An application of nonstandard analysis to category theory

by

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Abstract. In this paper we introduce a novel application of category theory, namely, to the theory of triples in category theory. In particular, we show that the category of compact Hausdorff space is tripleable over sets. The technique makes use of second enlargements.

1. Introduction. Nonstandard analysis has had applications to many different areas of mathematics. In this paper, we introduce a novel application, namely, to the theory of triples in category theory. Specifically, we shall prove a main result of [5]. Namely; that the category of compact Hausdorff spaces is tripleable over sets.

The methods of nonstandard analysis are especially suitable for the study of the category of compact Hausdorff spaces. For example, instead of dealing with ultrafilters on the set of ultrafilters, we deal with points in a second enlargement. Actually, it was especially the discovery of Theorem 5 that clinched the view that nonstandard analysis is the "right" approach. This was further vindicated by the proof of Lemma 5 in the main theorem.

A large part of this paper can be read with a minimum of familiarity of category theory. In fact, the main result which gives a strange characterization of compact Hausdorff spaces in terms of second enlargements will be stated in a form which makes no mention of category theory. This is done in section 5 which can be read independently of the rest of the paper. Hopefully the result is of some interest in this form although the main interest lies in its connection with [5].

The basic references in category theory are [1] where the basic results were first proved, and [2], an expository survey. Incidentally, the lucid style of the author and the nature of the material makes [2] surprisingly easy to read in spite of its being in a language foreign to many readers. [2] contains excellent examples for motivating this subject which are difficult to find elsewhere.

As stated before, in order to read this paper it is not necessary to be familiar with [1] or [2] in advance, in fact, a nonstandard analyst who is interested can use the example studied in this paper as motivation for reading [1] or [2] afterwards.

2. The Stone-Čech compactification. It is well-known that to every set S there corresponds a compact Hausdorff space βS containing S with the property that for every compact Hausdorff space X, every map from S into X can be uniquely extended to a continuous map from βS into X. In fact βS is the space of ultrafilters on S and S is regarded as a subset of βS by identifying each point p of S with the unique ultrafilter containing (p). In the language of category theory one says that the functor β from sets to compact Hausdorff spaces is left adjoint to the forgetful functor from compact Hausdorff spaces to sets.

Here, we study the adjoint situation by nonstandard methods. Consider an enlargement of a structure containing S. Then it is classical by now that βS can be obtained as a quotient space of S^*. In fact for $x \in S^*$ let $e(x) = \{A \subset S\colon x \in A^*\}$. Then $e(x)$ is an ultrafilter and e is a many-one epic map. When several sets are being simultaneously considered we shall use the notation ${}^eS$. If sets of the form A^* are taken as a base for a topology on S^* then the quotient topology with respect to e is the usual topology on BS. A direct proof that S^* is compact using transfer and a suitable concurrent relation is also well-known. Here we give a proof using second enlargements. First, we prove a lemma of independent interest.

LEMMA. *Let S^{**} be an enlargement of S^*. Then the monad $m(x)$ of a point $x \in S^*$ is given as*: $\{y\colon \forall A \subset S (x \in A^* \to y \in A^{**})\}$.

Remark. This says that x and y correspond to the same ultrafilter on S. Note that the implication may be replaced by an equivalence.

Proof. This is obvious by definition of monad.

THEOREM 1. *S^* is compact.*

The proof is now trivial. Since S^{**} is an enlargement of S every point $y \in S^{**}$ corresponds to an ultrafilter on S. If $x \in S^*$ corresponds to the same ultrafilter then $y \in m(x)$. Thus every point is nearstandard.

Since βS is a quotient of S^*, βS is also compact.

The same lemma may also be used to show that βS is Hausdorff. S^* is, of course, not Hausdorff. In fact, if $e(x) = e(y)$ then x and y have the same monads.

e extends to an epic map from S^{**} to $(\beta S)^*$. Then if $x \in S^*$, $e[m(x)] = m[e(x)]$. (This uses the fact that $e(x) = e(y)$ implies that x and y are contained in the same open sets.) Now $e(x) \neq e(y)$ implies by the lemma that $m(x) \cap m(y) = \emptyset$. Hence $m[e(x)] \cap m[e(y)] = e[m(x)] \cap e[m(y)] = \emptyset$. The last statement follows from the fact $m(x)$ and $m(y)$ are complete inverse images with respect to e.

The possibility of defining βS by nonstandard methods goes back almost to the beginning of the study of nonstandard analysis (e.g. see [3] and [4]). Here, we are primarily interested in the rest of the adjoint situation. We show next the convenient fact that from a nonstandard viewpoint β operates on maps essentially by transfer.

