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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Basic properties of h-regular local Noetherian rings

by

A. Tyc (Toruń)

Abstract. A commutative local Noetherian ring R is called *h-regular* if its maximal ideal is generated by a sequence $u_1, \ldots, u_{n_1}$ such that $u_i \neq 0$, $i = 1, \ldots, n$ and $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$, $k = 1, \ldots, n$, where h_k is the minimum of integers $m \geqslant 0$ such that $u_k^m = 0$ (if $u_k^m \neq 0$ for all m, then we put $h_k = \infty$ and $u_k^{\infty-1} = 0$).

In the paper basic properties of such rings are investigated. One proves, in particular, that if R is an h-regular local ring, then so are its completion $\hat{R}$ and every localization of R at a prime ideal. Moreover, under this assumption, the associated graded algebra $\mathrm{Gr}(R) = R/m \oplus m/m^2 \oplus \ldots$ is h-regular, i.e. the ideal $\bigoplus_{i>0} m^i/m^{i+1}$ is generated by an h-regular sequence of homogeneous elements. Also a characterization of h-regular local complete rings is given.

Introduction. Let R denote a commutative ring with identity. Recall that a sequence $u_1, \ldots, u_n$, $u_i \in R$, is called *regular in* R if $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1})$ for $k = 1, \ldots, n$ ($(u_1, \ldots, u_{k-1}) = 0$ for $k = 1$). In [8] the notion of a regular sequence has been generalized in the following way: let $u_1, \ldots, u_n$ be as above and let $h(u_i) = h_i$ be the minimum of integers $m \geqslant 0$ such that $u_i^m = 0$ (if there is no such an integer, put $h(u_i) = \infty$ and $u_i^\infty = u_i^{\infty-1} = 0$). The sequence $u_1, \ldots, u_n$ is said to be *h-regular in* R if $(u_1, \ldots, u_n) \neq R$ and $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$ for $k = 1, \ldots, n$. A local Noetherian (commutative) ring A is called *h-regular* if its unique maximal ideal is generated by an h-regular sequence.

The aim of this paper is an investigation of basic properties of h-regular local Noetherian rings. In particular, we show that if R is an h-regular local Noetherian ring then so is its completion $\hat{R}$, and the associated algebra $\mathrm{Gr}(R) = R/m \oplus m/m^2 \oplus \ldots$, where m is the maximal ideal of R, is an h-regular graded R/m-algebra in the sense of [8]. Also, a full description of complete h-regular local Noetherian rings is given.

In the whole paper the word "ring" means a commutative ring with identity. All local rings are assumed to be Noetherian.

Properties of h-regular sequences and h-regular local rings. Let R be a ring and let $u \in R$. The height of u (shortly $h(u)$) is the minimum of integers $m \geqslant 0$ such that $u^m = 0$. If $u^m \neq 0$ for all m, then we put $h(u) = \infty$.

1. Definition. A sequence $u_1, \ldots, u_n$, $0 \neq u_i \in R$, is called *h-regular in R* if the following conditions hold:

1° $(u_1, \ldots, u_n) \neq R$,

2° $(u_1, \ldots, u_{k-1})\colon (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$ [1], $k = 1, \ldots, n$, $h_k = h(u_k)$.

2. Definition. A local ring R is said to be *h-regular* if its unique maximal ideal is generated by an h-regular sequence.

It is clear that every regular local ring is an h-regular local ring and that every h-regular local ring without non-zero nilpotent elements is a regular local ring. In what follows we show (see Theorem 19) that if R is a regular local ring, $u_1, \ldots, u_n$ is a regular sequence of generators of the maximal ideal in R and $h_1, \ldots, h_n$ is a sequence of elements of the set $N^\infty = N \cup \{\infty\}$ (N will denote, as usual, the set of positive integers), then $R/(u_1^{h_1}, \ldots, u_n^{h_n})$ is an h-regular local ring.

3. Remark. It was proved in [8] that if $u_1, \ldots, u_n$ is an h-regular sequence in a local ring R then so is $u_{\sigma(1)}, \ldots, u_{\sigma(n)}$ for each permutation σ of the set $\{1, \ldots, n\}$. Moreover, it is easy to verify that each h-regular sequence $u_1, \ldots, u_n$ is a minimal set of generators of the ideal $(u_1, \ldots, u_n)$.

In the sequel we frequently use the following

4. Lemma. *If $u_1, \ldots, u_n$ is an h-regular sequence in a ring R, then for each i, $1 \leqslant i \leqslant n$, $u_1, \ldots, u_i$ is an h-regular sequence in R and the images $\bar{u}_{i+1}, \ldots, \bar{u}_n$ of u's in $\bar{R} = R/(u_1, \ldots, u_i)$ form an h-regular sequence in $\bar{R}$ such that $h(\bar{u}_j) = h(u_j)$, $j = i+1, \ldots, n$. Conversely, if there exists an i, $1 \leqslant i \leqslant n$, such that $u_1, \ldots, u_i$ is an h-regular sequence in R, $\bar{u}_{i+1}, \ldots, \bar{u}_n$ is an h-regular sequence in $\bar{R}$ and $h(\bar{u}_j) = h(u_j)$ for $j = i+1, \ldots, n$, then $u_1, \ldots, u_n$ is an h-regular sequence in R.*

The proof of the lemma is easy and is left to the reader.

5. Proposition. *Suppose $f\colon R \to R'$ is a homomorphism of rings such that R' is a flat R-module. Then, for each h-regular sequence $u_1, \ldots, u_n$ in R with $h(u_i) = h_i$, $v_1 = f(u_1), \ldots, v_n = f(u_n)$ is an h-regular sequence in R' whenever $(v_1, \ldots, v_n) \neq R'$.*

Proof. By the assumption, for all $i = 1, \ldots, n$

$$0 \to (u_1, \ldots, u_{i-1}, u_i^{h_i-1}) \hookrightarrow R \xrightarrow{u_i} R/(u_1, \ldots, u_{i-1})$$

is an exact sequence of R-modules. Hence the sequences

$$0 \to (u_1, \ldots, u_{i-1}, u_i^{h_i-1}) \otimes_R R' \to R \otimes_R R' \xrightarrow{u_i \otimes 1} R/(u_1, \ldots, u_i) \otimes_R R'$$

are exact because R' is a flat R-module. However, $I \otimes_R R' = f(I)R'$ for each ideal $I \subset R$ (again by the flatness of R'), and so it follows that

$$0 \to (v_1, \ldots, v_{i-1}, v_i^{h_i-1}) \hookrightarrow R' \xrightarrow{v_i} R'/(v_1, \ldots, v_i)$$

is an exact sequence for $i = 1, \ldots, n$. Consequently $(v_1, \ldots, v_{i-1})\colon (v_i) = (v_1, \ldots, v_{i-1}, v_i^{h_i-1})$ for all i, whence, making use of the inequalities $h(v_j) \leqslant h(u_j)$ and $(v_1, \ldots, v_n) \neq R'$, one easily gets $h(v_j) = h_j$. The proposition is proved.

Let R be a local ring with the maximal ideal m and let $\hat{R}$ denote the m-adic completion of R. Since $\hat{R}$ is a flat R-module and every set of generators of the ideal m is a set of generators of the maximal ideal $\hat{m}$ of $\hat{R}$, the above proposition yields

6. Corollary. *If R is an h-regular local ring, then so is its completion $\hat{R}$.*

Now one can also prove

7. Proposition. *If R is an h-regular local ring, then so is the localization $R_{\mathfrak{N}}$ of R at each prime ideal $\mathfrak{N} \subset R$.*

Proof. In virtue of the assumption the maximal ideal m of R is generated by an h-regular sequence $U = (u_1, \ldots, u_n)$. By Remark 3 we may assume that $u_1, \ldots, u_p$ is the maximal subsequence of U consisting of nilpotent elements. Then clearly $u_1, \ldots, u_p$ are in $\mathfrak{N}$ and $u_1' = u_1/1, \ldots, u_p' = u_p/1$ form an h-regular sequence in $R_{\mathfrak{N}}$ in view of Proposition 5. Hence, in order to prove that R is h-regular, it is sufficient to show, by using Lemma 4, that $R^* = R_{\mathfrak{N}}/(u_1', \ldots, u_p')$ is a regular local ring. It is obvious that $R^* = (R/(u_1, \ldots, u_p))_{\bar{\mathfrak{N}}}$ where $\bar{R}$ is the image of $\mathfrak{N}$ under the natural homomorphism $R \to \bar{R} = R/(u_1, \ldots, u_p)$. Furthermore, $\bar{R}$ is a regular local ring by Lemma 4. The conclusion now follows from the well-known fact that the localization at each prime ideal of a regular local ring is also a regular local ring.

