On closed graph theorems in topological spaces and groups

by

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Abstract. Let T, X be topological spaces and $f\colon T\to X$. Given a continuous pseudo-metric $p\colon X\times X\to R^+$, we define a certain function $p_f\colon T\times T\to R^+$ (Definition 4). The continuity properties of the functions p_f at the points of the diagonal $\Delta(T)$ are strictly related to some closed graph theorems concerning f.

1. Introduction. Throughout the paper (unless explicitly stated) T is a topological space, X is a Tychonoff topological space, and f is a function on T to X (not necessarily continuous). Furthermore, P is a $\leqslant$-directed family of continuous pseudo-metrics for X generating its topology, and $\mathscr{Y}$ is the uniformity (with symmetric members) for X generated by P. For instance, P may be a gage for X, or, if (X, p) is a metric space, we may take $P = \{p\}$. For some basic topological notions the reader is referred to the monographs of Kelley [3] or Engelking [2]. Given any $p\in P$, we define a function p_f on $T\times T$ to R^+ (Definition 4) and study its main properties (Theorems 1 and 2). The continuity in the first variable of the functions p_f at the points of the diagonal $\Delta(T)$ is equivalent to the nearly-continuity of f (Theorem 3) and, consequently, is related to some known closed graph theorems concerning f (Theorem 4). The continuity in the second variable of the functions p_f turns out to be of no less interest — we prove a corresponding closed graph theorem (Theorem 5). In some special cases the assertions of Theorems 4 and 5 are comparable; then Theorem 5 requires weaker assumptions (Theorems 6 and 11). The most important case of this kind arises where T and X are topological groups, f is a homomorphism, and the members of P are left-invariant. Then every function p_f is also a left-invariant pseudo-metric (Theorem 10).

Under some hypotheses the homomorphism f is automatically nearly continuous, which immediately produces classical closed graph theorems.

2. Functions p_f^X. The letters t, u, v (respectively; x, y) will always stand for elements of T (resp., X), and the letters U, V (resp., Y) will stand for open sets in T (resp., X).

DEFINITION 1. For any p in P we define the corresponding function p_f^X on $T\times X$ to R^+ by the formula:

$$p_f^X(t,x) = \sup_{U\ni t}\inf_{t'\in U} p(f(t'),x), \quad t\in T,\ x\in X.$$

PROPOSITION 1. *For every $p\in P$, $t\in T$ and $x\in X$,*

(i) $p_f^X(t,f(t)) = 0$,

(ii) $p_f^X(t,x)\leqslant p(f(t),x)$,

(iii) $p_f^X(t,x) = \inf\sup_\sigma p(f(t_\sigma),x)$, *where the infimum is taken over all nets $\{t_\sigma\}$ in T converging to t.*

Proof. (i) and (ii). For any $U\ni t$ take $t'=t$.

(iii) Suppose that $U\ni t$ and $t\in\lim t_\sigma$. There exists a $t_{\sigma_0}\in U$, so that $\inf_{t'\in U} p(f(t'),x)\leqslant p(f(t_{\sigma_0}),x)\leqslant\sup_\sigma p(f(t_\sigma),x)$. This implies the inequality $\leqslant$. If $p_f^\sigma(t,x)<r$, then, as easily seen, there is a net $\{t_\sigma\}$ converging to t with. $\sup_\sigma p(f(t_\sigma),x)\leqslant r$. This yields the converse inequality.

Part (iii) of Proposition 1 was pointed out by Professor C. Ryll-Nardzewski

DEFINITION 2. Given $p\in P$, define the function f_p on T to R^+ by

$$f_p(t) = \inf_{U\ni t}\sup_{t'\in U} p(f(t'),f(t)), \quad t\in T.$$

The function f is *p-continuous at a point t* if and only if $f_p(t)=0$.

