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On absolutely kth continuous functions

by

A. G. Das and **B. K. Lahiri** (Kalyani)

Abstract. In [3] and [4] Russell derives some properties of the functions of bounded kth variation (BV_k functions). Here we introduce the notion of AC_k functions and obtain some relations with those of BV_k functions involving kth Riemann $*$-derivative. We also refine the definitions of BV_k and AC_k functions to obtain the classes of BV_k^+, BV_k^-, AC_k^+ and AC_k^- functions and then study various interrelations of these classes.

1. Preliminaries and definitions. A. M. Russell in [3] obtained the definition of functions of bounded kth variation (BV_k functions). In the definition there were certain restrictions which he removed in [4], where he investigated in detail the properties of functions of bounded kth variation. Prior to [3], [4], he obtained in [2] the properties of functions of second variation. In this paper we introduce the notion of AC_k functions and investigate their properties. Also from BV_k functions we derive the notions of BV_k^+ and BV_k^- functions and obtain their relations. In the sequel, we shall need the following definitions and results from [4].

Definition 1(a). Let f be a real-valued function defined on $[a, b]$ and let $x_0, x_1, \ldots, x_k$ be $k+1$ distinct points, not necessarily in the linear order, belonging to $[a, b]$. Define the *k-th divided difference of f* as

$$Q_k(f;\ x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k} \Big[f(x_i)\Big/\prod_{\substack{j=0\\ j\neq i}}^{k}(x_i-x_j)\Big].$$

Definition 2. Let $x, x_1, \ldots, x_k$ be $k+1$ distinct points in $[a, b]$. Suppose that $h_i = x_i - x$ when $i = 1, 2, \ldots, k$ and that

$$0<|h_1|<|h_2|<\ldots<|h_k|.$$

Then define the *k-th Riemann $*$-derivative* by

$$D^k f(x) = k! \lim_{h_k\to 0} \lim_{h_{k-1}\to 0} \ldots \lim_{h_1\to 0} Q_k(f;\ x, x_1, \ldots, x_k),$$

if the iterated limit exists. The right and the left Riemann $*$-derivative are defined in the obvious way.

This definition has certain connections with the kth Riemann derivative as discussed in [1].

We shall call a subdivision of $[a, b]$ at $x_0, x_1, \ldots, x_n$ a π-subdivision of $[a, b]$ when $a \leqslant x_0 < x_1 < \ldots < x_n \leqslant b$ and denote it by $\pi(x_0, x_1, \ldots, x_n)$.

DEFINITION 3. The *total k-th variation of f in* $[a, b]$ is defined by

$$V_k[f;\ a, b] = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) |Q_k(f;\ x_i, x_{i+1}, \ldots, x_{i+k})| .$$

If $V_k[f;\ a, b] < \infty$, we say that f is of *bounded k-th variation on* $[a, b]$ and write $f \in \mathrm{BV}_k[a, b]$. The summations over which the sup. is taken are called *approximating sums* for $V_k[f;\ a, b]$.

LEMMA 1. *$Q_k(f;\ x_0, x_1, \ldots, x_k) = 0$ for all choices of $x_0, x_1, \ldots, x_k$ iff f is a polynomial of degree $k-1$ at most.*

LEMMA 3. *$Q_k(f;\ x_0, x_1, \ldots, x_k)$ is independent of the order in which the points $x_0, x_1, \ldots, x_k$ appear.*

LEMMA 4. *If $x_0, x_1, \ldots, x_k$ are any $k+1$ distinct points of $[a, b]$, then*

$$(x_0 - x_k) Q_k(f;\ x_0, x_1, \ldots, x_k) = Q_{k-1}(f;\ x_0, \ldots, x_{k-1}) - Q_{k-1}(f;\ x_1, \ldots, x_k) .$$

THEOREM 3. *The addition of extra points of subdivision to an existing subdivision does not decrease the approximating sums for $V_k[f;\ a, b]$.*

THEOREM 6. *If $f \in \mathrm{BV}_k[a, c]$, $f \in \mathrm{BV}_k[c, b]$ and f has a $(k-1)$-th Riemann $*$-derivative at c, where $a < c < b$, then $f \in \mathrm{BV}_k[a, b]$ and*

$$V_k[f;\ a, b] \leqslant V_k[f;\ a, c] + V_k[f;\ c, b] .$$

2. Let $a \leqslant x_{1,0} < x_{1,1} < \ldots < x_{1,k-1} < x_{1,k} \leqslant x_{2,0} < x_{2,1} < \ldots < x_{2,k-1} < x_{2,k} \leqslant \ldots \leqslant x_{n,0} < x_{n,1} < \ldots < x_{n,k-1} < x_{n,k} \leqslant b$ be any subdivision of $[a, b]$. We say that the intervals $(x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$ form an *elementary system I*, say, in $[a, b]$. The system is denoted by $I(x_{i,1}, \ldots, x_{i,k-1})\colon (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$. The elementary system consisting of the intervals $(a, x_{1,0}), (x_{1,k}, x_{2,0}), \ldots, (x_{n,k}, b)$ is said to be the *elementary system complementary to I* and will be denoted by I_c.