Let R be a local ring with the maximal ideal m and the quotient field $k = R/m$. As usual, $\mathrm{Gr}(R)$ will denote the associated graded k-algebra $k \oplus m/m^2 \oplus \ldots$ We are now going to show that if R is an h-regular local ring, then $\mathrm{Gr}(R)$ is an h-regular graded k-algebra in the sense of [8], i.e. that its augmentation ideal $\bigoplus_{i>0} m^i/m^{i+1}$ is generated by an h-regular sequence of homogeneous elements. Let us start with the following

8. Proposition. *If u is an element of m and $h(u) \leqslant h$ for some $h \in N^\infty$, then the element $u^* = u + m^2 \in \mathrm{Gr}(R)$ is h-regular in $\mathrm{Gr}(R)$ and $h(u^*) = h$ if and only if u is h-regular in R, $h(u) = h$, and the natural homomorphism of graded k-algebras $p\colon \mathrm{Gr}(R) \to \mathrm{Gr}(R/(u))$ induces an isomorphism $p_*\colon \mathrm{Gr}(R)/(u^*) \to \mathrm{Gr}(R/(u))$.*

Proof. Suppose u^* is an h-regular element in $\mathrm{Gr}(R)$ and $h(u^*) = h$. First we show that u is h-regular in R and $h(u) = h$. Let $ru = 0$. Then $r \in m$ and certainly $(r+m^2)u^* = 0$. Hence $r = y_1' u^{h-1} + y_2$ for some $y_1' \in R$ and $y_2 \in m^2$, which implies $0 = ru = y_2 u$. Therefore $(y_2 + m^3)u^* = 0$ whence $y_2 = y_2' u^{h-1} + y_3$ with $y_3 \in m^3$. Consequently, $r = (y_1' + y_2')u^{h-1} + y_3$ and continuing this procedure we get sequences $r_n \in R$ and $y_n \in m^n$, $n = 2, 3, \ldots$ such that $r = r_n u^{h-1} + y_n$ for all n. It follows that $r \in \bigcap_n (Ru^{h-1} + m^n) = Ru^{h-1}$ by the Krull Intersection Theorem. Since $h(u) = h$ (otherwise $h(u^*) < h$!), u is h-regular in R.

Now we show that $p_*\colon \mathrm{Gr}(R)/(u^*) \to \mathrm{Gr}(R/(u))$ is an isomorphism. For this purpose it is sufficient to prove that for given n the kernel of the nth component of p:

$$p_n\colon \mathrm{Gr}(R)_n = m^n/m^{n+1} \to m^n + (u)/m^{n+1} + (u) = \mathrm{Gr}(R/(u))_n$$

[1] Throughout the paper the following conventions are in force: $\infty - i = \infty$ if $i < \infty$, $x^\infty = 0$, $x^0 = 1$, $(u_1, \ldots, u_m) = 0$ if $m = 0$.

is equal to $(\mathrm{Gr}(R)u^*)_n = \{au+m^{n+1};\ a\in m^{n-1}\}$. It is clear that $(\mathrm{Gr}(R)u^*)_n \subset \mathrm{Ker}(p_n)$. Let $b+m^{n+1}\in\mathrm{Ker}(p_n)$, i.e. $b+m^{n+1} = au+m^{n+1}$ and assume that $a\in m^i\setminus m^{i+1}$. If $i\geqslant n-1$, then $a\in m^{n-1}$ and $b+m^{n+1}\in(\mathrm{Gr}(R)u^*)_n$. Suppose $i<n-1$. In this case $(a+m^{i+1})u^* = b+m^{i+2} = 0$ because $b\in m^n\subset m^{i+2}$. Hence $a = a'u^{h-1}+a_1$ for some $a'\in R$ and $a_1\in m^{i+1}$, which gives $au = a_1 u$. Proceeding in this way we will finally find an $a''\in m^{n-1}$ such that $au = a''u$, whence $b+m^{n+1} = a''u+m^{n+1}\in(\mathrm{Gr}(R)u^*)_n$, as was to be shown. Thus we have proved the "if" part of the proposition. In order to prove "only if" one assumes that u is h-regular in R, $h(u) = h$, and that $p_*\colon \mathrm{Gr}(R)/(u^*)\to\mathrm{Gr}(R/(u))$ is an isomorphism. Moreover, take a homogeneous element $\bar a\in\mathrm{Gr}(R)$ such that $\bar a u^* = 0$. Then $\bar a = a+m^k$ for some k, $a\in m^{k-1}$ and $au\in m^{k+1}$. It follows that $au+m^{k+2}\in\mathrm{Ker}p_{k+1} = (\mathrm{Gr}(R)u^*)_{k+1}$ by the assumption, and therefore $au = a_k u+b_k$ where $a_k\in m^k$, $b_k\in m^{k+2}$. This means $(a-a_k)u\in m^{k+2}$, whence $(a-a_k)u+m^{k+3}\in\mathrm{Ker}p_{k+2} = (\mathrm{Gr}(R)u^*)_{k+2}$. So, there are $a_{k+1}\in m^{k+1}$ and $b_{k+1}\in m^{k+3}$ such that $au = a_{k+1}u+a_k u+b_{k+1} = a'_{k+1}u+b_{k+1}$ where $a'_{k+1}\in m^k$. Repeating this procedure, one gets elements $a'_{k+n}\in m^k$ and $b_{k+n}\in m^{k+n+2}$, $n = 0, 1, \ldots$, such that $au = a'_{k+n}u+b_{k+n}$ for each n. Hence $au\in\bigcap_n(m^k u+m^n) = m^k u$ by the Krull Intersection Theorem, i.e. $au = a'u$ for some $a'\in m^k$. Consequently, there is $a, b\in R$ such that $a = a'+bu^{h-1}$ because $(0)\colon(u) = (u^{h-1})$ by the assumption. Hence we conclude that $a+m^k = bu^{h-1}+m^k$ and it remains to find such a b which belongs to $m^{k-1+h+1}$. But this is a consequence of the lemma below (for $i = h-1$).

9. LEMMA. *If $(0)\colon(u) = (u^{h-1})$ and the natural homomorphism*

$$p_*\colon \mathrm{Gr}(R)/(u^*)\to\mathrm{Gr}(R/(u)),\qquad u^* = u+m^2,$$

is an isomorphism of graded k-algebras, then for each i, $0\leqslant i\leqslant h$, the natural homomorphism

$$p^i_*\colon \mathrm{Gr}(R)/(u^{*i})\to\mathrm{Gr}(R/(u^i)),\qquad p^1_* = p_*,$$

is also an isomorphism of graded k-algebras.

