Remarks. 1) To get a theorem similar to Mycielski's (see introduction) we can extend Theorem 5.3 to the case where the number of relations is countable. This follows by the countable additivity of the measure P in 2^T.

2) Combining Theorems 3.3 and 5.3 it follows that almost every tree function is independent with perfect range in $[0, 1]$.

3) Theorem 4.3 ensures that Theorem 5.3 remains valid if the Hausdorff $(-1/\log t)^n$-measure of R is zero.

4) Theorem 4.4 ensures that Theorem 5.3 remains valid if R is of measure zero with respect to the product measure $(h_0\text{-measure})^n$.

5) The mapping $\alpha \to f_\alpha$ from 2^T to the space $[0, 1]^C$ is continuous. For any Borel set $B \subseteq [0, 1]^C$ we define $\mu(B) = P(\{\alpha \in 2^T: f_\alpha \in B\})$. Under this Borel measure, and for any h-null $R \subseteq [0, 1]^n$, almost every $f \in [0, 1]^C$ is independent over R.

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Sequence of iterates of generalized contractions

by

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Abstract. The main purpose of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if T is a generalized contraction mapping of closed, bounded and convex subset of a uniformly convex Banach space into itself with nonempty fixed points set, then the mapping T_λ defined by $T_\lambda = \lambda I + (1-\lambda)T$, for any λ such that $0<\lambda<1$ is asymptotically regular. As a corollary of this we get the result of Schaeffer (Jbr. Deutch. Math. Verein. (1957), pp. 131–140). In Section 2, we prove for Hilbert spaces the mapping T_λ as defined above is a reasonable wanderer. As a corollary of this we obtain the result of Browder and Petryshyn (J. Math. Anal. and Appl. 20 (1967), pp. 197–228). Finally in Sections 3 and 4, we have obtained some results for the weak and strong convergence of sequence of iterates for mappings of this type.

Introduction. The main aim of this paper is to study some properties of generalized contraction mappings. In Section 1 we have shown that if T is a generalized contraction mapping of a closed, bounded and convex subset of a uniformly convex Banach space into itself with non-empty fixed point set, then the mapping T_λ defined by $T_\lambda = \lambda I + (1-\lambda)T$, for any λ such that $0<\lambda<1$ is asymptotically regular. In section, it is shown that if T is a generalized contraction self mapping of a closed, convex subset of Hilbert space with non-empty fixed point set, then the mapping T_λ defined as above is a reasonable wanderer with the same fixed point as T. Finally in Sections 3 and 4 we have obtained some results for the weak and strong convergence of sequence of iterates of such kind of mappings.

DEFINITION 1.1. Let C be a closed, bounded and convex subset of a Banach space X. A mapping T: $C \to C$ is said to be *nonexpansive* if

$$||Tx-Ty|| \leqslant ||x-y|| \quad \text{for all } x, y \text{ in } C.$$

DEFINITION 1.2. A mapping T: $C \to C$ is said to be *quasi-nonexpansive* if

$$||Tx-Ty|| \leqslant a||x-y|| + b||x-Tx|| + c||y-Ty||$$

for all x, y in C, $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a+b+c \leqslant 1$.

The following example shows that there are quasi-nonexpansive mappings which are not nonexpansive.

EXAMPLE 1.1. Let $X = [0,1]$ and let $Tx = \frac{1}{3}x$, for $0\leqslant x<1$ and $T(1) = \frac{1}{6}$, then T is quasi-nonexpansive, but it is not nonexpansive.

DEFINITION 1.3. A mapping $T\colon C\to C$ is said to be a *generalized contraction* if for any x, y in C we have

$$\|Tx-Ty\|\leqslant\max\{\|x-y\|: \tfrac{1}{2}(\|x-Tx\|+\|y-Ty\|); \tfrac{1}{2}(\|x-Ty\|+\|y-Tx\|\}.$$

The following example shows that there are generalized contraction mappings which are not quasi-nonexpansive mappings.

EXAMPLE 1.2. Let

$$M_1 = \{m/n\colon m = 0, 1, 3, 9, \ldots, n = 1, 4, 7, \ldots, 3k+1, \ldots\},$$
$$M_2 = \{m/n\colon m = 1, 3, 9, 27, \ldots, n = 2, 5, \ldots, 3k+2 \ldots\},$$

and let $M = M_1\cup M_2$ with the usual metric. Define $T\colon M\to M$ as follows

$$T(x) = \begin{cases} \frac{2}{3}x & \text{for } x \text{ in } M_1, \\ \frac{1}{3}x & \text{for } x \text{ in } M_2. \end{cases}$$