DEFINITION 1. The real-valued function $g(x)$ defined on $[a, b]$ is said to be *absolutely k-th continuous on* $[a, b]$ if for an arbitrary $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for any elementary system $I(x_{i,1}, \ldots, x_{i,k-1})\colon (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$, in $[a, b]$ with $\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \delta$ the relation

$$\sigma|I| = \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) |Q_k(g;\ x_{i,0}, x_{i,1}, \ldots, x_{i,k})| < \varepsilon$$

is satisfied. In this case we write $g \in \mathrm{AC}_k[a, b]$. The sum $\sum_{i=1}^{n} (x_{i,k} - x_{i,0})$ will be denoted by mI.

LEMMA 1. *If $g \in \mathrm{AC}_k[a, b]$, then g possesses the $(k-1)$-th Riemann $*$-derivative in (a, b).*

Proof. Let $a < c < b$ and $\varepsilon > 0$ be arbitrary. There exists a $\delta(\varepsilon) > 0$ such that the condition of the definition of $\mathrm{AC}_k[a, b]$ functions is satisfied with ε replaced by $\varepsilon/(k-1)!(k-1)$. We choose points $x_{p-k+1} < x_{p-k+2} < \ldots < x_{p-1} < x_p = c < x_{p+1} < \ldots < x_{p+k-1}$ such that $(x_{p+k-1} - x_{p-k+1}) < \delta$.

Choose a positive integer i such that $p-k+1 \leqslant i \leqslant p-1$ and consider the elementary system consisting of a single interval

$$I(x_{i+1}, \ldots, x_{i+k-1})\colon (x_i, x_{i+k}) .$$

Using Lemma 4 of [4], we get

$$\begin{aligned} &|Q_{k-1}(g;\ x_{i+1}, \ldots, x_{i+k}) - Q_{k-1}(g;\ x_i, \ldots, x_{i+k-1})| \\ &\qquad = (x_{i+k} - x_i) |Q_k(g;\ x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k})| < \varepsilon/(k-1)!(k-1) . \end{aligned}$$

Since i may assume $k-1$ values in $p-k+1 \leqslant i \leqslant p-1$, we see that the above inequality is true for any one of these values of i, viz. for $i = p-k+1, \ldots, p-1$.

It may be noted, however, that these $k-1$ intervals taken together do not form an elementary system, because the intervals are overlapping.

Combining now the $k-1$ inequalities, we obtain

$$|(k-1)!\, Q_{k-1}(g;\ x_i, \ldots, x_{i+k-1}) - (k-1)!\, Q_{k-1}(g;\ x_j, \ldots, x_{j+k-1})| < \varepsilon$$

for $i = p-k+1, \ldots, p-1, p$ and $j = p-k+1, \ldots, p-1, p$.

Hence $g(x)$ possesses the $(k-1)$-th Riemann $*$-derivative at c.

Note. With suitable modifications it may be shown that $D^k_+ g(a)$ and $D^k_- g(b)$ exist.

THEOREM 1. *If $g \in \mathrm{AC}_k[a, b]$, then $g \in \mathrm{BV}_k[a, b]$.*

Proof. There exists a $\delta(1) = \delta > 0$ such that for any elementary system $I(x_{i,1}, \ldots, x_{i,k-1})\colon (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$, in $[a, b]$ with $\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \delta$ the relation

$$(1) \qquad \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) |Q_k(g;\ x_{i,0}, \ldots, x_{i,k})| < 1$$

is satisfied.

The interval $[a, b]$ is broken up into a finite number of sub-intervals $[c_0, c_1], [c_1, c_2], \ldots, [c_{N-1}, c_N]$ $(a = c_0 < c_1 \ldots < c_N = b)$ such that $(c_{s+1} - c_s) < \delta$ for each $s = 0, 1, \ldots, N-1$.

We keep s fixed temporarily and consider any $\pi(x_0, x_1, \ldots, x_n)$ subdivision of $[c_s, c_{s+1}]$.