PROPOSITION 2. *For every $p\in P$, $t\in T$ and $x\in X$,*

$$p(f(t),x)\leqslant p_f^X(t,x)+f_p(t).$$

Hence, if f is continuous at t, then

$$p_f^X(t,x) = p(f(t),x) \quad \textit{for all } p\in P \textit{ and } x\in X.$$

Proof. The triangle inequality implies

$$p(f(t),x)\leqslant p(f(t''),x)+\sup_{t'\in U} p(f(t'),f(t)) \quad \text{for any } U\ni t \text{ and } t''\in T.$$

Hence

$$p(f(t),x)\leqslant \inf_{t''\in U} p(f(t''),x)+\sup_{t'\in U} p(f(t'),f(t)) \quad \text{for any } U\ni t.$$

Now we get

$$p(f(t),x)\leqslant p_f^X(t,x)+\sup_{t'\in U} p(f(t'),f(t)) \quad \text{for any } U\ni t,$$

which in turn implies the asserted inequality.

PROPOSITION 3. *For any $p\in P$, $t\in T$ and $x,y\in X$,*

(i) $p_f^X(t,x)\leqslant p_f^X(t,y)+p(x,y)$,

(ii) $|p_f^X(t,x)-p_f^X(t,y)|\leqslant p(x,y)$; *the function p_f^X is uniformly continuous in the second variable.*

Proof. (i) The triangle inequality yields

$$\inf_{t'\in U} p(f(t'),x)\leqslant p(f(t''),y)+p(x,y) \quad \text{for any } U\ni t \text{ and } t''\in T.$$

Hence

$$\inf_{t'\in U} p(f(t'),x)\leqslant p_f^X(t,y)+p(x,y) \quad \text{for any } U\ni t,$$

which gives the desired inequality.

Part (ii) follows immediately from (i).

PROPOSITION 4. *For any p in P, the function p_f^X is lower semicontinuous.*

Proof. Let $p_f^X(t,x)>r$; we must prove that there are $U\ni t$ and $Y\ni x$ such that $p_f^X(t',x')>r$ for all $t'\in U$ and $x'\in Y$. Let $p_f^X(t,x)>r'>r$. By Definition 1, there is a $U\ni t$ such that $p(f(t'),x)>r'$ for all $t'\in U$. The same argument shows that $p_f^X(t',x)\geqslant r'$ for $t'\in U$. By Proposition 3, for any $t'\in U$ and $x'\in Y=S(x,p,r'-r)$ (the sphere of p-radius $r'-r$ about x),

$$p_f^X(t',x')\geqslant p_f^X(t',x)-p(x,x')>r'-(r'-r)=r.$$

Let $G(f)$ denote the graph of f; $G(f)=\{(t,f(t)):\ t\in T\}$.

PROPOSITION 5. *A point (t,x) is in $\overline{G(f)}$ if and only if for every $p\in P$ the function p_f^X satisfies $p_f^X(t,x)=0$.*

Proof. Since P is directed by $\leqslant$, every open set $Y\ni x$ contains an open sphere $S(x,p,r)$, where $p\in P$ and $r>0$. Hence, the point (t,x) is in the closure of $G(f)$ if and only if

$$\forall_{p\in P}\ \forall_{r>0}\ \forall_{U\ni t}\ \exists_{t'\in U}\ p(f(t'),x)<r$$

if and only if

$$\forall_{p\in P}\ \forall_{r>0}\ p_f^X(t,x)<r.$$

DEFINITION 3. Let $t\in T$. The graph of f is *closed at t* if, for any $x\in X$, $(t,x)\in\overline{G(f)}$ implies $(t,x)\in G(f)$.

If f is continuous at t, then the graph of f is closed at t.

EXAMPLE 1. Let T, X and f be such that $\overline{G(f)}=T\times X$. The graph of f is not closed at any point t of T (unless X is one-point). By Proposition 5, $p_f^X\equiv 0$ for all p in P. This shows, in particular, that the continuity of all the functions p_f^X does not imply the continuity of f.

From Proposition 5 we get

PROPOSITION 6. *Let $t\in T$. Then*

(i) *The graph of f is closed at t if and only if for any $x\in X$, $p_f^X(t,x)=0$ for all $p\in P$ implies $x=f(t)$.*

(ii) *If $p_f^X(t,x)=p(f(t),x)$ for all $p\in P$ and $x\in X$, then the graph of f is closed at t.*

EXAMPLE 2. Let $T = X = R, f(t) = 1/t$ for $t \neq 0$ and $f(0) = 0$. Let p be the Euclidean metric for R, and $P = \{p\}$. Notice that $p_f^R(t, x) = p(f(t), x)$ for all $t, x \in R$. Nevertheless, f is not continuous at 0.