Let $S \xrightarrow{f} T$. Then f induces a map $\beta S \xrightarrow{\beta f} \beta T$ in the following manner. Let F be an element of βS. Then $\beta f(F) = \{A\colon f^{-1}(A) \in F\}$. This is the same as saying that $\beta f(F)$ is generated by the sets of the form $f(A)$ with $A \in F$. Equivalently, this definition can be expressed topologically by saying that βf is the unique continuous extension of the composition $S \xrightarrow{f} T \xrightarrow{i} \beta T$ where i is the usual embedding.

THEOREM 2. *The following diagram is commutative*

$$\begin{array}{ccc} S^* & \xrightarrow{e_S} & \beta S \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \beta f} \\ T^* & \xrightarrow[e_T]{} & \beta T \end{array}$$

Proof. Let $x \in S^*$. Suppose $A \in (Bf)e_S(x)$. Then $f^{-1}(A) \in e_S(x)$, hence $x \in [f^{-1}(A)]^*$. Now $\forall x[x \in f^{-1}(A) \to f(x) \in A]$ is obviously ture, so by transfer we obtain: $\forall x[x \in [f^{-1}(A)]^* \to f^*(x) \in A^*]$. Therefore $A \in e_T f^*(x)$. Since an ultrafilter cannot properly contain another ultrafilter this completes the proof.

The above result shows that βf can be defined in a nonstandard way; namely, as the quotient map induced from f^*. We next show that the functorial properties of β can be obtained directly by nonstandard means (i.e. without using properties of ultrafilters).

We say that $x, y \in S^*$ are indiscernible written $x \sim y$, if $(\forall A \in S)[x \in A^* \leftrightarrow y \in A^*]$. Then e can be regarded as the map which takes each element of S^* into its equivalence class under $\sim$, and βX as the quotient set. It is sometimes convenient to extend this relation to elements of different enlargements, e.g. the lemma says that the monad of a point p in S^* is the set of all points equivalent to p. (as enlargements of S).

Consider a second enlargement of a structure containing a small category of sets. (Very few sets are really necessary at one time. e.g. to state the functorial properties of transfer precisely three sets will suffice.) It is easy to see by transfer that the operator which takes each set S into its enlargement S^* and the map $S \xrightarrow{f} T$ into its extension $S^* \xrightarrow{f^*} T^*$ is a functor.

THEOREM 3. *$S^* \xrightarrow{f^*} T^*$ is continuous.*

Proof. We use second enlargements. First, we show that f^* preserves the equivalence relation defined above. (Although this follows from Theorem 2 we prefer to avoid dealing with ultrafilters). Suppose $x \sim y$ and $f^*(x) \in B^*$ where $B \subset T$. Then $x \in (f^*)^{-1}(B^*) = [f^{-1}(B)]^*$. Hence $y \in [f^{-1}(B)]^* = (f^*)^{-1}(B^*)$. Thus $f^*(y) \in B^*$. Since this is valid for all B we obtain $f^*(x) \sim f^*(y)$.

We now apply this to the second enlargement $S^{**} \xrightarrow{f^{**}} T^{**}$. If $x \in S^*$, then $m(x) = (y \in S^{**}\colon y \sim x)$ and similarly if $x \in T^*$ then $m(x) = (y \in T^{**}\colon y \sim x)$. Hence if $f^*(x_1) = x_2$, by the above result $f^{**}[m(x_1)] \subset m(x_2)$. Therefore f^* is continuous.

From the fact that f^* preserves the equivalence relation it follows that f^* induces a continuous map from βS into βT and that β is a functor.

Now let C be a compact Hausdorff space and C^* an enlargement of C regarded as a set.

THEOREM 4. *The standard point map* st *from C^* onto C is continuous.*

Note. The topology on C^* is independent of the topology on C although the standard point map, of course, depends on the latter topology.

Proof. Suppose $\mathrm{st}(x) \in U$ where U is open in C. Choose V open so that $\mathrm{st}(x) \in V \subset \bar{V} \subset U$. Since $\mathrm{st}(x) \in V$, $x \in V^* \subset \bar{V}^*$. If $y \in \bar{V}^*$ then $\mathrm{st}(y) \in \bar{V} \subset U$. Therefore $\{y\colon y \in \bar{V}^*\}$ is an open set containing x which is mapped into U.