The sets of intervals (x_i, x_{i+k}), $i \in A_r = \{r, k+r, 2k+r, \ldots \leqslant n\}$ and $r = 0, 1, 2, \ldots, k-1$ form k elementary systems

$$I_r(x_{i+1}, \ldots, x_{i+k-1})\colon (x_i, x_{i+k}) , \quad i \in A_r \text{ and } r = 0, 1, 2, \ldots, k-1 .$$

So, by (1), we get

$$\sum_{i\in A_r}(x_{i+k}-x_i)|Q_k(g;\,x_i,x_{i+1},\ldots,x_{i+k})|<1$$

for $r=0,1,2,\ldots,k-1$. Combining all the inequalities, we obtain

$$\sum_{i=0}^{n-k}(x_{i+k}-x_i)|Q_k(g;\,x_i,x_{i+1},\ldots,x_{i+k})|<k\,.$$

Since this is true for any π-subdivision of $[c_s, c_{s+1}]$, it follows that $V_k[g;\,c_s, c_{s+1}]\leqslant k$, so that $g\in \mathrm{BV}_k[c_s, c_{s+1}]$ for $s=0,1,\ldots,N-1$. Hence, by Theorem 6 of [4] and Lemma 1, we infer that $g\in\mathrm{BV}_k[a,b]$. This proves the theorem.

THEOREM 2. *If the k-th Riemann *-derivative of a function* $g(x)\in \mathrm{AC}_k[a,b]$ *is zero almost everywhere in* $[a,b]$, *then the function* $g(x)$ *is a polynomial of degree* $(k-1)$ *at most.*

To prove the theorem we require the following lemma.

LEMMA 2. *Under the suppositions of the above theorem*

$$V=V_k[g;\,a,b]=0\,.$$

Proof of the lemma. By Theorem 1, $g(x)\in\mathrm{BV}_k[a,b]$. Let $E=\{x\colon x\in(a,b)$ and $D^k g(x)=0\}$. Let $\varepsilon>0$ be arbitrary. By Lemma 3 of [4], $Q_k(g;\,x_i,x_{i+1},\ldots,x_{i+k})$ is independent of the order in which the points $x_i,x_{i+1},\ldots,x_{i+k}$ appear. Consequently, if $x\in E$, there exists a $\delta>0$ such that, for all choices of $2k$ points $\{x_i\}, -k\leqslant i\leqslant k$, $i\neq 0$, such that $x_{-k}<\ldots<x_{-1}<x<x_1<\ldots<x_k$ with $(x_k-x_{-k})<\delta$, the relation

$$|Q_k(x_i,x_{i+1},\ldots,x_{i+k})|<\varepsilon/2(k+1)(b-a) \tag{2}$$

is satisfied for $i=-k,\ldots,0$, where $x_0=x$, $Q_k(x_i,\ldots,x_{i+k})=Q_k(g;\,x_i,\ldots,x_{i+k})$.

Now $g\in\mathrm{AC}_k[a,b]$, and so, corresponding to $\varepsilon/2k$, there exists a $\delta'>0$ such that for every elementary system I in $[a,b]$ with $mI<\delta'$ we have

$$\sigma|I|<\varepsilon/2k\,. \tag{3}$$

It is clear that the closed intervals like $[x_{-1},x_1]$ associated with each $x\in E$ for which condition (2) is satisfied cover the set E in Vitali's sense. Hence we can select from them a finite number of pairwise disjoint closed intervals $d_i=[x_{i,-1},x_{i,1}]$ with $x_{i-1,k}<x_{i,-k}<\ldots<x_{i,-2}<x_{i,-1}<x_{i,0}<x_{i,1}<\ldots<x_{i,k}$; $i=0,1,\ldots,n$, where $x_{i,0}=x_i$, $x_{0,-k}=a$, $x_{n,k}=b$ such that

$$m^*[E-\sum_{i=0}^{n}d_i]<\delta'\,. \tag{4}$$

Since $x_{i-1,k}<x_{i,-k}$ for $i=0,1,\ldots,n$, it is clear that the intervals $[x_{i,-k},x_{i,k}]$, $i=0,1,\ldots,n$ are disjoint.

We consider any $\pi(y_0,y_1,\ldots,y_m)$ subdivision of $[a,b]$. We discuss the following possibilities:

(i) If $y_p\in[x_{i,-k},x_{i,k}]$ for some i, $0\leqslant p\leqslant m$, then y_p may or may not coincide with any $x_{i,j}$, $-k\leqslant j\leqslant k$. Since the relation (2) is satisfied for all choices of $x_{i,j}$ in $[x_{i,-k},x_{i,k}]$, we may take y_p coincident with a suitable $x_{i,j}$.