3. Functions p_f. Given $p \in P$, put $pf(u, v) = p(f(u), f(v))$ for $u, v \in T$; pf is a pseudo-metric for T. The function f is continuous at a point t if and only if for every p in P the corresponding pseudo-metric pf is continuous (jointly or in one of two variables) at the point (t, t). Now let us define functions which are the central object of our interest.

DEFINITION 4. For every $p \in P$ we define the corresponding function p_f on $T \times T$ to R^+ by the formula

$$p_f(u, v) = \sup_{U \ni u} \inf_{u' \in U} pf(u', v), \; u, v \in T.$$

Evidently, $p_f(u, v) = p_f^X(u, f(v))$ for all $u, v \in T$. Let us list some properties of the functions p_f which follow from the corresponding properties of the functions p_f^X.

THEOREM 1. *For every $p \in P$ and $u, v \in T$,*

(i) $p_f(u, u) = 0$; *if $p_f(u, v) = 0$ for all p in P and the graph of f is closed at u, then $f(u) = f(v)$,*

(ii) $p_f(u, v) = \inf\{\sup_\sigma pf(u_\sigma, v): u \in \lim u_\sigma\}$,

(iii) $p_f(u, v) \leqslant pf(u, v) \leqslant p_f(u, v) + f_p(u)$; *if f is continuous at u, th*[illegible]$_f(u, v) = pf(u, v)$ *for all $v \in T$ (not conversely),*

(iv) $|p_f(t, u) - p_f(t, v)| \leqslant pf(u, v)$,

(v) *the function p_f is lower semicontinuous in the first variable.*

Proof. Part (i) follows from Proposition 1(i) and Proposition 6(i); part (ii) from Proposition 1(iii); part (iii) from Proposition 1(ii), Proposition 2 and Example 2; part (iv) from Proposition 3(ii); and, finally, part (v) from Proposition 4.

Theorem 1(iii) shows that, if f is continuous, then $p_f \equiv pf$, and so p_f is a pseudo-metric for T $(p \in P)$. In general, the functions p_f need not be even symmetric.

EXAMPLE 3. Let T, X, p and P be as in Example 2. Put $f(t) = 0$ for $t \neq 0$ and $f(0) = 1$. Then $p_f(0, t) = 0$ for all $t \in R$, but $p_f(t, 0) = 1$ for all $t \neq 0$.

However, it appears that if a function p_f is symmetric, then the triangle inequality is automatically satisfied.

THEOREM 2. *Let $p \in P$. If the function p_f is symmetric, then p_f is a pseudometric.*

Proof. Let $t, u, v \in T$ and $r > p_f(u, v)$; it is sufficient to prove that $p_f(u, t) \leqslant p_f(t, v) + r$. Choose any $U \ni u$. By Definition 4, there is a $u' \in U$ with $pf(u', v) < r$. By the symmetry assumption, Theorem 1(iv) and the choice of u',

$$p_f(u', t) = p_f(t, u') \leqslant p_f(t, v) + pf(u', v) < p_f(t, v) + r.$$

By Definition 4 again, there exists a $u'' \in U$ such that

$$pf(u'', t) < p_f(t, v) + r.$$

Since U containing u was arbitrary, this yields the desired inequality.

Let us recall the definitions of nearly-openness and nearly-continuity, which are convenient in the field of closed graph and open mapping theorems (cf. Kelley & Namioka [4] or Schaefer [8]). A subset of a topological space is called *nearly open* if it is in the interior of its closure; a function f is called *nearly continuous* (resp.; *nearly open*) if the counter image (resp.; image) of any open set is nearly open. Pták [6] introduced the following "localized" definition of nearly-continuity.

DEFINITION 5. Let $t \in T$. The function f is *nearly continuous at* t if for every open neighborhood Y of $f(t)$, t is in the interior of the closure of $f^{-1}(Y)$.

Clearly, f is nearly continuous if and only if f is nearly continuous at every point.

DEFINITION 6. Let $t \in T$. A function g on $T \times T$ to R is continuous in the first (resp.; second) variable at (t, t) if the function $g(\cdot, t)$ (resp.; $g(t, \cdot)$) on T to R is continuous at t.