Proof. Applying induction on i, one can assume that $i>1$ and that the lemma is true for $i-1$. As above, it is sufficient to show that the kernel of the nth component

$$p^i_n\colon \mathrm{Gr}(R)_n = m^n/m^{n+1}\to m^n+(u^i)/m^{n+1}+(u^i) = \mathrm{Gr}(R/(u^i))_n$$

of the natural homomorphism of graded k-algebras $p^i\colon \mathrm{Gr}(R)\to\mathrm{Gr}(R/(u^i))$ is equal to $(u^{*i})_n = \{au^i+m^{n+1};\ a\in m^{n-i}\}$. The inclusion $(u^{*i})_n\subset\mathrm{Ker}p^i_n$ is trivial. Let $a\in m^n$ and let $\bar a = a+m^{n+1}\in\mathrm{Ker}p^i_n$. Then $\bar a = bu^i+m^{n+1}$ for some b such that $bu^i\in m^n$. Writing $\bar a = (bu)u^{i-1}+m^{n+1}$ we see that $\bar a\in\mathrm{Ker}p^{i-1}_n$. Hence $(bu)u^{i-1}+m^{n+1} = b_1 u^{i-1}+m^{n+1}$ where $b_1\in m^{n-i+1}$, which implies $(bu-b_1)u^{i-1}+m^{n+2}\in\mathrm{Ker}p^{i-1}_{n+1}$ as $(bu-b_1)u^{i-1}\in m^{n+1}$. Again by the induction hypothesis there exists a $b_2\in m^{n-i+2}$ such that $bu^i-b_1u^{i-1}-b_2u^{i-1}\in m^{n+2}$. Continuing in this way we may find $b'_s\in m^{n-i+1}$, $s = 1, 2, \ldots$ for which $bu^i-b'_s u^{i-1}\in m^{n+s}\subset m^s$. Therefore $bu^i\in\bigcap_s(m^{n-i+1}u^{i-1}+m^s) = m^{n-i+1}u^{i-1}$, i.e. there is a $b'\in m^{n-i+1}$ such that

$(bu-b')u^{i-1} = 0$. This implies $bu-b'\in(u^{h-i+1})\subset(u)$ in view of [8], Lemma 1.2. In particular, $b' = ru\in m^{n-i+1}$. Since p^1 is an isomorphism, $b'+m^{n-i+2} = b''u+m^{n-i+2}$ with $b''\in m^{n-i}$ because $b'+m^{n-i+2}\in\mathrm{Ker}p^1_{n-i+1}$. Consequently, $\bar a = bu^i+m^{n+1} = b'u^{i-1}+m^{n+1} = b''u^i+m^{n+1}\in(u^{*i})_n$ and thus the proof of the proposition is completed.

Now we are able to prove

10. THEOREM. *A local ring R is h-regular if and only if there exists a sequence $u_1, \ldots, u_n$ of generators of the maximal ideal m such that $u_1^*, \ldots, u_n^*$, $u_i^* = u_i+m^2$, is an h-regular sequence in the graded k-algebra $\mathrm{Gr}(R)$ and $h(u_j^*) = h(u_j)$ for $j = 1, \ldots, n$.*

Proof. Let R be an h-regular local ring with the maximal ideal m. We show by using induction on $n = e-\dim R$ (¹) that, for each h-regular sequence $u_1, \ldots, u_n$ of generators of the ideal m, $u_1^*, \ldots, u_n^*$ is an h-regular sequence in $\mathrm{Gr}(R)$ and that $h(u_j^*) = h(u_j)$, $j = 1, \ldots, n$. Case $n = 0$ is trivial. Suppose that $n>0$ and that our assertion is true for all h-regular local rings R' with $e-\dim R'<n$. Now let $e-\dim R = n$ and let $u_1, \ldots, u_n$ be an h-regular sequence of generators of the ideal m. First we prove that u_1^* is an h-regular element in $\mathrm{Gr}(R)$ and $h(u_1^*) = h_1$ where $h_1 = h(u_1)$. By Proposition 8 we need only to check that the natural epimorphism $p_*\colon \mathrm{Gr}(R)/(u_1^*)\to\mathrm{Gr}(R/(u_1))$ is an isomorphism. Take $\bar a = a+m^{k+1}\in\mathrm{Ker}(p_k)$ where $p_k\colon \mathrm{Gr}(R)_k = m^k/m^{k+1}\to m^k+(u_1)/m^{k+1}+(u_1) = \mathrm{Gr}(R/(u_1))_k$ is, as above, the kth component of p. Then $\bar a = bu_1+m^{k+1}$ where $bu_1\in m^k$. Hence there exist a $b'\in m^{k-1}$ and a form $f\in R[X_2, \ldots, X_n]$ of degree k such that

$$(*)\qquad bu_1 = b'u_1+f(u_2, \ldots, u_n)$$

and $\deg_{X_i}f<h_i$ for $h_i = h(u_i)$, $i = 2, \ldots, n$. Now consider the homomorphism of graded k-algebras $g\colon k[X_2, \ldots, X_n]/(X_2^{h_2}, \ldots, X_n^{h_n})\to\mathrm{Gr}(R/(u_1))$ given by $g(X_i) = \bar u_i+\bar m^2$, where $\bar u_i = u_i+(u_1)\in\bar R = R/(u_1)$ and $\bar m$ is the maximal ideal of the ring $\bar R$. In virtue of Lemma 4 and the induction assumption, $\bar u_2^*, \ldots, \bar u_n^*$ is an h-regular sequence in $\mathrm{Gr}(R/(u_1))$ with $h(\bar u_i^*) = h_i$, $i = 2, \ldots, n$, whence g is an isomorphism by [8], Lemma 1.8. Furthermore, if $\bar f$ denotes the reduction $\mathrm{mod}\, m$ of the form $f(X_2, \ldots, X_n)$ from the equality $(*)$, then $g(\bar f+(X_2^{h_2}, \ldots, X_n^{h_n})) = 0$ because $f(u_2, \ldots, u_n)\in(u_1)$. Since $\deg_{X_i}f<h_i$, $i = 2, \ldots, n$, it follows that $\bar f = 0$, i.e. $f(u_2, \ldots, u_n)\in m^{k+1}$. Consequently $\bar a = bu_1+m^{k+1} = b'u_1+m^{k+1}$ where $b'\in m^{k-1}$ and thus $\bar a\in\mathrm{Gr}(R)u_1^*$. From this we conclude that p_* is an isomorphism and, as a result, we know that u_1^* is an h-regular element in $\mathrm{Gr}(R)$. Besides, $u_2^*+(u_1^*), \ldots, u_n^*+(u_1^*)$ is an h-regular sequence in $\mathrm{Gr}(R)/(u_1^*)$ and $h(u_j^*+(u_1^*)) = h(u_j)$ because so is $\bar u_2^*, \ldots, \bar u_n^*$ and $p_*(u_j^*+(u_1^*)) = \bar u_j^*$. Now Lemma 4 implies that $u_1^*, \ldots, u_n^*$ is an h-regular sequence in $\mathrm{Gr}(R)$ and the implication $\Rightarrow$ is proved.

For the implication $\Leftarrow$ it is sufficient to show that if a sequence $u_1, \ldots, u_n$ satisfies the assumption of the theorem, then it is h-regular in R. But this easily follows by induction on n in virtue of Proposition 8. The theorem is proved.

(¹) Here and in what follows $e-\dim R = \dim_{R/m}(m/m^2)$.

11. Corollary. *If R is an h-regular local ring, then $\mathrm{Gr}(R)$ is an h-regular graded k-algebra. Precisely, if $u_1, \ldots, u_n$ is an h-regular sequence of generators of the maximal ideal m and $h(u_i) = h_i$, then $u_1^*, \ldots, u_n^*$, $u_i^* = u_i + m^2$, is an h-regular sequence of homogeneous generators of the augumentation ideal of the k-algebra $\mathrm{Gr}(R)$. Moreover, the homomorphism of graded k-algebras*

$$g\colon k[X_1, \ldots, X_n]/(X_1^{h_1}, \ldots, X_n^{h_n}) \to \mathrm{Gr}(R)$$

defined by the formula: $g(X_i) = u_i^*$, $i = 1, \ldots, n$, is an isomorphism.

Proof. This is a consequence of the proof of the implication $\Rightarrow$ in Theorem 10 and [8], Theorem 1.9.