(ii) If $x_{i,k}<y_p<x_{i+1,-k}$, then we can easily introduce a new interval and a set of points satisfying (2) and (4) and y_p coinciding with a suitable new point.

Thus in any case we may suppose that y_p, $0\leqslant p\leqslant m$, coincides with $x_{i,j}$ for some i and some j for $0\leqslant i\leqslant n$, $-k\leqslant j\leqslant k$.

Let j be any positive integer such that $1\leqslant j\leqslant k$. Consider the elementary system

$$I_j(x_{i-1,j+1},\ldots,x_{i,j-k-2})\colon (x_{i-1,j},x_{i,j-k-1})\,,\qquad i=1,2,\ldots,n$$

where $x_{i,-k-1}=x_{i-1,k}$ and $x_{i-1,k+1}=x_{i,-k}$.

As j ranges from 1 to k, we obtain k numbers of elementary systems

$$I_j(x_{i-1,j+1},\ldots,x_{i,j-k-2})\colon (x_{i-1\,j},x_{i,j-k-1})\,,\qquad i=1,2,\ldots,n$$

where $x_{i,-k-1}=x_{i-1,k}$ and $x_{i-1,k+1}=x_{i,-k}$.

From (3), it follows that

$$\sum_{i=1}^{n}Q_{i,j}=\sum_{i=1}^{n}(x_{i,j-k-1}-x_{i-1,j})|Q_k(x_{i-1,j},\ldots,x_{i,j-k-1})|<\varepsilon/2k$$

for $j=1,2,\ldots,k$ and so

$$\sum_{j=1}^{k}\sum_{i=1}^{n}Q_{i,j}<\varepsilon/2\,. \tag{5}$$

On the other hand, by using (2),

$$Q'_{i,j}=(x_{i,j+k}-x_{i,j})|Q_k(x_{i,j},\ldots,x_{i,j+k})|<\frac{\varepsilon(x_{i,j+k}-x_{i,j})}{2(k+1)(b-a)}\,,$$

$i=0,1,\ldots,n$ and $j=-k,\ldots,0$.

This gives

$$\sum_{j=-k}^{0}Q'_{i,j}<\frac{\varepsilon(x_{i,k}-x_{i,-k})}{2(b-a)}\,,\qquad i=0,1,\ldots,n\,.$$

Consequently

$$\sum_{i=0}^{n}\sum_{j=-k}^{0}Q'_{i,j}<\varepsilon/2\,. \tag{6}$$

Since the addition of extra points does not decrease the approximating sum for V {Theorem 3, [4]}, we have

$$\sum_{i=0}^{m-k}(y_{i+k}-y_i)|Q_k(y_i,y_{i+1},\ldots,y_{i+k})|\leqslant\sum_{j=1}^{k}\sum_{i=1}^{n}Q_{i,j}+\sum_{i=0}^{n}\sum_{j=-k}^{0}Q'_{i,j}<\varepsilon/2+\varepsilon/2=\varepsilon\,,$$

by (5) and (6).

Since π is any subdivision of $[a, b]$ and $\varepsilon>0$ is arbitrary, it follows that

(7) $$V = V_k[g;\ a, b] = 0.$$

Proof of the theorem. We consider k fixed points $a_0<a_1<\ldots<a_{k-1}$ of (a, b). We show that, for all choices of k points $\{y_i\}$, $i = 0, 1, \ldots, k-1$, such that $a\leqslant y_0<y_1<\ldots<y_{k-1}\leqslant b$ where $a = y_0$ and $b = y_{k-1}$ do not hold simultaneously,

(8) $$Q_{k-1}(y_0, y_1, \ldots, y_{k-1}) = \text{a constant} = Q_{k-1}(a_0, a_1, \ldots, a_{k-1}).$$

For any collection of $2k$ points $\{x_i\}$, $i = 0, 1, \ldots, 2k-1$ with $a\leqslant x_0<x_1<\ldots<x_{2k-1}\leqslant b$,

(9) $$\begin{aligned}&|Q_{k-1}(x_0, x_1, \ldots, x_{k-1})-Q_{k-1}(x_k, \ldots, x_{2k-1})|\\ &= |\{Q_{k-1}(x_0, \ldots, x_{k-1})-Q_{k-1}(x_1, \ldots, x_k)\}+\{Q_{k-1}(x_1, \ldots, x_k)-\\ &\quad -Q_{k-1}(x_2, \ldots, x_{k+1})\}+\ldots+\{Q_{k-1}(x_{k-1}, \ldots, x_{2k-2})-Q_{k-1}(x_k, \ldots, x_{2k-1})\}|\\ &\leqslant \sum_{i=0}^{k-1}|Q_{k-1}(x_i, \ldots, x_{i+k-1})-Q_{k-1}(x_{i+1}, \ldots, x_{i+k})|\\ &= \sum_{i=0}^{k-1}(x_{i+k}-x_i)|Q_k(x_i, x_{i+1}, \ldots, x_{i+k})|, \text{ using Lemma 4 of [4]}\\ &\leqslant V_k[g;\ a, b] = V,\end{aligned}$$

by Definition 3 of [4].

