1}, \ldots, x_{2,k} \), \( x_{3,1}, \ldots, x_{3,k} \), \( \ldots \) be a set of points in \([a, b]\). If \( g \in BV_{+}^k[a, b] \) (or \( BV_{-}^k[a, b] \)), then the series

\[
\sum_{i=1}^{n} (x_{i,k} - x_{i,0})Q_i(g; x_{i,0}, x_{i,1}, \ldots, x_{i,k})
\]

is convergent.

**Proof.** We prove the lemma in case \( g \in BV_{+}^k[a, b] \). The other case is analogous.

Let \( (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,k}) \), \( i = 1, 2, \ldots, \) be a subsequence of the sequence of all intervals \( \{(x_{0,1}, x_{1,0}), (x_{1,1}, x_{2,0}), \ldots \} \), \( x_{1,0} \), \( x_{1,1} \), \( x_{1,k} \) renamed as \( \xi_{1,i} \), \( \xi_{1,k} \), for each of which \( (\xi_{i,k} - \xi_{i,0})Q_i(g; \xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,k}) \geq 0 \). If \( n \) is a positive integer, then, since \( (\xi_{i,0}, \xi_{i,1}) \), \( i = 1, 2, \ldots, n \) form an elementary system in \([a, b]\), we have

\[
\sum_{i=1}^{n} (\xi_{i,k} - \xi_{i,0})Q_i(g; \xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,k}) \leq V_{+}^k[g; a, b].
\]

Since \( V_{+}^k[g; a, b] < +\infty \) and \( n \) may be arbitrary, it follows that

\[
\sum_{i=1}^{n} (\xi_{i,k} - \xi_{i,0})Q_i(g; \xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,k})
\]

is convergent.
Next, let \((\eta_{i,0}, \eta_{i,k})\), \(i = 1, 2, \ldots\) be a subsequence of \((x_{i,0}, x_{i,k})\), \(i = 1, 2, \ldots\) with \(x_{i,1}, \ldots, x_{i,k-1}\) renamed as \(\eta_{i,1}, \ldots, \eta_{i,k-1}\) for each of which
\[
\left((\eta_{i,k}-\eta_{i,0})Q(g; \eta_{i,0}, \ldots, \eta_{i,k})\right) < 0.
\]
For a fixed positive integer \(n\) we consider an elementary system
\[
I(\eta_{1,1}, \ldots, \eta_{k,1-1})/I(\eta_{1,0}, \eta_{1,0}), \quad i = 1, 2, \ldots, n \text{ in } [a, b].
\]
If \(I_e\) denotes the elementary system in \([a, b]\) complementary to \(I\), then \(I\) and \(I_e\) together form an elementary system in \([a, b]\), which we denote by
\[
J(x_{i,1}, \ldots, x_{i+i-1})/x_{i,0}, x_{i,1}, \ldots, x_{i+m-1} = b. \quad i \in A_e = \{r, k + r, 2k + r, \ldots, (m+1)k\}
\]
where \(x_0 = a\) and \(x_{(m+1)k} = b\). We consider \(k-1\) elementary systems
\[
J(x_{i,1}, \ldots, x_{i+i-1})/x_{i,0}, x_{i,1}, \ldots, x_{i+k} = b, \quad i \in A, \quad i = 1, 2, \ldots, n
\]
for each \(r = 1, 2, \ldots, k-1\), so that
\[
\sigma I + \sigma I_1 + \cdots + \sigma I_{k-1} = Q(g; x_{m+k}, \ldots, x_{m+k}) - Q(g; a, \ldots, x_{m-1}).
\]
Consequently, \(\sigma I \geq -2K - kV^*_k[g; a, b]\) and so
\[
\sum_{i=1}^{N} (\eta_{i,k} - \eta_{i,0})Q(g; \eta_{i,0}, \ldots, \eta_{i,k}) \geq -2K - kV^*_k[g; a, b].
\]
Since the left-hand expression is negative and \(n\) may be any positive integer, the series
\[
\sum_{i=1}^{N} (\eta_{i,k} - \eta_{i,0})Q(g; \eta_{i,0}, \ldots, \eta_{i,k})
\]
is convergent.

Because
\[
\sum_{i=1}^{N} (x_{i,k} - x_{i,0})Q(g; x_{i,0}, \ldots, x_{i,k})
\]
the lemma follows.

**Corollary.** Under the hypotheses of Lemma 4, for any positive integer \(n \geq 1\)
\[
\sum_{i=1}^{n} (x_{i,k} - x_{i,0})Q(g; x_{i,0}, \ldots, x_{i,k}) \leq (k+1)V^*_k[g; a, b] + 2K.
\]

**Proof.** It is seen that the right-hand quantities of (10) and (11) are independent of \(n\). So, by using (10), (11) and (12), the corollary follows.

**Lemma 5.** If \(g \in BV^*_k[a, b]\), then \(g \in BV^*_k[a, b]\) and conversely.

**Proof.** Suppose that \(g \in BV^*_k[a, b]\). We consider an elementary system
\[
I(x_{i,1}, \ldots, x_{i+k-1})/x_{i,0}, x_{i,1}, \ldots, x_{i,k}, \quad i = 1, 2, \ldots, n \text{ in } [a, b].
\]

By the corollary there exists a \(M > 0\), independent of the choice of elementary systems, such that
\[
\sigma I = \sum_{i=1}^{n} (x_{i,k} - x_{i,0})Q(g; x_{i,0}, \ldots, x_{i,k}) \geq -M,
\]
and so \(V^*_k[g; a, b] \geq -M\). Consequently, by Lemma 3, \(g \in BV^*_k[a, b]\). The other case is similar.