12. Corollary. *If R is an h-regular local ring, $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ are h-regular sequences of generators of the ideal m, then $n = m$ and $h(u_i) = h(v_i)$ for $i = 1, \ldots, n$ whenever $h(u_1) \leqslant \ldots \leqslant h(u_n)$ and $h(v_1) \leqslant \ldots \leqslant h(v_n)$.*

Proof. The equality $n = m$ follows from Remark 3. In order to prove the remaining equalities assume that $h(u_1) \leqslant \ldots \leqslant h(u_n)$, $h(v_1) \leqslant \ldots \leqslant h(v_n)$ and consider the diagram

$$\begin{array}{lcl} k[X_1, \ldots, X_n]/(X_1^{h_1}, \ldots, X_n^{h_n}) & \searrow^{g_1} & \\ & & \mathrm{Gr}(R) \\ k[Y_1, \ldots, Y_n]/(Y_1^{s_1}, \ldots, Y_n^{s_n}) & \nearrow_{g_2} & \end{array}$$

where $h_i = h(u_i)$, $s_i = h(v_i)$ and $g_1(X_i) = u_i + m^2$, $g_2(Y_i) = v_i + m^2$, $i = 1, \ldots, n$. By Corollary 11, g_1 and g_2 are isomorphisms of graded k-algebras whence there is an isomorphism of graded k-algebras

$$f\colon k[X_1, \ldots, X_n]/(X_1^{h_1}, \ldots, X_n^{h_n}) \to k[Y_1, \ldots, Y_n]/(Y_1^{s_1}, \ldots, Y_n^{s_n}).$$

We show by induction on n that the existence of f implies $h_j = s_j$ for $j = 1, \ldots, n$. If $n = 0$, there is nothing to do. Suppose that $n > 0$ and that our assertion is true for all graded k-algebras of the above form generated by less than n elements. Let us start with the proof that $h_1 = s_1$. Denote: $y_j = Y_j + (Y_1^{s_1}, \ldots, Y_n^{s_n})$, $j = 1, \ldots, n$, $y = f(X_1 + (X_1^{h_1}, \ldots, X_n^{h_n}))$ and write $y = \sum_1^n k_i y_i$, $k_i \in k$. Then $k_q \neq 0$ for some q and

$$(*) \qquad 0 = y^{h_1} = \sum (i_1, \ldots, i_n) k_1^{i_1} \ldots k_n^{i_n} y_1^{i_1} \ldots y_n^{i_n} + k_q^{h_1} y_1^{h_1}$$

where $(i_1, \ldots, i_n) = \dfrac{(i_1 + \ldots + i_n)!}{i_1! \ldots i_n!}$ and $i_1, \ldots, i_n$ run over all n-tuples such that $\sum_j i_j = h_1$, $0 \leqslant i_j \leqslant h_1$ and $i_q < h_1$. But this is possible only when $h_1 \geqslant s_1$ because otherwise the set $\{y_1^{i_1} \ldots y_n^{i_n};\ \sum i_j = h_1,\ i_j \leqslant h_1 < s_1 \leqslant s_j\}$ would be a part of a basis of the k-vector space $k[Y_1, \ldots, Y_n]/(Y_1^{s_1}, \ldots, Y_n^{s_n})$ and the equality $(*)$ would imply $k_q = 0$. Hence $h_1 = s_1$ as the role of h_1 and s_1 is symmetric. The argument following the equality $(*)$ explains also why $y = \sum_1^t y_i$ where t is such that $h_1 = s_1 = \ldots = s_t < s_{t+1}$. Changing, if necessary, the ordering of the sequence $1, \ldots, t$ we may assume $q = 1$, i.e. $k_1 \neq 0$. Then clearly $(y_1, \ldots, y_n) = (y, y_2, \ldots, y_n)$. Moreover, we claim that $y, y_2, \ldots, y_n$ is an h-regular sequence with $h(y) = h_1 = s_1$. To prove this, by applying Lemma 1.4 in [8] (to the sequence $y_2, \ldots, y_n, y$) and the equality $y = k_1 y_1 + \ldots + k_n y_n$, it suffices to show that $h(y) = s_1$. Case $y = k_1 y_1$ is trivial. Let $k_i \neq 0$ for some $i > 1$. Then from the equality $(*)$ with $q = 1$ we conclude that $(j, s_1 - j) k_1^j k_i^{s_1 - j} = 0$ for all $j = 1, \ldots, s_1 - 1$. This means that $(j, s_1 - j) = 0$ in k for the same j. It is well known that such equalities hold in the field k only when the characteristic of k is $p > 0$ and $s_1 = h_1 = p^r$, $r \geqslant 0$. Since $h(y_j) = s_1$ for $j = 1, \ldots, t$ it follows that $y^{s_1} = (\sum_1^t k_j y_j)^{p^r} = 0$, i.e. $h(y) \leqslant s_1$. On the other hand,

$$h(y + (y_2, \ldots, y_n)) = h(k_1 y_1 + (y_2, \ldots, y_n)) = s_1$$

by [8], Lemma 1.4, whence $h(y) = s_1$. Consequently, we know that $y, y_2, \ldots, y_n$ is an h-regular sequence of homogeneous generators of the augumentation ideal of the algebra $B = k[Y_1, \ldots, Y_n]/(Y_1^{s_1}, \ldots, Y_n^{s_n})$. Applying again [8], Lemma 1.4, Theorem 1.9, we infer from it that $B/(y) = k[Y_2, \ldots, Y_n]/(Y_2^{s_2}, \ldots, Y_n^{s_n})$. Similarly, if $A = k[X_1, \ldots, X_n]/(X_1^{h_1}, \ldots, X_n^{h_n})$, then $A/(\bar{X}_1) = k[X_2, \ldots, X_n]/(X_2^{h_2}, \ldots, X_n^{h_n})$. Furthermore, as $f(\bar{X}_1) = y$, the isomorphism $f\colon A \to B$ yields an isomorphisms of graded k-algebras $f'\colon A/(\bar{X}_1) \to B/(y)$, i.e. an isomorphism

$$f'\colon k[X_2, \ldots, X_n]/X_2^{h_2}, \ldots, X_n^{h_n}) \to k[Y_2, \ldots, Y_n]/(Y_2^{s_2}, \ldots, Y_n^{s_n}).$$

Now, using the induction hypothesis, we obtain $h_j = s_j$ for $j = 2, \ldots, n$. This completes the proof of the corollary.

13. Corollary. *A local ring R is h-regular if and only if its completion $\hat{R}$ is h-regular and there exists an h-regular (in $\hat{R}$) sequence $u_1, \ldots, u_n$ of generators of the maximal ideal $\hat{m}$ in $\hat{R}$ such that $u_i \in R$ for $i = 1, \ldots, n$.*

Proof. The "if" part follows from Corollary 6. The "only if" part is a consequence of Theorem 10 in view of the fact that $\mathrm{Gr}(R) = \mathrm{Gr}(\hat{R})$.

Corollaries 11 and 13 show that, if R is an h-regular local ring, then both the completion $\hat{R}$ and the graded k-algebra $\mathrm{Gr}(R)$ are h-regular. We prove below (see Example 14) that one of the examples of "bad" Noetherian rings in Nagata's book "Local rings" provides a local ring R such that its completion $\hat{R}$ (and hence $\mathrm{Gr}(\hat{R}) = \mathrm{Gr}(R)$) is h-regular whereas R itself is not h-regular.

14. Example ([6], Example 3.1, p. 206). Let k be a field of characteristic $p > 0$ such that $[k : k^p] = \infty$ and let $S = \{f = \Sigma f_i t^i \in k[[t]];\ \{f_i\}$ generate in k a finite extension of $k^p\}$. One can verify that S is a local ring and $\hat{S} = k[[t]]$. In particular, S is a regular local ring. Let $b_1, \ldots, b_n, \ldots$ be a sequence of elements of k linearly independent over k^p and let $b = \Sigma b_i t^i$ (observe that $b \notin S$). By definition

$$R = S[X]/(X^p - d) \quad \text{where} \quad d = b^p \in S.$$

Then $\hat{R} = \hat{S}[[X]]/(X^p - d) = k[[t, X]]/(X-b)^p \simeq k[[t, Y]]/(Y^p)$, which shows that $\hat{R}$ is h-regular because $t+(Y^p)$, $Y+(Y^p)$ is an h-regular sequence of generators of the maximal ideal of the ring $k[[t, Y]]/(Y^p)$. On the other hand, R is not h-regular since otherwise R, being an integral domain, would be regular, which is impossible because $\hat{R}$ is not regular.