Then T is generalized contraction. Indeed, if both x and y are in M_1 or in M_2, then $\|Tx-Ty\|\leqslant\frac{2}{3}\|x-y\|$. Therefore $\|Tx-Ty\|\leqslant\max\{\|x-y\|;\ \frac{1}{2}(\|x-Tx\|+\|y-Ty\|);\ \frac{1}{2}(\|x-Ty\|+\|y-Tx\|)\}$. If x in M_1 and y in M_2, then for $x = 1$, $y = \frac{1}{2}$ we have

$$\|Tx-Ty\| = \tfrac{2}{3}-\tfrac{1}{6} = \tfrac{1}{2}.$$

Also

$$\max\{\|x-y\|; \tfrac{1}{2}(\|x-Tx\|+\|y-Ty\|); \tfrac{1}{2}(\|x-Ty\|+\|y-Tx\|)\}$$
$$= \max\{\tfrac{1}{2}; \tfrac{1}{2}(\tfrac{1}{3}+\tfrac{1}{3}); \tfrac{1}{2}(\tfrac{5}{6}+\tfrac{1}{6})\} = \max\{\tfrac{1}{2}; \tfrac{1}{3}; \tfrac{1}{2}\} = \tfrac{1}{2}.$$

Thus

$$\|Tx-Ty\|\leqslant\max\{\|x-y\|; \tfrac{1}{2}(\|x-Tx\|+\|y-Ty\|); \tfrac{1}{2}(\|x-Ty\|+\|y-Tx\|)\}.$$

However, T is not quasi-nonexpansive. Indeed, taking $a = b = c = \frac{1}{3}$ we have

$$\tfrac{1}{3}\{\|x-y\|+\|x-Tx\|+\|y-Ty\|\} = \tfrac{1}{3}\{\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{2}\} = \tfrac{7}{18},$$

it follows that

$$\|Tx-Ty\|\nleqslant\tfrac{1}{3}\{\|x-y\|+x-Tx\|+\|y-Ty\|\}.$$

DEFINITION 1.4. A Banach space X is said to be *uniformly convex* if given $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\|x-y\|\geqslant\varepsilon$ for $\|x\|\leqslant 1$ and $\|y\|\leqslant 1$ implies $\|\frac{1}{2}(x+y)\|\leqslant 1-\delta(\varepsilon)$.

DEFINITION 1.5. Let X be a Banach space and C be a closed, convex subst of X. A mapping $T\colon C\to C$ is called *asymptotically regular* at x if and only if $\|T^nx-T^{n+1}x\|\to 0$ as $n\to\infty$.

THEOREM 1.1. *Let D be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X. Let $T\colon D\to D$ be a generalized contraction mapping. Let us suppose that $F = \{x \text{ in } D\colon Tx = x\}$ is nonempty. Then the mapping T_λ defined by $T_\lambda = \lambda T+(1-\lambda)I$ for any λ such that $0<\lambda<1$ is asymptotically regular with the same fixed point as T.*

Proof. It is clear that $F(T) = F(T_\lambda)$, where $F(T)$ and $F(T_\lambda)$ are the fixed point sets of T and T_λ respectively. Indeed, x in $F(T)$ implies $Tx = x$. Thus $T_\lambda x = \lambda x+(1-\lambda)x = x$, hence x in $F(T_\lambda)$, i.e. $F(T)$ is contained in $F(T_\lambda)$. Conversly y in $F(T_\lambda)$ implies $T_\lambda y = y = \lambda Ty+(1-\lambda)y$, which implies $\lambda Ty = \lambda y$, or $Ty = y$, Thus $F(T_\lambda)$ is contained in $F(T)$. Hence $F(T) = F(T_\lambda)$.

Let x_0 in D, $x_{n+1} = T_\lambda(x_n)$, $n = 0, 1, 2, \ldots$ Since $T_\lambda x-x = \lambda(x-Tx)$ for x in D, it is enough to show that $\|x_n-Tx_n\|\to 0$ as $n\to\infty$. Now let x in D and y in $F(T)$, hence in $F(T_\lambda)$, then

$$(1)\qquad \|Tx-y\| = \|Tx-Ty\|$$
$$\leqslant \max\{\|x-y\|; \tfrac{1}{2}(\|x-Tx\|+\|y-Ty\|); \tfrac{1}{2}(\|x-Ty\|+\|y-Tx\|)\}$$
$$= \max\{\|x-y\|; \tfrac{1}{2}\|x-Tx\|; \tfrac{1}{2}(\|x-y\|+\|y-Tx\|)\}.$$

Now

$$\|Tx-y\|\leqslant\tfrac{1}{2}\|x-Tx\|\leqslant\tfrac{1}{2}(\|x-y\|+\|y-Tx\|)$$

implies

$$\|Tx-y\|\leqslant\|x-y\|.$$

Thus from (1) we obtain

$$(2)\qquad \|Tx-y\|\leqslant\|x-y\|.$$

Also

$$(3)\qquad \|y-T_\lambda x)) = \|y-(1-\lambda)x-\lambda Tx\| = \|(1-\lambda)(y-x)+\lambda(y-Tx\|$$
$$\leqslant (1-\lambda)\|x-y\|+\lambda\|y-Tx\|\leqslant(1-\lambda)\|x-y\|+\lambda\|x-y\|$$
$$= \|y-x\|.$$