**Lemma 6.** If \(g \in BV^*_k[a, c] \text{ and } BV^*_k[c, b]\), where \(a < c < b\), then \(g \in BV^*_k[a, b]\) and conversely.

**Proof.** Suppose that \(g \in BV^*_k[a, c] \text{ and } BV^*_k[c, b]\). Let
\[
I(x_{i,1}, \ldots, x_{i+k-1})/x_{i,0}, x_{i,1}, \quad i = 1, 2, \ldots, n
\]
be any elementary system in \([a, b]\). We consider the following cases:

(a) If \(x_{m+k} < c\) then \(\sigma I \leq V^*_k[g; c, b]\).

(b) If \(x_{m+k} < c\) then \(\sigma I \leq V^*_k[g; a, c]\).

(c) If \(x_{m+k} < c\) then \(\sigma I \leq V^*_k[g; c, b]\).

(d) Let \(x_{m+k} < c\) then \(\sigma I \leq V^*_k[g; c, b]\).

The converse part is clear. By Definition 3 it easily follows that
\[
V^*_k[g; a, c] + V^*_k[g; c, b] < V^*_k[g; a, b].
\]

**Theorem 3.** If \(g \in BV^*_k[a, b]\) (or \(BV^*_k[a, b]\)), then \(g \in BV^*_k[a, b]\) and
\[
V^*_k[g; a, b] < V^*_k[g; a, b].
\]
Proof. Suppose that \( g \in BV^+_k [a, b] \), then by Lemma 5, \( g \in BV^+_k [a, b] \). We consider any \( \pi (x_0, x_1, ..., x_n) \) subdivision of \([a, b]\), \( n > k \). Then

\[
\sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q(g; x_i, x_{i+1}, ..., x_{i+k})| = \sum_{i \in A^+_r} + \sum_{i \in A^-_r} - \sum_{i \in A^+_r} - \sum_{i \in A^-_r}
\]

where \( A \) contains the suffixes \( r, k, 2k, r, ..., \leq n \) for \( r = 0, 1, ..., k-1 \).

We now consider each \( A \) to be the union of two sets of suffixes \( A^+_r \) and \( A^-_r \) such that \( i \in A^+_r \) if \( (x_{i+k} - x_i) Q(g; x_i, x_{i+1}, ..., x_{i+k}) \geq 0 \) and \( i \in A^-_r \) if \( (x_{i+k} - x_i) Q(g; x_i, x_{i+1}, ..., x_{i+k}) < 0 \).

Then, from (13) we get

\[
\sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q(g; x_i, x_{i+1}, ..., x_{i+k})| = \sum_{i \in A^+_r} + \sum_{i \in A^-_r} - \sum_{i \in A^+_r} - \sum_{i \in A^-_r}
\]

We thus obtain 2k elementary systems like

\[
I^+_r (x_{i+1}, ..., x_{i+k-1}): (x_i, x_{i+k}), \quad i \in A^+_r,
\]

\[
I^-_r (x_{i+1}, ..., x_{i+k-1}): (x_i, x_{i+k}), \quad i \in A^-_r
\]

where \( r = 0, 1, ..., k-1 \).

Hence, from (14), we get

\[
\sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q(g; x_i, x_{i+1}, ..., x_{i+k})| = \sigma I^+_0 + ... + \sigma I^+_m - \sigma I^-_m - ... - \sigma I^-_{m-1}
\]

\[
\leq k \{ V^+_k [g; a, b] - V^-_k [g; a, b] \}
\]

Since \( \pi (x_0, x_1, ..., x_n) \) is arbitrary, it follows that

\[
V_k [g; a, b] \leq k \{ V^+_k [g; a, b] - V^-_k [g; a, b] \}
\]

This proves the theorem.

**Theorem 4.** If \( g \in AC^+_k [a, b] \) (or \( AC^-_k [a, b] \)), then \( g \in BV_k [a, b] \).

Proof. We prove the theorem in case \( g \in AC^+_k [a, b] \). The other case is analogous.

There exists a \( \delta (1) = \delta > 0 \) such that, for every elementary system \( I \) in \([a, b]\), we have

\[
\sigma I \leq 1 \quad \text{whenever} \quad mI < \delta.
\]

We subdivide \([a, b]\) into a finite number of subintervals \([c_0, c_1], [c_1, c_2], ..., [c_{N-1}, c_N]\) (\( a = c_0 < c_1 < ... < c_N = b \)) such that \( c_{r+1} - c_r < \delta \) for each \( r = 0, 1, ..., N-1 \).

For any elementary system \( I (x^{(r)}_0, ..., x^{(r)}_{k-1}): (x^{(r)}_0, x^{(r)}_k) \) in \([c_r, c_{r+1}]\), we have, from (15) \( \sigma I < 1 \). Consequently \( V^+_k [g; c_r, c_{r+1}] \leq 1 \). This implies, by Definition 4 and Lemma 3, that \( g \in BV^+_k [c_r, c_{r+1}] \). By Lemma 6, it therefore follows that \( g \in BV^+_k [a, b] \). Hence, by Theorem 3, \( g \in BV_k [a, b] \). This proves the theorem.