Since $x \sim y \to \mathrm{st}(x) = \mathrm{st}(y)$, st induces a continuous map from βC onto C. From the point of view of ultrafilters it is immediate that this is the map which takes each ultrafilter into the point it converges to.

We can now easily show that βS has the universal property. Let C be a compact Hausdorff space and let f be a map from S into C. Since S is dense in βS there is at most one continuous extension of f into a map from βS into C. On the other hand by Theorems 3 and 4 st. f is continuous and clearly extends f.

3. The triple associated with B. Having shown that β satisfies the required universal property, we are ready to study the associated triple. (A triple is also called a standard construction or a monad. For nonstandard analysts it is, of course, advisable to avoid the latter term for this concept!) We begin with a result which can be understood without any prior knowledge of the concept of a triple. This result, which states how a certain mapping on ultrafilters translates to a mapping on enlargements, illustrates the advantages of nonstandard methods.

Let F be an ultrafilter on βS. Then we define an element $\mu F \in \beta S$ as follows: $A \in uF$ iff $(G\colon A \in G) \in F$ (μ as will be pointed out later is an ingredient of the triple.) We also define a map μ' from S^{**} into S^* as follows: Let $x \in S^{**}$ and choose $y \in S^*$ such that $x \sim y$. Then $\mu' x = y$. (The existence of a choice for y causes a certain inconvenience. e.g. We do not make μ' into a functor. However, this is not a serious problem. We now show that μ' is essentially the same as μ. Precisely we have

THEOREM 5.

$$\begin{array}{ccccc} S^{**} & \overset{e_{S^*}}{\to} & (\beta S)^* & \overset{\beta S}{\to} & \beta(\beta S) \\ {\scriptstyle \mu'}\downarrow & & & & {\scriptstyle \mu}\downarrow \\ S^* & & \xrightarrow{\qquad e_s \qquad} & & \beta S \end{array} \quad \textit{commutes.}$$

Proof. Let $x \in S^{**}$ and $A \in e_S \mu' x$. Then $\mu' x \in A^*$. Hence $x \in A^{**}$ since $\mu' x$ is indiscernible with x. Now let H be the set of all ultrafilters containing A. Then $e_s A^* = H$. By transfer $e_{s^*} A^{**} = H^*$. Hence $e_{s^*} x \in H^*$. Therefore $H \in e_{\beta S} e_{s^*} x$. Finally, by definition of μ, $A \in \mu e_{\beta S} e_{s^*} x$. Since A was arbitrary in $e_s u' x$ this completes the proof.

The lack of a one-one correspondence between S^* and βS is a slight nuisance which is accentuated in a second enlargement. We cannot simply say that $\mu' = \mu$ although with abuse of terminology this is essentially what Theorem 5 says.

There is one point of caution required in studying second enlargements. A set S as a subset of S^* gets enlarged to an elementary extension $^*S$ of S. Although $^*S$ is elementarily equivalent to S^* they may not even have the same cardinality. The embedding in S^{**} is necessarily different, in fact, $S^* \cap {}^*S = S$. (As an illustrative example, when one takes an ultrapower of an ultrapower of the integers with respect to a nonprincipal ultrafilter on a countable set (these are not enlargements) then S^{**} is a cofinal extension of S^* but an end extension of $^*S$.)

Nevertheless it is still possible to get by with an abuse of terminology e.g. a map from S^{**} to $^*S$ may be regarded as a map from S^{**} to S^* if the image in $^*S$ is replaced by an indescirnable element in S^*. This is possible since what really matters is the image under $\sim$.

Although Theorem 5 can be understood without the concept of a triple the latter is needed to appreciate the significance of μ and hence of μ'. A triple on a category C consists of three ingredients. A functor F from C into C, a natural transformation η from the identity into F and a natural transformation μ from F^2 into F, satisfying the axioms $u \cdot F\eta = u \cdot \eta F = 1$ and $\mu \cdot F\mu = \mu \cdot \mu F$. A pair of adjoint functors gives rise to a triple. For the details see [1] or [2]. Here it will suffice to say what the ingredients become in our special case. F is β regarded as a functor from sets to sets or more precisely β followed by the forgetful functor. ηs is the inclusion $S \to \beta S$. μs is the map $\beta\beta S \to \beta S$ defined earlier. μ can also be defined as the unique continuous extension of the identity map $\beta S \to \beta S$ to $\beta\beta S \to \beta S$.

It is now possible to prove a theorem which bears the same relationship to Theorem 5 that Theorem 3 bears to Theorem 2, i.e., we can get μ directly by nonstandard methods.