The next two definitions concern the case where T is a Tychonoff space and $\mathcal{U}$ is a uniformity (with symmetric members) for T. Here, $S(t, U)$ denotes the sphere of the radius U about t; $S(t, U)$ $\{u \in T: (t, u) \in U\}$. $\mathcal{Y}$ denotes the uniformity for X generated by P.

DEFINITION 7. The function f is *uniformly nearly continuous* if

$$\forall_{Y \in \mathcal{Y}} \; \exists_{U \in \mathcal{U}} \; \exists_{t \in T} \; S(t, U) \subset \overline{f^{-1}(S(f(t), Y))}.$$

DEFINITION 8. A function g on $T \times T$ to R is uniformly continuous at the points of the diagonal $\Delta(T)$ if

$$\forall_{\varepsilon > 0} \; \exists_{U \in \mathcal{U}} \; \forall_{t \in T} \; \forall_{u, v \in S(t, U)} \; |g(t, t) - g(u, v)| < \varepsilon.$$

Evidently, the continuity of f (or uniform nearly-continuity of f) implies nearly-continuity of f; uniform continuity of g at the points of $\Delta(T)$ implies continuity of g at the points of $\Delta(T)$. There is a strict connection between the nearly-continuity of f and the continuity of the corresponding functions p_f, described by the following

THEOREM 3. (i) *Let $t \in T$. The function f is nearly continuous at t if and only if for every $p \in P$ the function p_f is continuous in the first variable at the point (t, t).*

(ii) *Let T be a Tychonoff space and $\mathcal{U}$ a uniformity for T. The function f is uniformly nearly continuous if and only if for every $p \in P$ the function p_f is uniformly continuous at the points of the diagonal $\Delta(T)$.*

Proof. (i) The following successive conditions are equivalent:

f is nearly continuous at t;

$$\forall_{p \in P} \; \forall_{\varepsilon > 0} \; t \in \operatorname{Int} \overline{f^{-1}(S(f(t), p, \varepsilon))};$$

$$\forall_{p \in P} \; \forall_{\varepsilon > 0} \; \exists_{U \ni t} \; \forall_{u \in U} \; \forall_{U' \ni u} \; \exists_{u' \in U'} \; p(f(u'), f(t)) < \varepsilon;$$

$$\forall_{p \in P} \; \forall_{\varepsilon > 0} \; \exists_{U \ni t} \; \forall_{u \in U} \; p_f(u, t) < \varepsilon;$$

$\forall_{p \in P}$ p_f is continuous in the first variable at (t, t).

(ii) The following successive conditions are equivalent:

f is uniformly nearly continuous;

$$\underset{p\in P}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{U\in\mathcal{U}}{\exists}\ \underset{t\in T}{\forall}\ S(t,U)\subset\overline{f^{-1}\big(S(f(t),p,\varepsilon)\big)};$$

$$\underset{p\in P}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{U\in\mathcal{U}}{\exists}\ \underset{u\in T}{\forall}\ \underset{v\in S(u,U)}{\forall}\ \underset{U'\ni v}{\forall}\ \underset{v'\in U'}{\exists}\ v'\in f^{-1}\big(S(f(t),p,\varepsilon)\big);$$

$$\underset{p\in P}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{U\in\mathcal{U}}{\exists}\ \underset{\substack{u,v\in T\\(u,v)\in U}}{\forall}\ p_f(u,v)<\varepsilon;$$

$$\underset{p\in P}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{V\in\mathcal{U}}{\exists}\ \underset{t\in T}{\forall}\ \underset{u,v\in S(t,V)}{\forall}\ p_f(u,v)<\varepsilon\ \text{(take } V\circ V\subset U);$$

$\underset{p\in P}{\forall}$ p_f is uniformly continuous at the points of $\Delta(T)$.

The property dual to that appearing in Theorem 3(i) — concerning the continuity in the *second* variable of the functions p_f — depends on the uniformity $\mathcal{Y}$ generated by the family P.

PROPOSITION 7. *Let $t\in T$. The function f has the property that for every p in P the corresponding function p_f is continuous in the second variable at the point (t,t) if and only if for every $Y\in\mathcal{Y}$, t is in the interior of the set $\{u\in T\colon t\in\overline{f^{-1}\big(S(f(u),Y)\big)}\}$.*

The proof is similar to that of Theorem 3(i).