So the sequence $\{\|y-x_n\|\}$ is bounded by $M = \|y-x_0\|$. If $y = x_n$ for some n, then from (3), $\{x_n\}$ converges to y and the proof is complete. So we may assume that $y\neq x_n$ for all $n = 0, 1, 2, \ldots$ Suppose $\lambda\leqslant\frac{1}{2}$. Now

$$(4)\qquad \|y-x_{n+1}\| = \|\lambda(y-x_n+y-Tx_n)+(1-2\lambda)(y-x_n)\|$$
$$\leqslant \lambda\|y-x_n+y-Tx_n\|+(1-2\lambda)\|y-x_n\|$$
$$= (\|y-x_n\|)\lambda\left\|\frac{y-x_n+y-Tx_n}{\|y-x_n\|}\right\|+(1-2\lambda)\|y-x_n\|.$$

Let

$$(5)\qquad a = (y-x_n)/\|y-x_n\| \quad\text{and}\quad b = (y-Tx_n)/\|y-x_n\|,$$

then

$$\|a+b\| = \|y-x_n+y-Tx_n/\|y-x_n\|$$

and $\|a\|\leqslant 1$, $\|b\|\leqslant 1$. Thus from (4) we get

$$(6)\qquad \|y-x_{n+1}\|\leqslant 2\lambda\|\tfrac{1}{2}(a+b)\|\,\|y-x_n\|+(1-2\lambda)\|y-x_n\|$$
$$= \{2\lambda\|\tfrac{1}{2}(a+b)\|+(1-2\lambda)\}\|y-x_n\|.$$

Since X is uniformly convex

$$\delta(\varepsilon) = \inf\{1-||\tfrac{1}{2}(x+y)||:\ ||x||\leqslant 1,\ ||y||\leqslant 1,\ ||x-y||\geqslant\varepsilon\}$$

is positive for ε in $(0, 2]$. Also $\delta(0) = 0$. From (5) we have

$$||\tfrac{1}{2}(a+b)||\leqslant 1-\delta(||x_n-Tx_n||/||y-x_n||).$$

Since δ is monotonically nondecreasing on $[0, 2]$

(7) $$||\tfrac{1}{2}(a+b)||\leqslant 1-\delta(||x_n-Tx_n||/M).$$

Thus from (6) and (7) we have

(8) $$\begin{aligned}||y-x_{n+1}|| &\leqslant 2\lambda||y-x_n||(1-\delta(||x_n-Tx_n||/M)+(1-2\lambda)||y-x_n||\\ &= \{2\lambda-2\lambda\delta(||x_n-Tx_n||/M)(1-2\lambda)\}||y-x_n||\\ &= (1-2\lambda\delta(||x_n-Tx_n||/M)||y-x_n||.\end{aligned}$$

From (8) and induction we obtain

(9) $$||y-x_{n+1}||\leqslant\prod_{j=0}^{n}(1-2\lambda\delta(||x_j-Tx_j||/M)M.$$

Suppose $\{||x_n-Tx_n||\}$ does not converge to zero. Then there exists a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ such that $\{||x_{k(n)}-Tx_{k(n)}||\}$ converges to some constant α in $(0,\infty)$. Since δ is monotonically nondecreasing and $1-2\lambda\delta(||x_j-Tx_j||/M)$ belongs to $[0, 1]$ for each j, we have from (9) for sufficiently large n

$$||y-x_{k(n+1)}||\leqslant 1-2\lambda\delta(\alpha/2M)^n M.$$

So $\{x_{k(n)}\}$ converges to y, but then from (2) we get the convergence of $\{Tx_{k(n)}\}$ to y. Therefore $\{||x_{k(n)}-Tx_{k(n)}||\}$ converges to zero, a contradiction to the choice of α. If $\lambda\geqslant\frac{1}{2}$ then $1-\lambda\leqslant\frac{1}{2}$, we can apply the same kind of argument as above by replacing (5) as

$$\begin{aligned}||y-x_{n+1}|| &= ||(1-\lambda)(y-x_n+y-Tx_n)+(2\lambda-1)(y-Tx_n)||\\ &\leqslant(1-\lambda)||y-x_n+y-Tx_n||+(2\lambda-1)||y-Tx_n||\\ &=(1-\lambda)||y-x_n||\,||\tfrac{1}{2}(a+b)||+(2\lambda-1)||y-Tx_n||.\end{aligned}$$

By interchanging the roles of λ and $1-\lambda$ we can obtain as earlier a contradiction. Thus T_λ is asymptotically regular.

Remark 1.3. A theorem similar to our Theorem 1.1 for nonexpansive mappings was proved by Schaefer {15} and for quasi-nonexpansive mappings by the present author {19}.