With abuse of terminology $FS = S^*$ and ηS is the inclusion $S \to S^*$. It is also easy to see that μ' is continuous. In fact $(u')^{-1} A^* = A^{**}$. This fact together with the continuity of the horizontal maps in the diagram for Theorem 5 leads to an alternative proof of Theorem 5 (using only the continuity of u rather than its explicit definition). The routine details are left to the reader.

It is also possible to show directly by nonstandard means that F, η, and μ satisfy the axioms for a triple without using the fact that they arise out of a pair of adjoint functors. Since one of the axioms requires a third enlargement compounding the nuisance referred to earlier and since the details are tedious though straightforward and finally since this is outside our main development anyway we leave the proof to the interested reader.

4. The Eilenberg-Moore category associated with B. It was shown in [1] that a triple gives rise to a pair of adjoint functors. In fact given a category E with a triple $\{F, \eta, \mu\}$ we define a new category D as follows: The objects are maps $FA \overset{\xi}{\to} A$ such that (1) $\xi\eta = 1$ and (2) $\xi \cdot F\xi = \xi \cdot \mu$. The morphisms are maps $A \overset{f}{\to} B$ such that $f\xi = \xi Ff$. Then a pair of adjoint functors can be defined on C and D which induces the given triple. A category which arises in such a manner from the category of sets is said to be tripleable over sets.

The category D seems to be highly contrived at first. This is the point where [2], especially Chapter 4, is highly recommended for the examples which are a help in appreciating the significance of D. We include an example here for motivation.

The category of universal algebras of a given type is a canonical example: A universal algebra gives rise to a triple as follows. Given a set S, FS is the set of words generated by S, ηS is the inclusion of S into FS, and μS corresponds to the simplification of a word on words on S to a word on S. For example for abelian groups if $S = \{x, y\}$ then $FS[\{mx+ny\}\colon m, n \in Z]$ and for a typical element x of FFS e.g. $5(2x+3y)+4(3x+5y)$, $ux = 22x+35y$. A map $FS \xrightarrow{\xi} S$ is essentially an algebra structure on S. Condition (1) says that a trivial word is mapped into itself. Condition (2) is a compatibility condition which says that the two following ways of evaluating a word on a word give the same answer. First simplify and then evaluate, evaluate the inside words first ad then evaluate the outside word on the answers obtained. The morphisms correspond to homomorphisms.

This heuristically illustrates the fact that the category D arising from the triple is precisely the given category. Thus the triple can be regarded as being a concise way of giving the operations and relations that define a type of an algebra and an object $FS \xrightarrow{\xi} S$ as a concise way of giving an algebra of the given type. (ξ combines all the operations at once. Note also that the relations are built in e.g. FS is different for groups and for abelian groups.

The main result of [5] is that the category of compact Hausdorff spaces also arises as such a category D. This is a precise way of expressing a heuristic fact familiar from general topology, namely that the category of compact Hausdorff spaces although apparently topologically defined, behaves very much like a category of algebras.

By Theorems 2 and 5 the category D can be expressed in an especially succinct manner. The objects are maps $S^* \xrightarrow{\xi} S$ which are constant on indiscernibles. (Note that this is the same as a map $\beta S \to S$.) (1) says that ξ is a retraction, i.e. the identity on S. (2) says roughly that if $x \in S^{**}$ and $y \in S^*$ is indiscernible with x then $\xi(y) = \xi \cdot \xi^*(x)$. (The latter statement requires some caution in line with previous remarks. Strictly speaking ξ^* maps S^{**} into ${}^*S$, thus $\xi \cdot \xi^*(x)$ makes no sense. However, since ξ is constant on indiscernibles, ξ can also be regarded as a map from ${}^*S$ into S more precisely, in the next section we shall replace $\xi \cdot \xi^*$) by $\xi \cdot \nu \cdot \xi^*$ where ν maps an element of ${}^*S$ into an indiscernible in S^*. The morphisms are maps $S \xrightarrow{f} T$ satisfying $f\xi = \xi \cdot f^*$.

5. The Main Theorem. We can now state the main result in a form which does not use concepts in category theory. Let S^{**} be a second enlargement of S. For the benefit of readers who joined us in this section we recall the definition that x is indiscernible with y (written $x \sim y$) iff $(\forall A \in S)$ $(x \in A^*$ iff $y \in A^*)$. This definition can be extended to apply to elements of different enlargements, in particular to elements of S^{**} and S^* and to elements of S^* and the enlargement ${}^*S$ of S in S^{**} (note that $S^* \neq {}^*S$). We now fix two mappings μ and ν where μ maps each element in S^{**} to an indiscernible in S^* and ν maps each element in ${}^*S$ to an indiscernible in S^* the choice of the mappings does not matter. We can now state the main theorem.