The next example indicates the existence of non h-regular local rings R which are complete (even Artinian) and for which $\mathrm{Gr}(R)$ is an h-regular graded k-algebra.

15. Example. Let k be a field of characteristic 2 and let $R = k[[X, Y]]/X^2 - XY^2, Y^4)$. It is easy to see that $\mathrm{Gr}(R) = k[X_1, X_2]/(X_1^2, X_2^4)$ and that there is no element $u \in R$ with $h(u) = 2$. It follows from Corollary 12 that R is not h-regular.

Corollary 12 justifies the following

16. Definition. A local ring R is called $(h_1, \ldots, h_n)$*-regular*, $h_i \in N^\infty$, $h_1 \leqslant \ldots \leqslant h_n$, if there exists an h-regular sequence $u_1, \ldots, u_n$ of generators of the ideal m such that $h(u_i) = h_i$ for $i = 1, \ldots, n$.

It is obvious that each h-regular local ring R is $(h_1, \ldots, h_n)$-regular for some sequence $h_1, \ldots, h_n$. Moreover, if R is $(h_1, \ldots, h_n)$-regular, then so are its completion $\hat{R}$ and the graded k-algebra $\mathrm{Gr}(R)$.

If R is an h-regular local ring, then by Remark 3 every h-regular sequence of generators of the maximal ideal m is a minimal set of generators of m. The next proposition decides when a given minimal sequence of generators of m is h-regular.

17. Proposition. *If R is an $(h_1, \ldots, h_n)$-regular local ring, then a minimal sequence $v_1, \ldots, v_n$ of generators of the ideal m is h-regular in R if and only if $h(v_j) \leqslant h_j$, $j = 1, \ldots, n$, provided that $h(v_1) \leqslant \ldots \leqslant h(v_n)$.*

Proof. Taking into account Corollary 12, it is enough to show that a minimal sequence $v_1, \ldots, v_n$ of generators of m such that $h(v_j) \leqslant h_j$ and $h(v_1) \leqslant \ldots \leqslant h(v_n)$ is h-regular in R. It will be done (as usual in this paper) by induction on n. Case $n = 1$ is easy and can be proved for each $v_1 \in m \setminus m^2$. Suppose that $n > 1$, that the assertion is true for all h-regular local rings whose maximal ideal is generated by less than n elements and that the maximal ideal m of the ring under consideration is generated by an h-regular sequence $u_1, \ldots, u_n$ with $h(u_i) = h_i$. Then $u_1 = \Sigma b_i v_i$ for some $b_i \in R$ and there is an r such that $b_r \notin m$. Hence in the ring $\bar{R} = R/(u_1)$ we have $\bar{m} = (\bar{u}_2, \ldots, \bar{u}_n) = (\bar{v}_1, \ldots, \hat{\bar{v}}_r, \ldots, \bar{v}_n)$ (1) where $\bar{m}$ denotes the maximal ideal of the ring $\bar{R}$. It is easy to verify that $\bar{v}_1, \ldots, \hat{\bar{v}}_r, \ldots, \bar{v}_n$ is a minimal set of generators of $\bar{m}$. Moreover, $h(\bar{v}_i) \leqslant h(v_i) \leqslant h_i \leqslant h_{i+1} = h(\bar{u}_{i+1})$ for $i \leqslant r$ and $h(\bar{v}_j) \leqslant h(v_j) \leqslant h_j = h(\bar{u}_j)$ for $j > r$. Since $\bar{R}$ is an $(h_2, \ldots, h_n)$-regular local ring by Lemma 4 it follows by the induction assumption that $\bar{v}_1, \ldots, \hat{\bar{v}}_r, \ldots, \bar{v}_n$ is an h-regular sequence in $\bar{R}$. In particular, $h(\bar{v}_i) = h(\bar{u}_{i+1}) = h_{i+1}$ for $i < r$ and $h(\bar{v}_j) = h(\bar{u}_j) = h_j$ for $j > r$ in view of Corollary 12. Hence $h(\bar{v}_i) = h(v_i) = h_1$ for $i < r$ and $h(\bar{v}_j) = h(v_j) = h_j$ for $j > r$ because $h_r = h(\bar{v}_{r-1}) \geqslant h_{r-1} = h(\bar{v}_{r-2}) \geqslant \ldots \geqslant h_2 = h(\bar{v}_1) \geqslant h_1$ and $h_1 \leqslant \ldots \leqslant h_n$. Now, applying Lemma 4, conclude that $u_1, v_1, \ldots, \hat{v}_r, \ldots, v_n$ is an h-regular sequence in R,

(1) $(x_1, \ldots, \hat{x}_i, \ldots, x_n) \overset{\mathrm{df}}{=} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

whence $\tilde{R} = R/(v_1, \ldots, \hat{v}_r, \ldots, v_n)$ is an (h_1)-regular local ring. Furthermore, it is obvious that $\tilde{v}_r = v_r + (v_1, \ldots, \hat{v}_r, \ldots, v_n) \in \tilde{m} \setminus \tilde{m}^2$ where $\tilde{m}$ is the maximal ideal of $\tilde{R}$. Hence (case $n = 1$) we infer that $\tilde{v}_r$ is an h-regular element in $\tilde{R}$ and $h(\tilde{v}_r) = h_1 = h_r$. This gives $h(\tilde{v}_r) = h(v_r)$ and consequently, again by Lemma 4, $v_1, \ldots, v_n$ is an h-regular sequence in R. The proposition follows.

18. Corollary. *If R is an $(h_1, \ldots, h_n)$-regular local ring, then a minimal sequence $v_1, \ldots, v_n$ of generators of the ideal m such that $h(v_1) \leqslant \ldots \leqslant h(v_n)$ is h-regular in R if and only if $h(v_j) = h_j$ for $j = 1, \ldots, n$.*

19. Theorem. *If R is an $(h_1, \ldots, h_n)$-regular local ring and $u_1, \ldots, u_n$ is an h-regular sequence of generators of the ideal m such that $h(u_i) = h_i$, $i = 1, \ldots, n$, then for each sequence $k_1 \leqslant \ldots \leqslant k_n$, $k_j \in N^\infty$, $R/(u_1^{k_1}, \ldots, u_n^{k_n})$ is an $(h'_s, \ldots, h'_n)$-regular local ring where s is the first number such that $k_s > 1$ and $h'_i = \min(h_i, k_i)$.*

In the proof of this theorem the following lemma is very useful:

20. Lemma. *If $u_1, \ldots, u_m$ is an h-regular sequence in a ring S and $h(u_j) = h_j$, then for each sequence $k_1, \ldots, k_{m-1}$, $k_j \in N^\infty$, and for each j, $0 \leqslant j < h_m$ the following implication holds:*

$$(*) \qquad a u_m^j \in (u_1^{k_1}, \ldots, u_{m-1}^{k_{m-1}}) \Rightarrow a \in (u_1^{k_1}, \ldots, u_{m-1}^{k_{m-1}}, u_m^{h_m - j}).$$

Proof. It is clear that we may assume $k_i < h_i$ for $i = 1, \ldots, m$. Moreover, one can guess that, for the proof, induction on m will be used. If $m = 1$, then the lemma is a reformulation of [8], Lemma 1.2. Let $m > 1$ and let $a u_m^j \in (u_1^{k_1}, \ldots, u_{m-1}^{k_{m-1}})$. Then the sequence $\bar{u}_2, \ldots, \bar{u}_m$, $\bar{u}_i = u_i + (u_1)$, is an h-regular sequence in $R/(u_1)$ with $h(\bar{u}_m) = h_m$ and by the induction hypothesis

$$a = a_m u_m^{h_m - j} + \sum_{i=2}^{m-1} a_i u_i^{k_i} + b u_1 \quad \text{for some } a_i,\ b \in R.$$

Multiplying this equality by u_m^j, we obtain

$$\sum a_i u_i^{k_i} u_m^j + b u_1 u_m^j = a u_m^j = \sum_{i=1}^{m-1} s_i u_i^{k_i} \quad \text{for some } s_i \in R.$$

Therefore $(b u_m^j - s_1 u_1^{k_1 - 1}) u_1 \in (u_2^{k_2}, \ldots, u_{m-1}^{k_{m-1}})$, whence $b u_m^j - s_1 u_1^{k_1 - 1} \in (u_1^{h_1 - 1}, u_2^{k_2}, \ldots, u_{m-1}^{k_{m-1}})$. Repeating the above arguments for b instead of a, we conclude that $b \in (u_m^{h_m - j}, u_2^{k_2}, \ldots, u_{m-1}^{k_{m-1}}, u_1)$. Hence $a \in (u_m^{h_m - j}, u_2^{k_2}, \ldots, u_{m-1}^{k_{m-1}}, u_1^2)$. Continuing in this way, one gets the required result.