4. Closed graph theorems. A topological space T is called (Kelley [3]) *metrically topologically complete* if there is a complete metric for T generating the given topology.

THEOREM 4 (cf. [9], [5] and [1]). *Let (X,p) be a complete metric space. Suppose that at least one of the following three conditions is satisfied*:

(a) *T is metrically topologically complete,*

(b) *the graph of f is metrically topologically complete in its relative product topology,*

(c) *the counter image of any compact set is compact.*

Then the following three conditions are equivalent:

(i) *f is continuous,*

(ii) *the graph of f is closed and f is nearly continuous,*

(iii) *the graph of f is closed and the function p_f is continuous in the first variable at every point of the diagonal $\Delta(T)$.*

Proof. (i)⇔(ii) Parts (a) and (b) are due to Weston [9] and Pettis [5]; in this form they are given in [5] — the proof is based on a very interesting result of [9]. Part (c) is a special case of the recent result of Byczkowski and Pol [1], which asserts the same for any space X topologically complete (in the sense of Čech).

(ii)⇔(iii) follows from Theorem 3(i).

Our central result, Theorem 5, shows that the dual statement — concerning the continuity in the *second* variable of the function p_f — is also true. It is worth noting that Theorem 4 cannot be "localized" — the assumptions at a single point are not sufficient for the implications (ii)⇒(i) or (iii)⇒(i). Let us also emphasize that in Theorem 5 we need no assumptions like (a), (b) or (c) of Theorem 4; in particular, T is an arbitrary topological space.

THEOREM 5. *Let (X,p) be a complete metric space. Let $t\in T$. The function f is continuous at t if and only if the graph of f is closed at t and the function p_f is continuous in the second variable at the point (t,t).*

During our participation in the Fourth Prague Topological Symposium we became acquainted with the nondiscrete induction theorem due to Professor V. Pták [7]. Let us formulate, as a lemma, a special case of that useful result, which enables us to simplify our original proof of Theorem 5. Here, for any $A\subset X$ and $r>0$, $S(A,r)$ denotes the open sphere of p-radius r about A.

LEMMA (cf. [7]). *Let (X,p) be a complete metric space. Let $Z(r)$, $r\in(0,1)$, be closed subsets of X such that $Z(r)\subset Z(r')$ for $r<r'$. Let $Z(0)$ denote the intersection of all $Z(r)$, $r\in(0,1)$. If*

$$Z(r)\subset S(Z(r/2),r)\quad \textit{for each } r\in(0,1),$$

then

$$Z(r)\subset S(Z(0),2r)\quad \textit{for each } r\in(0,1).$$

Proof ([7]). Let $x\in Z(r)$. Since $x\in S(Z(r/2),r)$, there is an $x_1\in Z(r/2)$ with $p(x,x_1)<r$. Since $x_1\in S(Z(r/4),r/2)$, there is an $x_2\in Z(r/4)$ with $p(x_1,x_2)<r/2$. Since $x_2\in S(Z(r/8),r/4)$, there is an $x_3\in Z(r/8)$ with $p(x_2,x_3)<r/4$. Continuing this process, we obtain a p-Cauchy sequence $\{x_n\}$; let $y=\lim\limits_{n\to\infty}x_n$. $\{Z(r/2^n)\}$ is a decreasing sequence of closed sets and $x_n\in Z(r/2^{n+1})$, so that y is in $\bigcap\limits_{n=1}^{\infty}Z(r/2^n)$ $=Z(0)$. Now

$$p(x,y)\leqslant p(x,x_1)+p(x_1,x_2)+p(x_2,x_3)+\ldots<2r.$$

Hence x is in $S(Z(0),2r)$.