DEFINITION 2.1. Let H be a Hilbert space and C be a closed, convex subset of H. A mapping $T\colon C\to C$ is said to be *reasonable wanderer* in C if starting at any point x_0 in C, its successive steps $x_n = T^n x_0$ $(n = 1, 2, 3, \ldots)$ are such that the sum of square of their lengths is finite, i.e.

$$\sum_{n=0}^{\infty}||x_{n+1}-x_n||^2<\infty.$$

THEOREM 2.1. *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let $T\colon C\to C$ be a generalized contraction mapping. Suppose F, the fixed point set of T in C is nonempty. Let $T_\lambda = \lambda I+(1-\lambda)T$ for any given λ with $0<\lambda<1$, then T_λ is a reasonable wanderer from C into C with the same fixed point as T.*

Proof. For any x in C, set $x_n = T_\lambda^n x$ and let y be a fixed point of T and, hence of T_λ. Then

(10) $$\begin{aligned}x_{n+1}-y &= \lambda x_n+(1-\lambda)Tx_n-\lambda y-(1-\lambda)y\\ &= \lambda(x_n-y)+(1-\lambda)(Tx_n-y).\end{aligned}$$

Now

(11) $$\begin{aligned}||Tx_n-y|| &= ||Tx_n-Ty||\\ &\leqslant\max\{||x_n-y||;\ \tfrac{1}{2}(||x_n-Tx_n||+||y-Ty||);\ \tfrac{1}{2}(||x_n-Ty||+\\ &\qquad+||y-Tx_n||)\}\\ &= \max\{||x_n-y||;\ \tfrac{1}{2}||x_n-Tx_n||;\ \tfrac{1}{2}(||x_n-y||+||y-Tx_n||)\}.\end{aligned}$$

Since

$$||Tx_n-y||\leqslant\tfrac{1}{2}||x_n-Tx_n||\leqslant\tfrac{1}{2}(||x_n-y||+||y-Tx_n||)$$

implies

$$||Tx_n-y||\leqslant||x_n-y||.$$

Thus from (11) we have

(12) $$||Tx_n-y||\leqslant||x_n-y||.$$

For any constant a, we have

(13) $$a(x_n-Tx_n) = a(x_n-y+y-Tx_n) = a(x_n-y)-a(Tx_n-y).$$

Using (10) we get

(14) $$\begin{aligned}||x_{n+1}-y||^2 &= \lambda^2||x_n-y||^2+(1-\lambda)^2||Tx_n-y||^2+2\lambda(1-\lambda)(Tx_n-y,\ x_n-y)\\ &\leqslant\lambda^2||x_n-y||^2+(1-\lambda)^2||x_n-y||^2+2\lambda(1-\lambda)(Tx_n-y,\ x_n-y)\\ &= \{\lambda^2+(1-\lambda)^2\}||x_n-y||^2+2\lambda(1-\lambda)(Tx_n-y,\ x_n-y).\end{aligned}$$

Using (13) we get

(15) $$\begin{aligned}a^2||x_n-Tx_n||^2 &= a^2||x_n-y||^2+a^2||Tx_n-y||^2-2a^2(Tx_n-y,\ x_n-y)\\ &\leqslant 2a^2||x_n-y||^2-2a^2(Tx_n-y,\ x_n-y).\end{aligned}$$

Adding (14) and (15) we obtain

(16) $$\begin{aligned}&||x_{n+1}-y||^2+a^2||x_n-Tx_n||^2\\ &\quad\leqslant\{2a^2+\lambda^2+(1-\lambda)^2\}||x_n-y||^2+2\{\lambda(1-\lambda)-a^2\}(Tx_n-y,\ x_n-y).\end{aligned}$$

If we assume that a is such that $a^2 \leqslant \lambda(1-\lambda)$, then from (16) and using Cauchy–Schwarz inequality we get

$$(17)\quad \|x_{n+1}y\|^2+a^2\|x_n-Tx_n\|^2 \leqslant \{2a^2+\lambda^2+(1-\lambda)^2\}\|x_n-y\|+\{2\lambda(1-\lambda)-2a^2\}\|x_n-y\|^2 = \|x_n-y\|^2 .$$

Letting $a^2 = \lambda(1-\lambda)>0$ and summing up (17) from $n=0$ to $n=N$ we get

$$\lambda(1-\lambda)\sum_{n=0}^{N}\|x_n-Tx_n\|^2 \leqslant \sum_{n=0}^{N}\{\|x_n-y\|^2-\|x_{n+1}-y\|^2\} \leqslant \|x_0-y\|^2-\|x_{N+1}-y\|^2 \leqslant \|x_0-y\|^2 .$$

Hence $\sum_{n=0}^{\infty}\|x_n-Tx_n\|^2<\infty$. Since $x_{n+1}-x_n = (1-\lambda)(Tx_n-x_n)$, we obtain

$$\lambda(1-\lambda)\sum_{n=0}^{\infty}\|x_n-Tx_n\|^2 = \lambda(1-\lambda)\sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^2}\|x_n-x_{n+1}\|^2 = \frac{\lambda}{1-\lambda}\sum_{n=0}^{\infty}\|x_{n+1}-x_n\|^2 \leqslant \|x_0-y\|^2 .$$

Hence

$$\sum_{n=0}^{\infty}\|x_{n+1}-x_n\|^2 \leqslant \frac{1-\lambda}{\lambda}\|x_0-y\|^2<\infty ,$$

i.e. T_λ is reasonable wanderer.