Once the lemma is shown, we can prove the theorem. For this purpose it is sufficient to show that $\bar{u}_s, \ldots, \bar{u}_n$, $\bar{u}_i = u_i + (u_1^{k_1}, \ldots, u_n^{k_n})$, is an h-regular sequence in $R/(u_1^{k_1}, \ldots, u_n^{k_n})$ and $h(\bar{u}_i) = h'_i$ for $i = s, \ldots, n$. It is obvious that $h(\bar{u}_i) \leqslant h'_i$, $i = s, \ldots, n$. If $q = h(\bar{u}_i) < h'_i$ for some i, i.e. $u_i^q \in (u_1^{k_1}, \ldots, u_n^{k_n})$, then

$$u_i^q (1 - a u_i^{k_i - q}) \in (u_1^{k_1}, \ldots, \widehat{u_i^{k_i}}, \ldots, u_n^{k_n})$$

and by Lemma 20 we have $1 \in (u_1^{k_1}, \ldots, u_{i-1}^{k_{i-1}}, u_i^{k_i - q}, u_{i+1}^{k_{i+1}}, \ldots, u_n^{k_n}) \subset (u_1, \ldots, u_n)$ because $q < h'_i \leqslant k_i$. This contradiction shows that $h(\bar{u}_i) = h'_i$ for $i = s, \ldots, n$. Now, fix i, $s \leqslant i \leqslant n$, and take $\bar{a} \in \bar{R} = R/(u_1^{k_1}, \ldots, u_n^{k_n})$ such that $\bar{a}\bar{u}_i \in (\bar{u}_s, \ldots, \bar{u}_{i-1})$, i.e. $au_i \in (u_1, \ldots, u_{i-1}, u_i^{k_i}, \ldots, u_n^{k_n})$ (recall that $k_1 = \ldots = k_{s-1} = 1$). Then there is a $b \in R$ such that $(a - bu_i^{k_i-1})u_i \in (u_1, \ldots, u_{i-1}, u_{i+1}^{k_{i+1}}, \ldots, u_n^{k_n})$, whence $a - bu_i^{k_i-1} \in (u_1, \ldots, u_{i-1}, u_i^{h_i-1}, u_{i+1}^{k_{i+1}}, \ldots, u_n^{k_n})$ on the basis of Lemma 20 applied to the h-regular sequence $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, u_i$ and $k'_1 = \ldots = k'_{i-1} = 1$, $k'_j = k_j$, $j = i+1, \ldots, n$. Since $(u_1, \ldots, u_{i-1}, u_i^{h'_i-1}, u_{i+1}^{k_{i+1}}, \ldots, u_n^{k_n}) = (u_1, \ldots, u_{i-1}, u_i^{k_i-1}, u_i^{h_i-1}, u_{i+1}^{k_{i+1}}, \ldots, u_n^{k_n})$, it follows that $\bar{a} \in (\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_i^{h'_i-1})$, and thus the theorem is proved.

21. Corollary. *If R is a regular local ring, then for each regular sequence $u_1, \ldots, u_n$ of genarators of the maximal ideal m and for each sequence $k_1 \leqslant \ldots \leqslant k_n$, $k_i \in N^\infty$, $R/(u_1^{k_1}, \ldots, u_n^{k_n})$ is an $(k_s, \ldots, k_n)$-regular local ring where s is the first number such that $k_s > 1$.*

22. Theorem. *If a local ring R' is a homomorphic image of a regular local ring R, then it is $(h_1, \ldots, h_n)$-regular if and only if there exists a regular sequence $u_1, \ldots, u_m$ of generators of the maximal ideal m in R such that*

$$R' \simeq R/(u_{n+1}, \ldots, u_m, u_1^{h_1}, \ldots, u_n^{h_n}).$$

Proof. By the above corollary we only have to prove the implication $\Rightarrow$. Let R be a regular local ring and let $p\colon R \to R'$ be an epimorphism of rings, which exists in virtue of the assumption. Moreover, let $I = \operatorname{Ker} p$ and let $n = e\text{-}\dim R'$. If $I \not\subset m^2$ (m is the maximal ideal of R), then there is $u_{n+1} \in I \setminus m^2$ and p induces an epimorphism of rings $p'\colon R/(u_{n+1}) \to R'$ such that $R/(u_{n+1})$ is again a regular local ring. Repeating this procedure one can find a sequence $u_{n+1}, \ldots, u_m$, $u_i \in I$, such that $\bar{R} = R/(u_{n+1}, \ldots, u_m)$ is a regular local ring and the kernel of the epimorphism $\bar{p}\colon R \to R'$ induced by p is contained in the square of the maximal ideal of the ring $\bar{R}$. Now we are done by the following

23. Lemma. *Assume that R is a regular local ring with the maximal ideal m, R' is an $(h_1, \ldots, h_n)$-regular local ring with the maximal ideal m' and $p\colon R \to R'$ is an epimorphism of rings with $I = \operatorname{Ker} p \subset m^2$. Then for each h-regular sequence $u'_1, \ldots, u'_n$ of generators of the ideal m' with $h(u'_i) = h_i$ and for each sequence $u_1, \ldots, u_n$ in R such that $p(u_i) = u'_i$, $i = 1, \ldots, n$, $u_1, \ldots, u_n$ is a regular sequence of generators of the ideal m and the epimorphism*

$$\bar{p}\colon R/(u_1^{h_1}, \ldots, u_n^{h_n}) \to R'$$

induced by p is an isomorphism (note that $u_i^{h_i} \in \operatorname{Ker} p$).

Proof. Let $u'_1, \ldots, u'_n$ be an h-regular sequence of generators of the ideal m' such that $h(u'_j) = h_j$ and let $p(u_j) = u'_j$, $j = 1, \ldots, n$. Then clearly $m = (u_1, \ldots, u_n) + I = (u_1, \ldots, u_n) + m^2$ and by Nakayama Lemma $m = (u_1, \ldots, u_n)$. Moreover, $u_1, \ldots, u_n$ is a minimal (hence regular) sequence of generators of m because so is $u'_1, \ldots, u'_n$ for m'. We show by induction on n that $I = (u_1^{h_1}, \ldots, u_n^{h_n})$. Case $n = 0$ is trivial. Suppose $n > 0$. If $\bar{R} = R/(u_n)$, $\bar{R}' = R'/(u'_n)$ and $p_1\colon \bar{R} \to \bar{R}'$ is the epimorphism induced by p, then from the induction assumption it follows that $\operatorname{Ker} p_1 = (\bar{u}_1^{h_1}, \ldots, \bar{u}_{n-1}^{h_{n-1}})$, $\bar{u}_i = u_i + (u_n)$. Hence if $a \in I$ then

$$a = \sum_{i=1}^{n-1} a_i u_i^{h_i} + a_n u_n \quad \text{for some } a_j \in R,$$

which implies $0 = p(a) = p(a_n)p(u_n) = p(a_n)u'_n$. By Remark 3 u'_n is an h-regular element in R' so it results $p(a_n) = p(r)u_n'^{h_n-1} = p(ru_n^{h_n-1})$, i.e. $a_n = ru_n^{h_n-1} + a'$ with $a' \in I$. Consequently $a = \Sigma a_i u_i^{h_i} + (ru_n^{h_n-1} + a')u_n \in (u_1^{h_1}, \ldots, u_n^{h_n}) + mI$ and $I = (u_1^{h_1}, \ldots, u_n^{h_n}) + mI$ because $u_i'^{h_i} = 0$. The conclusion now follows by Nakayama Lemma.