Proof of Theorem 5. Put $Z(r)=\{x\in X\colon p_f^X(t,x)\leqslant r/2\}$ for $r\in(0,1)$. Proposition 3(ii) (or 4) implies that each $Z(r)$ is closed. Since $Z(0)=\{x\colon p_f^X(t,x)=0\}$ and the graph of f is closed at t, Proposition 6(i) shows that $Z(0)=\{f(t)\}$. The function p_f is assumed to be continuous in the second variable at (t,t), so that for each $r\in(0,1)$ there is an open $U_r\ni t$ such that for any $u\in U_r$, $p_f^X(t,f(u))$ $=p_f(t,u)<r/2$. Thus $f(U_r)\subset Z(r)$ for each r. Given $x\in Z(r)$, we have $p_f^X(t,x)<r$, and so, by the definition of p_f^X, there exists a $t'\in U_{r/2}\subset f^{-1}(Z(r/2))$ with $p(f(t'),x)<r$; x is in $S(Z(r/2),r)$. We may apply the lemma, which yields

$$f(U_r)\subset Z(r)\subset S(f(t),2r)\quad \text{for each } r\in(0,1).$$

This proves the continuity of f at t.

We do not know if the induction theorem can also be applied to obtain a simple proof of Theorem 4.

THEOREM 6. *Let T be a Tychonoff space and $\mathcal{U}$ a uniformity for T. Let (X, p) be a complete metric space. The following three conditions are equivalent*:

(i) *f is uniformly continuous*;

(ii) *the graph of f is closed and f is uniformly nearly continuous*;

(iii) *the graph of f is closed and the function p_f is uniformly continuous at the points of the diagonal $\Delta(T)$.*

Proof. The equivalence of (ii) and (iii) is an immediate consequence of Theorem 3.

(i)⇒(iii) follows from Theorem 1(iii).

(iii)⇒(i) Since p_f is uniformly continuous at the points of $\Delta(T)$, for each $r \in (0, 1)$ there is an open $U^r \in \mathcal{U}$ such that for any $t \in T$ and $u \in U_r = S(t, U^r)$, $p_f(t, u) < r/2$. In consequence, the proof of Theorem 5 yields the uniform continuity of f.

In case T is a metrizable space, Theorem 6 follows from the closed graph theorem of Pták [7] (the last result concerns more general objects than functions, namely relations).

5. Some generalizations. Let us introduce, for brevity, the following

DEFINITION 9. Let $t \in T$. The graph of f is *p-complete at t* (where $p \in P$) if, for any net $\{t_\sigma\}$ convergent to t and such that $\{f(t_\sigma)\}$ is a p-Cauchy net, $\lim_\sigma pf(t_\sigma, t) = 0$. The graph of f is *P-complete at t* if the above holds for every pseudo-metric p in P, and P-complete if it is P-complete at any point of T.

If the graph of f is P-complete at t, then it is closed at t. The equivalence holds provided that (X, p) is a complete metric space and $P = \{p\}$. The word "complete" cannot be omitted:

PROPOSITION 8. *Let (X, p) be a metric space. Suppose that, for any metrizable space T, the graph of any function $f: T \to X$ is p-complete whenever it is closed. Then X is complete.*

Proof. Let $\tilde{X}$ denote the completion of X; assume, to get a contradiction, that there is a point x_0 in $\tilde{X} \setminus X$. Consider the set $T = X \cup \{x_0\}$ with the metrizable relative topology. Put $f(x) = x$ for $x \in X$ and $f(x_0) = x_1 \in X$. The graph of the function f on T to X is closed, but is not p-complete at $x_0 \in T$.

We omit the easy proof of the following

PROPOSITION 9. *Fix $p \in P$. Let $(\tilde{X}, \tilde{p})$ denote the completion of the quotient metric space associated with (X, p). For any $t \in T$ let $\tilde{f}(t)$ be the equivalence class of $f(t)$ in $\tilde{X}$; $\tilde{f}$ is a function on T to $\tilde{X}$. Then*

(i) *the graph of $\tilde{f}$ is closed at a point t if and only if the graph of f is p-complete at t,*

(ii) *$\tilde{p}\tilde{f} \equiv pf$ and $\tilde{p}_{\tilde{f}} \equiv p_f$.*

The next three theorems are direct consequences of Theorems 4, 5, 6 (respectively) and of Proposition 9 (for any $p \in P$, we apply those theorems to the function $\tilde{f}: T \to \tilde{X}$ as in Proposition 9).