Remark 2.1. A similar theorem for nonexpansive mapping was proved by Browder and Petryshyn [2]. Unfortunately the theorem is not stated correctly there. The mapping T_λ should be defined by $T_\lambda = \lambda I+(1-\lambda)T$ instead of $T_\lambda = I+(1-\lambda)T$.

LEMMA 3.1. *Let H be a Hilbert space and C be a nonempty, convex subset of H. Let T: $C\to C$ be a generalized contraction mapping. Suppose F, the fixed point set of T in C is nonempty, then F is convex.*

Proof. We may assume that F consists of more than one point; otherwise the result is proved. Let x, y be in F. It is enough to show that $z = \lambda x+(1-\lambda)y$, $0<\lambda<1$ belongs to F. Since T is generalized contraction, from (2) we have

$$\|Tz-x\| \leqslant \|z-x\| \quad\text{and}\quad \|Tz-y\| \leqslant \|z-y\| .$$

Now $z-x = \lambda x+(1-\lambda)y-x = -(1-\lambda)(x-y)$. Hence $x-z = (1-\lambda)(x-y)$, and $z-y = \lambda(x-y)$. Thus we obtain

$$\|x-y\| \leqslant \|x-Tz\|+\|Tz-y\| \leqslant \|x-z\|+\|z-y\| = (1-\lambda)\|x-y\|+\lambda\|x-y\| = \|x-y\| .$$

Hence

$$\|x-Tz\|+\|Tz-y\| = \|x-Tz+Tz-y\| .$$

If $x-Tz = 0$, then $\|Tz-y\| = \|x-y\| \leqslant \|z-y\| = \lambda\|x-y\|$, whence $1\leqslant\lambda$, which is not true. Similarly $Tz-y = 0$ implies $1\leqslant 1-\lambda$, whence $\lambda\leqslant 0$, which is not true. Since H is strictly convex, therefore there exists $\alpha>0$ such that $Tz-x = \alpha(y-Tz)$, whence $Tz = (1-\beta)x+\beta y$, where $\beta = \alpha/(1+\alpha)$. We have $Tz-x = \beta(y-x)$ and so

$$\beta\|y-x\| = \|Tz-x\| \leqslant \|z-x\| = (1-\lambda)\|x-y\|$$

which gives $\beta\leqslant 1-\lambda$. Using $Tz-y = (1-\beta)(x-y)$, a similar argument gives $\beta\geqslant 1-\lambda$. Thus $\beta = 1-\lambda$ and so $Tz = \lambda x+(1-\lambda)y = z$, i.e. z belongs to F.

LEMMA 3.2 ([5], Proposition 2.5, pp. 53). *Let X be a Banach space and g a convex continuous real-valued function on X. Then g is weakly lower semicontinuous.*

LEMMA 3.3 ([5], Proposition 1.4, pp. 32). *Let X be a topological space and C be a compact subset of it. Let g: $X\to R$ be a lower semicontinuous function in X. Then there exists x_0 in C such that $g(x_0) = \inf_{x\in C} g(x)$.*

DEFINITION 3.1. Let X be a Banach space. A mapping T: $X\to X$ is said to be *demiclosed* if for any sequence $\{x_n\}$ such that $x_n\rightharpoonup x$ (i.e. x_n converges weakly to x) and $Tx_n\to y$, then $y = Tx$.

THEOREM 3.1. *Let H be a Hilbert space and T be a generalized contraction asymptotically regular mapping of H into itself. Suppose T is continuous and $I-T$ is demiclosed. Let F, the fixed point set of T in H be nonempty. Then for each x_0 in H, the sequence of iterates $\{T^n x_0\}$ converges weakly to a point of F.*

Proof. Since F is nonempty we see that a ball B about some fixed point and containing x_0 is mapped into itself by T; consequently B contains the sequence of iterates $T^n x_0$. So we may restrict ourselves to mappings of a ball into itself. It follows from Lemma 3.1 that F is convex. The continuity of T implies that F is closed. Thus F being closed, bounded and convex is weakly compact.

Define in F the following mapping g: $F\to R^+$ (R^+ = nonnegative real numbers)

$$(18)\qquad g(y) = \inf_n \|T^n x_0-y\| = \lim_{n\to\infty}\|T^n x_0-y\| .$$

(In (18) lim = inf, because the sequence $\{\|T^n x_0-y\|\}$ is nonincreasing, please see (20)). The mapping g so defined is continuous. Indeed,

$$g(z) = \lim\|T^n x_0-z\| \leqslant \lim\|T^n x_0-y\|+\|y-z\| = g(y)+\|y-z\|$$

from this inequality it follows that $|g(y)-g(z)| \leqslant \|y-z\|$. On the other hand, g is a convex function. In fact

$$\begin{aligned} g(\lambda y+(1-\lambda)z) &= \lim\|T^n x_0-(\lambda y+(1-\lambda)z\| \\ &= \lim\|\lambda T^n x_0-\lambda y+(1-\lambda)(T^n x_0-z)\| \\ &\leqslant \lambda\lim\|T^n x_0-y\|+(1-\lambda)\lim\|T^n x_0-z\| \\ &= \lambda g(y)+(1-\lambda)g(z) . \end{aligned}$$