Suppose that $y_{k-1}<b$ and choose y_{k+i}, $i = 0, 1, \ldots, k-1$ such that

$$\max(a_{k-1}, y_{k-1})<y_k<\ldots<y_{2k-1}\leqslant b.$$

Applying (9) to the subdivision $\{x_i\}$, $i = 0, 1, \ldots, 2k-1$ where $x_i = a_i$ $(0\leqslant i\leqslant k-1)$ and $x_i = y_i$ $(k\leqslant i\leqslant 2k-1)$, we obtain

$$|Q_{k-1}(a_0, a_1, \ldots, a_{k-1})-Q_{k-1}(y_k, y_{k+1}, \ldots, y_{2k-1})|\leqslant V.$$

By using (9) again

$$|Q_{k-1}(a_0, a_1, \ldots, a_{k-1})-Q_{k-1}(y_0, y_1, \ldots, y_{k-1})|\leqslant 2V.$$

And so by Lemma 2

$$Q_{k-1}(a_0, a_1, \ldots, a_{k-1}) = Q_{k-1}(y_0, y_1, \ldots, y_{k-1}).$$

If $x_0, x_1, \ldots, x_k$ is any collection of $k+1$ distinct points of $[a, b]$, then, using Lemma 4 of [4] and relation (8), we have $Q_k(x_0, x_1, \ldots, x_k) = 0$.

The theorem now follows from Lemma 1 of [4].

We consider an elementary system of $[a, b]$

$$I(x_{i,1}, \ldots, x_{i,k-1}):\ (x_{i,0}, x_{i,k}),\quad i = 1, 2, \ldots, n$$

and write

$$\sigma I = \sum_{i=1}^{n}(x_{i,k}-x_{i,0})\,Q_k(g;\ x_{i,0}, x_{i,1}, \ldots, x_{i,k})$$

where $mI = \sum_{i=1}^{n}(x_{i,k}-x_{i,0})$.

Definition 2. The function $g(x)$ is said to be *absolutely k-th continuous from above on* $[a, b]$ if for an arbitrary $\varepsilon>0$ there exists a $\delta>0$ such that for any elementary system $I(x_{i,1}, \ldots, x_{i,k-1})$: $(x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$, in $[a, b]$ with $mI<\delta$ the relation $\sigma I<\varepsilon$ is satisfied. It is said to be *absolutely k-th continuous from below* if the relation $\sigma I>-\varepsilon$ holds when $mI<\delta$. If $g(x)$ is absolutely kth continuous from above (or from below) on $[a, b]$, we write $g\in \mathrm{AC}_k^+[a, b]$ (or $g\in \mathrm{AC}_k^-[a, b]$).

Definition 3. The least upper bound and the greatest lower bound of the aggregate $\{\sigma I\}$ of sums σI for all possible elementary systems I in $[a, b]$ are called, respectively, the *positive* and the *negative k-th variation of* $g(x)$ *in* $[a, b]$ and are designated by $V_k^+[g;\ a, b]$ and $V_k^-[g;\ a, b]$.

Henceforth we shall assume that $(k-1)$-th divided differences of g in $[a, b]$ are bounded in absolute value by K.

Lemma 3. $V_k^+[g;\ a, b]\geqslant 0$ *and* $V_k^-[g;\ a, b]\leqslant 0$.

The proof is omitted.

Definition 4. If $V_k^+[g;\ a, b]<+\infty$, we say that $g\in \mathrm{BV}_k^+[a, b]$ and if $V_k^-[g;\ a, b]>-\infty$, then $g\in \mathrm{BV}_k^-[a, b]$.

Lemma 4. *Let* $x_{1,0}<x_{1,1}<\ldots<x_{1,k-1}<x_{1,k}\leqslant x_{2,0}<x_{2,1}<\ldots<x_{2,k-1}<x_{2,k}\leqslant\ldots$ *be a set of points in* $[a, b]$. *If* $g\in \mathrm{BV}_k^+[a, b]$ (*or* $\mathrm{BV}_k^-[a, b]$), *then the series*

$$\sum_i(x_{i,k}-x_{i,0})|Q_k(g;\ x_{i,0}, \ldots, x_{i,k})|$$

is convergent.