24. Corollary. *Suppose R is an $(h_1, \ldots, h_n)$-regular local complete ring with the maximal ideal m and the quotient field $k = R/m$ if characteristic p (p can be 0). In equal characteristic case $R = k[[X_1, \ldots, X_n]]/(X_1^{h_1}, \ldots, X_n^{h_n})$. In non-equal characteristic case there exists a local complete discrete valuation ring A with the maximal ideal generated by p such that $R = A[[X_1, \ldots, X_n]]/(p - f, X_1^{h_1}, \ldots, X_n^{h_n})$ for some $f \in (X_1, \ldots, X_n) \subset A[[X_1, \ldots, X_n]]$ (observe that $p - f, X_1, \ldots, X_n$ is a regular sequence of generators of the maximal ideal of the ring $A[[X_1, \ldots, X_n]]$.*

Proof. In equal characteristic case R contains the field k and the natural homomorphism of k-algebras $g\colon k[[X_1, \ldots, X_n]] \to R$ given by $g(X_i) = u_i$, where $u_1, \ldots, u_n$ is an h-regular sequence of generators of m is an epimorphism by the completness of R. Moreover, $\operatorname{Ker}(g) \subset (X_1, \ldots, X_n)^2$ as $e\text{-}\dim(R) = n = e\text{-}\dim(k[[X_1, \ldots, X_n]])$. In virtue of Lemma 23 it follows that

$$R \simeq k[[X_1, \ldots, X_n]]/(X_1^{h_1}, \ldots, X_n^{h_n}).$$

In the non-equal characteristic case there exist ([2], Th. 12) a local complete discrete valuation ring A with the maximal ideal generated by p and a homomorphism $g'\colon A \to R$ such that the induced homomorphism of the quotient fields $\bar{g}'\colon A/(p) \to k$ is an isomorphism. Hence the homomorphism $g\colon A[[X_1, \ldots, X_n]] \to R$, defined by the formulas $g|_A = g'$, $g(X_i) = u_i$ for u_i's as above, is an epimorphism of rings. In particular, there is an $f \in (X_1, \ldots, X_n)$ such that $g(f) = p \in m$, whence $p - f \in \operatorname{Ker}(g)$. Furthermore, if $g_1\colon S = A[[X_1, \ldots, X_n]]/(p - f) \to R$ denotes the epimorphism induced by g, then $\operatorname{Ker} g_1 \subset m_1^2$ where m_1 is the maximal ideal of the ring S because $e\text{-}\dim R = n = e\text{-}\dim S$. We have also $g(\bar{X}_i) = u_i$ for $i = 1, \ldots, n$. Consequently, by Lemma 23,

$$R \simeq S/(\bar{X}_1^{h_1}, \ldots, \bar{X}_n^{h_n}) \simeq A[[X_1, \ldots, X_n]]/(p - f, X_1^{h_1}, \ldots, X_n^{h_n}),$$

as was to be shown.

25. Remark. Theorem 22 shows that from the structural point of view the following question is of great importance:

Question. Is every h-regular local ring a homomorphic image of a regular local ring?

The answer to this question is unknown to the author.

26. THEOREM. *If k is a field and R is a local k-algebra of finite type such that R/m (m being, as usual, the maximal ideal of R) is a separable extension of k, then R is an $(h_1, \ldots, h_n)$-regular local ring if and only if there exist a regular local ring S and an $r \leqslant n$ such that*

$$R = S[[X_1, \ldots, X_r]]/(X_1^{h_1}, \ldots, X_r^{h_r}).$$

Proof. The "only if" part is a consequence of Theorem 22. For the proof of the "if" part take an h-regular sequence $u_1, \ldots, u_n$ of generators of m and denote by N the ideal $(u_1, \ldots, u_r)$ where r is the maximal number such that $h_r < \infty$. Then $S = R/N$ is a regular local ring by Lemma 4 (recall $h_1 \leqslant \ldots \leqslant h_n$) and its quotient field, being isomorphic to R/m, is a separable extension of k. By [4], Th. 6.3, Ex. 1.5, Def. 1.1, it follows that the structural homomorphism of rings $k \to S$ is formally smooth, i.e. for every k-algebra B and every nilpotent ideal J in B and every homomorphism of k-algebras $f\colon S \to B/J$ there exists a homomorphism of k-algebras $g'\colon S \to B$ such that the diagram

$$\begin{array}{ccc} & & S \\ & \swarrow^{g'} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[p]{} & B/J \end{array}$$

where p is the natural projection, is commutative. In particular, putting $B = R$, $J = N$ and $f = 1$, one obtains a homomorphism of k-algebras $g'\colon S \to R$ such that $pg' = 1_S$. The homomorphism g' permits us to define a homomorphism of k-algebras $g\colon \bar{S} = S[[X_1, \ldots, X_r]]/(X_1^{h_1}, \ldots, X_r^{h_r}) \to R$ as follows: $g(s) = g'(s)$ for $s \in S$, $g(X_i) = u_i$ for $i = 1, \ldots, r$. We claim that g is an isomorphism. Using the equality $pg' = 1$, one can easily show by induction on m that $R \subset \operatorname{Im} g + N^m$ for $m = 0, 1, \ldots$ ($N^0 = R$). Since N is a nilpotent ideal, this implies that g is an epimorphism. The injectivity of g will be proved by induction on $r = r(R)$ (note that r is an invariant of R because $r = e - \dim R - \operatorname{Dim} R$ where $\operatorname{Dim} R$ denotes the Krull dimention of R). If $r = 0$ then the injectivity of $g = g'$ is a consequence of the equality $pg' = 1$. Suppose that $r > 0$ and that for all h-regular local k-algebras R' of finite type with the separable (over k) quotient field and $r(R') < r$ all homomorphisms constructed analogously to the homomorphism g (g depends on $u_1, \ldots, u_n$ and g') are injective. Now, if $g(\bar{a}) = 0$ for some $\bar{a} \in S[[X_1, \ldots, X_r]]/(X_1^{h_1}, \ldots, X_r^{h_r})$, then $\bar{a}$ has a unique representative $a \in S[[X_1, \ldots, X_r]]$ such that $a = \sum_{i=1}^{h_r - 1} s_i(X_1, \ldots, X_{r-1}) X_r^i$ and $\deg_{X_i} s_j < h_i$ for $i = 1, \ldots, r-1$ and $j = 0, \ldots, h_r - 1$. Consider the following commutative diagram:

$$\begin{array}{ccc} S[[X_1, \ldots, X_r]]/(X_1^{h_1}, \ldots, X_r^{h_r}) & \xrightarrow{g} & R \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p_r} \\ S[[X_1, \ldots, X_{r-1}]]/(X_1^{h_1}, \ldots, X_{r-1}^{h_{r-1}}) & \xrightarrow{\bar{g}} & R/(u_r) \end{array},$$

where $q(X_i) = X_i + (X_1^{h_1}, \ldots, X_{r-1}^{h_{r-1}})$, $i = 1, \ldots, r-1$, $q(X_r) = 0$, p_r is the natural projection and $\bar{g}(X_i) = \bar{u}_i = u_i + (u_r)$ for $i = 1, \ldots, r-1$. Observe that $\bar{R} = R/(u_r)$ is an $(h_1, \ldots, h_{r-1}, h_{r+1}, \ldots, h_n)$-regular local ring with the same quotient field as R and with $r(\bar{R}) = r-1$. Furthermore, $S(\bar{R}) = \bar{R}/(\bar{u}_1, \ldots, \bar{u}_{r-1}) = R/(u_1, \ldots, u_r) = S$ and $p'\bar{g}|_S = 1$, where $p' = \bar{R} \to S$ denotes the natural projection. Consequently, applying the induction hypothesis, one knows that $\bar{g}$ is an injection. Since $0 = p_r g(\bar{a}) = \bar{g}q(\bar{a})$, it follows that

$$0 = q(\bar{a}) = s_0(X_1, \ldots, X_{r-1}) + (X_1^{h_1}, \ldots, X_{r-1}^{h_{r-1}})$$

and further $s_0 = 0$ as $\deg_{X_i} s_0 < h_i$ for $i = 1, \ldots, r-1$. This implies $0 = g(\bar{a}) = g(\Sigma s_i X_r^{i-1}) u_r$, whence $g(\Sigma s_i X_r^{i-1}) \in (u_r)$ because u_r is an h-regular element in R. Therefore

$$\bar{g}q(\sum s_i X_r^{i-1}) = p_r g(\sum s_i X_r^{i-1}) = 0,$$

i.e. $q(\Sigma s_i X^{i-1}) = 0$, which gives, as above, $s_1 = 0$. Repeating this reasoning, one gets $s_0 = \ldots = s_{h_r - 1} = 0$. As a result, we obtain $\bar{a} = 0$ and thus the injectivity of g is proved. The theorem follows.