Thus using Lemma 3.2 we see that g is weakly lower semicontinuous. Now applying Lemma 3.3 we conclude that there exists a point u in F such that

$$g(u) = \alpha = \inf_{y\in F} g(y).$$

Now we claim that u is unique. In fact, suppose this not so, i.e. there exists another point v in F such that $g(v) = \alpha$. Since g is convex, for $0 \leqslant \lambda \leqslant 1$ we have

$$g(\lambda u+(1-\lambda)v) \leqslant \lambda g(u)+(1-\lambda g(v) = \lambda\alpha+(1-\lambda)\alpha = \alpha.$$

Thus

$$\begin{aligned}\alpha \geqslant g(\lambda u+(1-\lambda)v) &= \inf\|T^n x_0-(\lambda u+(1-\lambda)v)\| \\ &= \inf\|\lambda(T^n x_0-u)+(1-\lambda)(T^n x_0-v)\| \\ &\geqslant \lambda\inf\|T^n x_0-u\|+(1-\lambda)\inf\|T^n x_0-v\| \\ &= g(u)+(1-\lambda)g(v) = \lambda\alpha+(1-\lambda)\alpha = \alpha.\end{aligned}$$

Hence, $g(\lambda u+(1-\lambda)v) = \alpha$. Since u is in F and T is generalized contraction, it follows that

$$\begin{aligned}(19)\quad \|T^n x_0-u\| &= \|T^n x_0-Tu\| \leqslant \max\{\|T^{n-1}x_0-u\|;\tfrac{1}{2}(\|T^{n-1}x_0-T^n x_0\| \\ &\quad +\|u-Tu\|;\tfrac{1}{2}(\|u-T^n x_0\|+\|T^{n-1}x_0-T^n u\|)\} \\ &= \max\{\|T^{n-1}x_0-u\|;\tfrac{1}{2}\|T^{n-1}x_0-T^n x_0\|;\tfrac{1}{2}(\|u-T^n x_0\|+ \\ &\quad +\|T^{n-1}x_0-u\|)\}.\end{aligned}$$

Now

$$\|T^n x_0-u\| \leqslant \tfrac{1}{2}\|T^{n-1}x_0-T^n x_0\| \leqslant \tfrac{1}{2}(\|T^{n-1}x_0-u\|+\|u-T^n x_0\|$$

implies

$$\|T^n x_0-u\| \leqslant \|T^{n-1}x_0-u\|.$$

Thus from (19) we get

$$(20)\qquad \|T^n x_0-u\| \leqslant \|T^{n-1}x_0-u\|.$$

Similarly for v in F it follows that

$$(21)\qquad \|T^n x_0-v\| \leqslant \|T^{n-1}x_0-v\|.$$

So, the sequences $\{\|T^n x_0-u\|\}$ and $\{\|T^n x_0-v\|\}$ are nonincreasing. Therefore

$$\|x_n-u\| = \|T^n x_0-u\| \to \alpha \quad \text{and} \quad \|x_n-v\| = \|T^n x_0-v\| \to \alpha.$$

Thus from the uniform convexity of H, we conclude that $\|(x_n-u)-(x_n-v)\| \to 0$, i.e. $u = v$.

Finally it remains to show that the sequence $\{T^n x_0\}$ converges weakly to u. Suppose not, then by the reflexivity of H and the boundedness of the sequence $\{T^n x_0\}$, there exists a convergent subsequence of $\{T^{n(j)}x_0\}$ whose limit say z is different from u. Since T is asymptotically regular, it follows that the sequence $\{(I-T)(T^{n(j)}x_0)\}$ tends to zero as $n\to\infty$. Since by hypothesis $I-T$ is demiclosed, $(I-T)z = 0$, i.e. z a fixed point of T. We claim that $z = u$. Indeed, we have

$$\begin{aligned}\|T^{n(j)}x_0-u\|^2 &= \|T^{n(j)}x_0-z+z-u\|^2 \\ &= \|T^{n(j)}x_0-z\|^2+\|z-u\|^2+2Re(T^{n(j)}x_0-z,\, z-u).\end{aligned}$$

Taking limits we obtain

$$g(u) = g(z)+\|z-u\|$$

which is possible only if $z = u$. Thus the theorem.

THEOREM 3.2. *Let X be a reflexive Banach space and T an asymptotically regular generalized contraction mapping from X into itself. Suppose T is continuous and $I-T$ is demiclosed. Let $F(T)$, the fixed point set of T in X be nonempty. Then, for each x_0 in X, every subsequence of $\{T^n x_0\}$ contains a further subsequence which converges weakly to a fixed point of T. In particular, if $F(T)$ consists of precisely one point then the whole sequence $\{T^n x_0\}$ converges to this point.*

Proof. Let y be in $F(T)$. Since T is generalized contraction, it follows as in the derivation of (20) that $\|T^n x_0-y\| \leqslant \|x_0-y\|$. So the sequence $\{T^n x_0\}$ is bounded. Thus it follows from the reflexivity of X that every subsequence of $\{T^{n(j)}x_0\}$ contains a further subsequence, which we again denote by $\{T^{n(j)}x_0\}$ such that $T^{n(j)}x_0 \to y$. Now we show that y is a fixed point of T. Indeed, since $T^{n(j)}x_0 \to y$, it follows that $(I-T)T^{n(j)}x_0 \to (I-T)y$. On the other hand since T is asymptotically regular it follows that

$$(I-T)T^{n(j)}x_0 = T^{n(j)}x_0-T^{n(j)+1}x_0 \to 0 \quad \text{as } n\to\infty.$$

Thus $(I-T)y = 0$, i.e. y is fixed point of T. If $F(T)$ contains only one point y then the whole sequence must converge to y.