Proof. We prove the lemma in case $g\in \mathrm{BV}_k^+[a, b]$. The other case is analogous.

Let $\{(\xi_{i,0}, \xi_{i,k})\}$, $i = 1, 2, \ldots$ be a subsequence of the sequence of all intervals $\{(x_{i,0}, x_{i,k})\}$, $i = 1, 2, \ldots$, with the intermediate points $x_{i,1}, \ldots, x_{i,k-1}$ renamed as $\xi_{i,1}, \ldots, \xi_{i,k-1}$, for each of which $[(\xi_{i,k}-\xi_{i,0})\,Q_k(g;\ \xi_{i,0}, \ldots, \xi_{i,k})]\geqslant 0$. If n is a positive integer, then, since $(\xi_{i,0}, \xi_{i,k})$, $i = 1, 2, \ldots, n$ form an elementary system in $[a, b]$, we have

(10) $$\sum_{i=1}^{n}(\xi_{i,k}-\xi_{i,0})\,Q_k(g;\ \xi_{i,0}, \ldots, \xi_{i,k})\leqslant V_k^+[g;\ a, b].$$

Since $V_k^+[g;\ a, b]<+\infty$ and n may be arbitrary, it follows that

$$\sum_i(\xi_{i,k}-\xi_{i,0})\,Q_k(g;\ \xi_{i,0}, \ldots, \xi_{i,k})$$

is convergent.

Next, let $\{(\eta_{i,0}, \eta_{i,k})\}$, $i = 1, 2, \ldots$ be a subsequence of $\{(x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots$ with $x_{i,1}, \ldots, x_{i,k-1}$ renamed as $\eta_{i,1}, \ldots, \eta_{i,k-1}$ for each of which

$$[(\eta_{i,k}-\eta_{i,0})\, Q_k(g; \eta_{i,0}, \ldots, \eta_{i,k})]<0 .$$

For a fixed positive integer n we consider an elementary system

$$I(\eta_{i,1}, \ldots, \eta_{i,k-1}):(\eta_{i,0}, \eta_{i,k}), \quad i = 1, 2, \ldots, n \text{ in } [a, b].$$

If I_c denotes the elementary system in $[a, b]$ complementary to I, then I and I_c together form an elementary system in $[a, b]$, which we denote by

$$J(\alpha_{i+1}, \ldots, \alpha_{i+k-1}):(\alpha_i, \alpha_{i+k}), \quad i = 0, k, 2k, \ldots, (m-1)k, mk$$

where $\alpha_0 = a$ and $\alpha_{(m+1)k} = b$. We consider $k-1$ elementary systems

$$J_r(\alpha_{i+1}, \ldots, \alpha_{i+k-1}):(\alpha_i, \alpha_{i+k}), \quad i \in A_r = \{r, k+r, 2k+r, \ldots \leqslant (m+1)k\}$$

for each $r = 1, 2, \ldots, k-1$, so that

$$\sigma I+\sigma I_c+\sigma J_1+\ldots+\sigma J_{k-1} = Q_{k-1}(g; \alpha_{mk+1}, \ldots, \alpha_{(m+1)k})-Q_{k-1}(g; \alpha_0, \alpha_1, \ldots, \alpha_{k-1}) .$$

Consequently, $\sigma I \geqslant -2K-kV_k^+[g; a, b]$ and so

$$\sum_{i=1}^{n} (\eta_{i,k}-\eta_{i,0})\, Q_k(g; \eta_{i,0}, \ldots, \eta_{i,k}) \geqslant -2K-kV_k^+[g; a, b] . \tag{11}$$

Since the left-hand expression is negative and n may be any positive integer, the series

$$\sum_i (\eta_{i,k}-\eta_{i,0})\, Q_k(g; \eta_{i,0}, \ldots, \eta_{i,k})$$

is convergent.

Because

$$\sum_i (x_{i,k}-x_{i,0})\,|Q_k(g; x_{i,0}, \ldots, x_{i,k})| \tag{12}$$
$$= \sum_i (\xi_{i,k}-\xi_{i,0})\, Q_k(g; \xi_{i,0}, \ldots, \xi_{i,k})-\sum_i (\eta_{i,k}-\eta_{i,0})\, Q_k(g; \eta_{i,0}, \ldots, \eta_{i,k}) ,$$

the lemma follows.