THEOREM 7. *Let T be a metrically topologically complete space. The following three conditions are equivalent*:

(i) *f is continuous*;

(ii) *the graph of f is P-complete and f is nearly continuous*;

(iii) *the graph of f is P-complete and for every $p \in P$ the function p_f is continuous in the first variable at any point of the diagonal $\Delta(T)$.*

THEOREM 8. *Let $t \in T$. The function f is continuous at t if and only if the graph of f is P-complete at t and for every $p \in P$ the function p_f is continuous in the second variable at the point (t, t).*

THEOREM 9. *Let T be a Tychonoff space and $\mathcal{U}$ a uniformity for T. The following three conditions are equivalent*:

(i) *f is uniformly continuous*;

(ii) *the graph of f is P-complete and f is uniformly nearly continuous*;

(iii) *the graph of f is P-complete and for every $p \in P$ the function p_f is uniformly continuous at the points of the diagonal $\Delta(T)$.*

6. Case of topological groups. In this section T and X are Hausdorff topological groups, the members of P are left-invariant, and f is a homomorphism.

THEOREM 10. *For every $p \in P$ the function p_f is a left-invariant pseudo-metric for T and*

$$p_f(u, v) = \sup_{\substack{U \ni u \\ V \ni v}} \inf_{\substack{u' \in U \\ v' \in V}} pf(u', v') \quad \text{for all } u, v \in T.$$

Proof. Let g denote the function on $T \times T$ to R^+ defined by the right side of the asserted equality. For any $U \ni u$ and $V \ni v$ we have

$$p_f(u, v) \geqslant \inf_{\substack{u' \in U \\ v' \in V}} pf(u', v').$$

This yields the inequality $p_f \geqslant g$. To prove the converse one, take any open set U containing u. Choose $U' \ni u$ and $V' \ni v$ so that $V'V'^{-1}U' \subset U$. Given $u' \in U'$ and $v' \in V'$, put $u'' = vv'^{-1}u'$; $u'' \in V'V'^{-1}U' \subset U$ and

$$pf(u'', v) = p\big(f(v)f(v')^{-1}f(u'), f(v)\big) = pf(u', v').$$

Hence

$$\inf_{u'' \in U} pf(u'', v) \leqslant \inf_{\substack{u' \in U' \\ v' \in V'}} pf(u', v') \leqslant g(u, v),$$

which implies the inequality $p_f \leqslant g$. Since the function g is symmetric, so is p_f. By Theorem 3, p_f is a pseudo-metric. Finally, p_f is left-invariant:

$$\begin{aligned} p_f(tu, tv) &= \sup_{U \ni tu} \inf_{u' \in U} pf(u', tv) = \sup_{U' \ni u} \inf_{t^{-1}u' \in U'} pf(u', tv) \\ &= \sup_{U' \ni u} \inf_{u'' \in U'} pf(tu'', tv) \\ &= \sup_{U' \ni u} \inf_{u'' \in U'} pf(u'', v) = p_f(u, v). \end{aligned}$$

COROLLARY. *The following four conditions are equivalent:*

(i) *f is nearly continuous at the identity e;*

(ii) *for every $p \in P$ the function p_f is continuous in the first (or second) variable at the point (e, e);*

(iii) *f is uniformly nearly continuous;*

(iv) *for every $p \in P$ the function p_f is uniformly continuous at the points of the diagonal $\Delta(T)$.*

Now, taking into account Corollary, let us see what results from Theorems 4 and 5. Theorem 5 provides more information:

THEOREM 11 (Kelley [3], Problem R on p. 213). *Let X be a metrizable topological group which is complete relative to its left uniformity. The homomorphism f is continuous if and only if the graph of f is closed and f is nearly continuous.*

Similarly, Theorem 8 gives a more accurate result than Theorem 7:

THEOREM 12. *The homomorphism f is continuous if and only if the graph of f is P-complete and f is nearly continuous.*

Finally, let us recall some assumptions under which the homomorphism f is automatically nearly continuous:

(1) T is of the second category and X has the Lindelöf covering property (cf. Kelley [3], Problem R on p. 213),

(2) T is of the second category and $f(T)$ is separable (cf. Weston [9], Theorem 3 on p. 345).

(3) T is of the second category and T and X are linear topological spaces over the field of rationals (cf. ibidem),

(4) T and X are locally convex spaces, T is barrelled and f is linear (cf. Kelley & Namioka [4], Problem E on p. 106).

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