Remark 3.1. A theorem similar to our Theorem 3.1 for nonexpansive mapping was proved by Opial [10], and a theorem similar to our Theorem 3.2 was obtained by Browder and Petryshyn [1].

In the sequel we will prove some theorems for the strong convergence of sequence of iterates for the generalized contraction mapping. We will assume that T is continuous in the present section.

THEOREM 4.1. *Let X be a Banach space and T a generalized contractive asymptotically regular mapping of X into itself. Suppose that $F(T)$, the fixed point set of T in X is nonempty. Let us further assume that T satisfies the following condition:*

(A) *$I-T$ maps bounded closed sets into closed sets.*

Then, for any point x_0 in X, the sequence $\{T^n x_0\}$ converges strongly to some point in $F(T)$.

Proof. Let y be a fixed point of T. Since T is generalized contraction, it follows that

$$\|T^{n+1}x_0-y\| \leqslant \|T^n x_0-y\|, \quad n = 1, 2, 3, \ldots$$

So the sequence $\{T^n x_0\}$ is bounded. Let D be the strong closure of $\{T^n x_0\}$. By condition (A) it follows that $(I-T)(D)$ is closed. This together with the fact that T is asymptotically regular implies that zero belongs to $(I-T)(D)$. So there exists a z in D such that $(I-T)z = 0$. But this implies that either $z = T^n x_0$ for some n, or there exists a sequence $\{T^{n(j)} x_0\}$ converging to z. Since z is a fixed point of T, we can then conclude that in either case the whole sequence $\{T^n x_0\}$ converges to z.

COROLLARY 4.1. *Let X be a uniformly convex Banach space and T a generalized contraction mapping of X into itself. Suppose $F(T)$, the fixed point set of T in X is nonempty and T satisfies the following condition*:

(A) *$I-T$ maps closed, bounded sets into closed sets.*

Then, for each point x_0 in X, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \lambda x_n + (1-\lambda) T x_n, \quad 0<\lambda<1$$

converges strongly to a fixed point of T.

Proof. Let λ be such that $0<\lambda<1$. Let $T_\lambda = \lambda I + (1-\lambda)T$. It follows from Theorem 1.1 that T_λ is asymptotically regular. T satisfies condition (A) if and only if T_λ also does. Indeed, we just observe that $I-T_\lambda = (1-\lambda)(I-T)$. Let us observe that T_λ is not generalized contraction, however, for any y in $F(T)$ it follows from (3) that $\|T_\lambda x - y\| \leqslant \|x-y\|$. From this we can conclude that the sequence $\{T_\lambda^n x_0\}$ is bounded, hence the corollary follows from Theorem 4.1.

DEFINITION 4.1. A continuous mapping T from a Banach X into itself is said to be *demicompact* if every bounded sequence $\{x_n\}$ such that $\{(I-T)(x_n)\}$ converges strongly, contains a strongly convergent subsequence $\{x_{n(j)}\}$.

Remark 4.1. It follows from Proposition II.4 ([5], pp. 47) that a demicompact mapping T of a Banach space X into itself satisfies condition (A). Thus we have the following corollary:

COROLLARY 4.2. *Let X be a uniformly convex Banach space. Let T be a generalized contractive demicompact mapping of X into itself. Then, for each point x_0 in X, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \lambda x_n + (1-\lambda) T x_n, \quad 0<\lambda<1$$

converges strongly to a fixed point of T.

Remark 4.2. Theorem 4.1, Corollary 4.1 and Corollary 4.2 for nonexpansive mappings were proved by Browder and Petryshyn [1].

As our final result we prove following:

THEOREM 4.2. *Let X be strictly convex Banach space, and D be compact convex subset of X. Let $T\colon D\to D$ be a continuous generalized contraction. Then the fixed point set $F(T)$ of T is nonempty and compact, moreover for any x_0 in D and any λ such that $0<\lambda<1$, $\{T_\lambda^n x_0\}$ converges to a fixed point of T, where*

$$T_\lambda x = (1-\lambda)x + \lambda T x, \ x \text{ in } X.$$

Proof. By the continuity of T and the Schauder–Tychonoff theorem, it follows that $F(T)$, the fixed point set of T is compact and nonempty. Let $n\geqslant 0$, $x_n = T_\lambda^n x_0$. Since D is compact, $\{x_n\}$ has a convergent subsequence $\{x_{k(n)}\}$ which converges to some point x in D. We need to show that x is a fixed point of T. From (2) it follows that $\{\|x_n - y\|\}$, where y is a fixed point of T is monotonically non-increasing. So by the countinuity of norm $\|\ \|$ and T_λ we have

$$\|x-y\| = \lim_{n\to\infty} \|x_{k(n+1)} - y)\| \leqslant \lim_{n\to\infty} \|x_{k(n)+1} - y\| = \lim_{n\to\infty} \|T(x_{k(n)}) - y\| = \|T_\lambda x - y\|. \tag{22}$$