COROLLARY. *Under the hypotheses of Lemma 4, for any positive integer* $n \geqslant 1$

$$\sum_{i=1}^{n} (x_{i,k}-x_{i,0})\,|Q_k(g; x_{i,0}, \ldots, x_{i,k})| \leqslant (k+1)V_k^+[g; a, b]+2K .$$

Proof. It is seen that the right-hand quantities of (10) and (11) are independent of n. So, by using (10), (11) and (12), the corollary follows.

LEMMA 5. *If* $g \in \mathrm{BV}_k^+[a, b]$, *then* $g \in \mathrm{BV}_k^-[a, b]$ *and conversely.*

Proof. Suppose that $g \in \mathrm{BV}_k^+[a, b]$. We consider an elementary system

$$I(x_{i,1}, \ldots, x_{i,k-1}):(x_{i,0}, x_{i,k}), \quad i = 1, 2, \ldots, n \text{ in } [a, b] .$$

By the corollary there exists an $M>0$, independent of the choice of elementary systems, such that

$$\sigma I = \sum_{i=1}^{n} (x_{i,k}-x_{i,0})\, Q_k(g; x_{i,0}, \ldots, x_{i,k}) \geqslant -M ,$$

and so $V_k^-[g; a, b] \geqslant -M$. Consequently, by Lemma 3, $g \in \mathrm{BV}_k^-[a, b]$. The other case is similar.

LEMMA 6. *If* $g \in \mathrm{BV}_k^+[a, c]$ *and* $\mathrm{BV}_k^+[c, b]$, *where* $a<c<b$, *then* $g \in \mathrm{BV}_k^+[a, b]$ *and conversely. Further*

$$V_k^+[g; a, b] \geqslant V_k^+[g; a, c]+V_k^+[g; c, b] .$$

Proof. Suppose that $g \in \mathrm{BV}_k^+[a, c]$ and $\mathrm{BV}_k^+[c, b]$. Let

$$I(x_{i,1}, \ldots, x_{i,k-1}):(x_{i,0}, x_{i,k}), \quad i = 1, 2, \ldots, n$$

be any elementary system in $[a, b]$. We consider the following cases:

(a) If $x_{1,0} \geqslant c$ then $\sigma I \leqslant V_k^+[g; c, b]$.

(b) If $x_{n,k} \leqslant c$ then $\sigma I \leqslant V_k^+[g; a, c]$.

(c) If $x_{m,k} \leqslant c \leqslant x_{m+1,0}$ $(m<n)$, I can be presented as the sum of two elementary systems, I_1 in $[a, c]$ and I_2 in $[c, b]$, and so

$$\sigma I = \sigma I_1+\sigma I_2 \leqslant V_k^+[g; a, c]+V_k^+[g; c, b] .$$

(d) Let $x_{m,0}<c<x_{m,k}$, $1 \leqslant m \leqslant n$. Then the intervals $(x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, m-1$, form an elementary system I_1 in $[a, c]$ and $(x_{i,0}, x_{i,k})$, $i = m+1, \ldots, n$ form an elementary system I_2 in $[c, b]$. So

$$\begin{aligned}
\sigma I = &\sum_{i=1}^{m-1} (x_{i,k}-x_{i,0})\, Q_k(g; x_{i,0}, \ldots, x_{i,k})+\\
&+\sum_{i=m+1}^{n} (x_{i,k}-x_{i,0})\, Q_k(g; x_{i,0}, \ldots, x_{i,k})+\\
&+(x_{m,k}-x_{m,0})\, Q_k(g; x_{m,0}, \ldots, x_{m,k})\\
&\leqslant V_k^+[g; a, c]+V_k^+[g; c, b]+Q_{k-1}(g; x_{m,1}, \ldots, x_{m,k})\\
&-Q_{k-1}(g; x_{m,0}, \ldots, x_{m,k-1}), \text{ by Lemma 4 of [4]}\\
&\leqslant V_k^+[g; a, c]+V_k^+[g; c, b]+2K .
\end{aligned}$$

Since I is any elementary system in $[a, b]$, considering all the cases, we obtain $g \in \mathrm{BV}_k^+[a, b]$.

The converse part is clear. By Definition 3 it easily follows that

$$V_k^+[g; a, c]+V_k^+[g; c, b] \leqslant V_k^+[g; a, b] .$$

THEOREM 3. *If* $g \in \mathrm{BV}_k^+[a, b]$ (*or* $\mathrm{BV}_k^-[a, b]$), *then* $g \in \mathrm{BV}_k[a, b]$ *and*

$$V_k[g; a, b] \leqslant k\{V_k^+[g; a, b]-V_k^-[g; a, b]\} .$$

Proof. Suppose that $g \in \mathrm{BV}_k^+[a,b]$, then by Lemma 5, $g \in \mathrm{BV}_k^-[a,b]$. We consider any $\pi(x_0, x_1, \ldots, x_n)$ subdivision of $[a,b]$, $n>k$. Then

$$\sum_{i=0}^{n-k}(x_{i+k}-x_i)|Q_k(g; x_i, x_{i+1}, \ldots, x_{i+k})| = \sum_{i\in A_0} + \sum_{i\in A_1} + \ldots + \sum_{i\in A_{k-1}}, \tag{13}$$

where A_r contains the suffixes $r, k+r, 2k+r, \ldots \leqslant n$ for $r = 0, 1, \ldots, k-1$.