Now we formulate the notion of an h-regular sequence in homological terms. Recall for this purpose that a sequence $u_1, \ldots, u_n$ in a ring R is called p-ordered (see [8]) if $h(u_j) = \infty$ for $j = 1, \ldots, p$ and $h(u_j) < \infty$ for $j = p+1, \ldots, n$. If $u_1, \ldots, u_n$ is a sequence in a ring R (not necessarily p-ordered for some p), let $E(u_1, \ldots, u_n)$ denote the Koszul complex of $u_1, \ldots, u_n$, i.e. the exterior algebra on the free R-module $RT_1 \oplus \ldots \oplus RT_n$ with the differential d given by $d(T_i) = u_i$ and grading defined by $\deg T_i = 1$, $i = 1, \ldots, n$. Moreover, if the sequence $u_1, \ldots, u_n$ is p-ordered, let $\mathrm{Eh}(u_1, \ldots, u_n)$ denote the graded differential R-algebra $E(u_1, \ldots, u_n) \otimes_R \Gamma(RS_{p+1} \oplus \ldots \oplus RS_n)$ where $\Gamma(RS_{p+1} \oplus \ldots \oplus RS_n)$ is the algebra with divided powers on the free R-module $RS_{p+1} \oplus \ldots \oplus RS_n$, $\deg(S_j) = 2$, and the differential d is given by $d(S_j) = u_j^{h_j - 1} T_j$ with $h_j = h(u_j)$ (in the notation of [8] $E(u_1, \ldots, u_n) = R\langle T_1, \ldots, T_n, dT_i = u_i\rangle$, $\mathrm{Eh}(u_1, \ldots, u_n) = E(u_1, \ldots, u_n)\langle S_{p+1}, \ldots, S_n, dS_j = u_j^{h_j - 1} T_j\rangle$; for details see [8]). From Proposition 2.5 in [8] applied to $A = A_0 = R$ we get the following

27. COROLLARY. *If $u_1, \ldots, u_n$ is a p-ordered sequence in a local ring R, $u_i \in m$ and $\mathrm{Eh} = \mathrm{Eh}(u_1, \ldots, u_n)$, then the following conditions are equivalent:*

(i) *$u_1, \ldots, u_n$ is an h-regular sequence in R,*

(ii) *Eh is a free resolution of the cyclic R-module $R/(u_1, \ldots, u_n)$,*

(iii) *$H_1(\mathrm{Eh}) = H_2(\mathrm{Eh}) = 0$.*

The next proposition is a generalization of [3], Prop. 1.

28. PROPOSITION. *If $u_1, \ldots, u_n$ is an h-regular sequence in a ring R (not necessarily Noetherian) and $f\colon S \to R$ (S does not have to be Noetherian too) is a homomorphism of rings such that the composition $pf\colon S \to R \to R/(u_1, \ldots, u_n)$, where p is the natural projection, is injective, then the homomorphism of rings*

$$g\colon T = S[X_1, \ldots, X_n]/(X_1^{h_1}, \ldots, X_n^{h_n}) \to R, \quad h_j = h(u_j),$$

defined by $g(s) = f(s)$ for $s \in S$, $g(X_i) = u_i$ for $i = 1, \ldots, n$, is also injective. Moreover, if R, S are local (Noetherian) rings and $R/(u_1, \ldots, u_n)$ is a flat S-module ($R/(u_1, \ldots, u_n)$ is an S-module by pf: $S \to R/(u_1, \ldots, u_n)$), then R is a flat T-module.

Proof. The proof of the injectivity of g is a slight modification of the proof of the injectivity of the homomorphism g considered in the proof of Theorem 26 and we omit it. Assume that R, S are local rings and $R/(u_1, \ldots, u_n)$ is a flat S-module. In order to show that R is a flat T-module we use the criterion of flatness ([1], Chap. III, § 5, n° 2, Th. 1(iii)) applied to the ring T, the ideal $J = (\bar{X}_1, \ldots, \bar{X}_n) \subset T$ and the T-module R. We have to prove that:

(i) R/SR is a flat T/J-module.

(ii) $\mathrm{Tor}_1^T(T/J, R) = 0$.

(iii) For any finitely generated ideal I in T $I \otimes_T R$ is a T-module separate in J-adic topology.

(i) is a consequence of the assumption because $R/SR = R/(u_1, \ldots, u_n)$ and $T/J = S$. For the equality $\mathrm{Tor}_1^T(T/J, R) = 0$ observe that by [7], Th. 4, $\mathrm{Eh}(\bar{X}_1, \ldots, \bar{X}_n)$ is a free resolution of the T-module $S = T/J$. Moreover, it is easy to see that $\mathrm{Eh}(\bar{X}_1, \ldots, \bar{X}_n) \otimes_T R = \mathrm{Eh}(u_1, \ldots, u_n)$. Hence

$$\mathrm{Tor}_1^T(T/J, R) = H_1(\mathrm{Eh}(\bar{X}_1, \ldots, \bar{X}_n) \otimes_T R) = H_1(\mathrm{Eh}(u_1, \ldots, u_n)) = 0$$

in view of Corollary 27. It remains to show that, for each finitely generated ideal I in T, $I \otimes_T R$ is separate in J-adic topology, i.e. $\bigcap_m J^m(I \otimes_T R) = 0$. But

$$\bigcap_m J^m(I \otimes_T R) \subseteq \bigcap_m (I \otimes_T R) L^m$$

where $L = (u_1, \ldots, u_n)$ and $I \otimes_T R$ is regarded as a (right) R-module in a natural way. Since $I \otimes_T R$ is clearly a finitely generated R-module, the required result follows from [1], Chap. III, § 3, n° 2. This completes the proof of the proposition.

29. Remark. If R is a complete local ring, then the ring T in the above proposition can be replaced by $S[[X_1, \ldots, X_n]]/(X_1^{h_1}, \ldots, X_n^{h_n})$.

30. Corollary. *If R is a local ring containing a field k and $u_1, \ldots, u_n$ is an h-regular sequence in R with $h(u_i) = h_i$, then for each h-regular sequence $f_1, \ldots, f_m$ in T (with $S = k$) such that $f_j(0) = 0$ for all j, $f_1(u_1, \ldots, u_n), \ldots, f_m(u_1, \ldots, u_n)$ is an h-regular sequence in R and $h(f_j(u_1, \ldots, u_n)) = h(f_j)$ for $j = 1, \ldots, m$. In particular, if $u_1, \ldots, u_n$ is a regular sequence in R, then for each regular sequence $f_1, \ldots, f_m$ of polynomials from $k[X_1, \ldots, X_n]$ such that $f_j(0) = 0$, $f_1(u_1, \ldots, u_n), \ldots, f_m(u_1, \ldots, u_n)$ is a regular sequence in R.*

Proof. This is a consequence of Proposition 5 and Proposition 28.

31. Corollary (from the proof). *Let R and R' be local rings with the maximal ideals m and m', respectively. If f: $R \to R'$ is a homomorphism of rings such that $f(m) \subset m'$ and J is an ideal in R, then R' is a flat R-module whenever $R'/f(J)R'$ is a flat R/J-module and* $\mathrm{Tor}_1^R(R/J, R') = 0$.

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