By (22) and (3) we obtain

$$\|T_\lambda x - y\| = \|x-y\|. \tag{23}$$

Moreover,

$$\|T_\lambda x - y\| = \|(1-\lambda)x + \lambda T x - y\| = \|(1-\lambda)(x-y) + \lambda(Tx-y)\| \leqslant (1-\lambda)\|x-y\| + \lambda\|x-y\| = \|x-y\|. \tag{24}$$

Combining (23) and (24) we conclude that all inequalities in (24) are equalities. So

$$\|(1-\lambda)(x-y) + \lambda(Tx-y)\| = (1-\lambda)\|x-y\| + \lambda\|Tx-y\| \tag{25}$$

and

$$\|Tx-y\| = \|x-y\|. \tag{26}$$

By (25) and strict convexity of X, either $x = y$, or $Tx - y = t(x-y)$ for some $t>0$. From (26) it follows that $t = 1$. Thus $Tx - y = x - y$, or $x = Tx$. Hence x is a fixed point of T. It follows from (24) that the sequence $\{\|x_n - x\|\}$ is monotonically non-increasing, hence $\{x_n\}$ converges to x.

Remark 4.3. The above result was proved by Edelstein [6] for nonexpansive mappings.

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Berechnung einiger Poincaré-Reihen

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Resümee. In dieser Arbeit werden mit den Methoden von Shamash (siehe J. of Algebra 17, 19) Poincaré–Reihen P_R gewisser lokaler Ringe berechnet. Es wird zunächst die Theorie von Shamash auf kommutative, endlich dimensionale k-Algebren ausgedehnt, eine obere Abschätzung der Poincaré–Reihe durch eine rationale Reihe und Kriterien für die Gleichheit in dieser Abschätzung angegeben. Als Anwendung davon werden Poincaré–Reihen von einigen k-Algebren der Gestalt $k[X_1, \dots, X_n]/\mathfrak{A}$ bestimmt, wobei $\mathfrak{A}$ ein Ideal ist, das von Monomen in den Unbestimmten X_i erzeugt wird. Außerdem wird gezeigt, daß jeder Cohen–Macauley–Ring, dessen Multiplizität kleiner oder gleich 5 ist, eine rationale Poincaré–Reihe besitzt. Schließlich wird für die Reduktion eines Ringes R modulo eines Elementes aus dem Sockel die Abschätzung der Poincaré–Reihe $P_{\bar{R}}$ des reduzierten Rings R: $P_{\bar{R}} \leqslant P_R(1-X^2P_R)^{-1}$ hergeleitet. Mit den benutzten Methoden läßt sich zeigen, daß für eine gewisse Klasse von Gorensteinringen in dieser Abschätzung Gleichheit gilt.

Das letztgenannte Resultat für beliebige artinsche Gorensteinringe wurde mit anderen Methoden von L. Avramov and G. Levin in den Stockholm Lecture Notes, No. 15, 1976, bewiesen.

Einleitung. Mit den Methoden, die Shamash in seinen Arbeiten [10], [11], [12] entwickelt hat, sollen hier die Poincaré–Reihen gewisser lokaler Ringe berechnet werden.

Im ersten Paragraphen wird die Theorie von Shamash kurz skizziert und gleichzeitig auf kommutative, endlich-dimensionale k-Algebren ausgedehnt. Im allgemeinen erhält man für die Poincaré–Reihen eine obere Abschätzung durch eine rationale Reihe, vgl. (1.2). In den Folgerungen (1.7) und (1.9) werden dann Kriterien angegeben, wann in (1.2) Gleichheit gilt.

In § 2 wird die Theorie auf Algebren mit monomialen Relationen angewandt, das heisst auf Algebren der Gestalt $k[X_1, \dots, X_n]/\mathfrak{A}$, wobei $\mathfrak{A}$ ein Ideal ist, das von Monomen in den Unbestimmten X_i erzeugt wird.

In § 3 zeigen wir, dass jeder Cohen–Macaulay–Ring, dessen Multiplizität kleiner oder gleich 5 ist, eine rationale Poincaré–Reihe besitzt.

Schliesslich untersuchen wir in § 4 das Verhalten der Poincaré–Reihe bei der Reduktion eines Rings modulo einem Element aus dem Sockel.

Ist (R, m) ein noetherscher lokaler Ring mit maximalem Ideal m und ist σ ein Element aus dem Sockel von R, dann gilt

$$P_{\bar{R}}(X) \leqslant P_R(X)\left(1-P_R(X)X^2\right)^{-1}, \tag{4.1}$$