We now consider each A_r to be the union of two sets of suffixes A_r^+ and A_r^- such that $i \in A_r^+$ if $(x_{i+k}-x_i)Q_k(g; x_i, x_{i+1}, \ldots, x_{i+k}) \geqslant 0$ and $i \in A_r^-$ if

$$(x_{i+k}-x_i)Q_k(g; x_i, x_{i+1}, \ldots, x_{i+k}) < 0 .$$

Then, from (13) we get

$$\sum_{i=0}^{n-k}(x_{i+k}-x_i)|Q_k(g; x_i, x_{i+1}, \ldots, x_{i+k})| = \sum_{i\in A_0^+} + \ldots + \sum_{i\in A_{k-1}^+} - \sum_{i\in A_0^-} - \ldots - \sum_{i\in A_{k-1}^-}. \tag{14}$$

We thus obtain $2k$ elementary systems like

$$I_r^+(x_{i+1}, \ldots, x_{i+k-1}):(x_i, x_{i+k}), \quad i \in A_r^+,$$
$$I_r^-(x_{i+1}, \ldots, x_{i+k-1}):(x_i, x_{i+k}), \quad i \in A_r^-$$

where $r = 0, 1, \ldots, k-1$.

Hence, from (14), we get

$$\sum_{i=0}^{n-k}(x_{i+k}-x_i)|Q_k(g; x_i, x_{i+1}, \ldots, x_{i+k})| = \sigma I_0^+ + \ldots + \sigma I_{k-1}^+ - \sigma I_0^- - \ldots - \sigma I_{k-1}^-$$
$$\leqslant k\{V_k^+[g; a, b] - V_k^-[g; a, b]\} .$$

Since $\pi(x_0, x_1, \ldots, x_n)$ is arbitrary, it follows that

$$V_k[g; a, b] \leqslant k\{V_k^+[g; a, b] - V_k^-[g; a, b]\} .$$

This proves the theorem.

THEOREM 4. *If* $g \in \mathrm{AC}_k^+[a,b]$ (*or* $\mathrm{AC}_k^-[a,b]$), *then* $g \in \mathrm{BV}_k[a,b]$.

Proof. We prove the theorem in case $g \in \mathrm{AC}_k^+[a,b]$. The other case is analogous.

There exists a $\delta(1) = \delta > 0$ such that, for every elementary system I in $[a,b]$, we have

$$\sigma I < 1 \quad \text{whenever} \quad mI < \delta . \tag{15}$$

We subdivide $[a,b]$ into a finite number of subintervals $[c_0, c_1], [c_1, c_2], \ldots, [c_{N-1}, c_N]$ $(a = c_0 < c_1 < \ldots < c_N = b)$ such that $c_{r+1} - c_r < \delta$ for each $r = 0, 1, \ldots, N-1$.

For any elementary system $I_r(x_{i,1}^{(r)}, \ldots, x_{i,k-1}^{(r)})$: $(x_{i,0}^{(r)}, x_{i,k}^{(r)})$ in $[c_r, c_{r+1}]$, we have, from (15) $\sigma I_r < 1$. Consequently $V_k^+[g; c_r, c_{r+1}] \leqslant 1$. This implies, by Definition 4 and Lemma 3, that $g \in \mathrm{BV}_k^+[c_r, c_{r+1}]$. By Lemma 6, it therefore follows that $g \in \mathrm{BV}_k^+[a,b]$. Hence, by Theorem 3, $g \in \mathrm{BV}_k[a,b]$. This proves the theorem.

References

[1] P. S. Bullen, A. *Criterion for n-convexity*, Pacific J. Math. 36 (1971), pp. 81–98.
[2] A. M. Russell, *Functions of bounded second variation and Stieltjes type integrals*, J. Lond. Math. Soc. 2 (2) (1970), pp. 193–208.
[3] — *On functions of bounded k-th variation*, ibid. 3 (2) (1971), pp. 742-746.
[4] — *Functions of bounded k-th variation*, Proc. Lond. Math. Soc. 26 (3) (1973), pp. 547–563.

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