MAIN THEOREM. *Let C^{**} be a second enlargement of a compact Hausdorff space: Then the standard part map* st. x *from C^* to C satisfies*

(1) $x \sim y \to \operatorname{st} x = \operatorname{st} y$,

(2) $x \in C \to \operatorname{st} x = x$,

(3) *if* $x \in C^{**}$ *then* $\operatorname{st} \cdot \mu(x) = \operatorname{st} \cdot \nu \cdot \operatorname{st}^*(x)$.

Conversely for any set S, any map ε from S^* to S satisfying the above 3 properties is the standard point map of a unique compact Hausdorff structure on S.

Proof. Since indiscernibles are, in particular, contained in the same open sets st. is constant on indiscernibles. Condition (2) is trivial. For condition (3) let $x \in C^{**}$ and let p be the standard part of x (note that C^{**} is an enlargement of C). Since μx is indiscernible with x then $\operatorname{st}\mu(x) = p$. Now let V be an open set containing p and let w be open so that $p \in W \subset \overline{W} \subset V$. Since $\overline{W}$ is closed, $\operatorname{st}(\overline{W})^* \subset \overline{W}$. Hence, by transfer, $\operatorname{st}^*(\overline{W}^{**}) \subset {}^*\overline{W}$. Therefore $\nu \operatorname{st}^*(\overline{W}^{**}) \subset \overline{W}^*$. Now $x \in W^{**}$ since p is the standard part of x. Hence $x \in \overline{W}^{**}$. Thus $\nu \operatorname{st}^*(x) \in \overline{W} \subset V$. Since V is an arbitrary open set containing p we have that $\operatorname{st}\nu\operatorname{st}^*(x) = p$. Hence $\operatorname{st}\mu(x) = \operatorname{st}\nu\operatorname{st}^*(x)$.

We now proceed to the converse. Uniqueness is clear, since it is well known that a set A is closed iff $\operatorname{st}(A^*) = A$. Thus the topology is uniquely determined by the mapping. The proof of existence is harder although we know what the topology must be (if there is such a topology). We define: A is closed iff $\xi(A^*) = A$ (or equivalently because of (2), $\xi(A^*) \subset A$). First, this is a topology on S. $\xi(\varphi^*) = \xi(\varphi) = \varphi$. Hence φ is closed. $\xi(X^*) \subset X$. Hence X is closed. Suppose A and B are closed, i.e., $\xi(A^*) \subset A$ and $\xi(B^*) \subset B$. Then

$$\xi[A \cup B]^* = \xi[A^* \cup B^*] = \xi(A^*) \cup \xi(B)^* \subset A \cup B.$$

Thus $A \cup B$ is closed. Now assume A_α is closed for all α, i.e., $\xi(A_\alpha^*) \subset A_\alpha$ for all α. Then $\xi[(\bigcap A_\alpha)^*] \subset \xi[\bigcap A_\alpha^*] \subset \bigcap \xi(A^*) \subset \bigcap A_\alpha$. Therefore $\bigcap A_\alpha$ is closed. Note that this argument fails for arbitrary unions (as it should!) since the first inclusion goes the wrong way. It is immediate that $\xi[\{p\}^*] = \xi\{p\} = \{p\}$. Thus we have a T_1 topology.

LEMMA 1. *$\xi(A^*)$ is closed.*

Note. This is the first time we need condition (3).

Proof. Let $x \in [\xi(A^*)]^*$. We must show that $\xi(x) \in \xi(A^*)$. Now ξ maps A^* onto $\xi(A^*)$ (by definition). Hence by transfer ξ^* maps A^{**} onto ${}^*[\xi(A^*)]$. Since $x \in [\xi(A^*)]^*$ there exists $r \in A^{**}$ such that $\xi^* r = x$. Then $\nu\xi^* r \sim x$. Hence by (1) $\xi\nu\xi^*(x) = \xi x$. Now by (3) $\xi x = \xi\nu\xi^*(r) = \xi\mu(r)$. $r \in A^{**}$ implies that $\mu(r) \in A^*$. Hence $\xi r \in \xi(A^*)$.

LEMMA 2. *$\varepsilon(A^*)$ is the closure of A.*

Note. Though this is not needed in the sequel. It is mentioned because it is immediate.

Proof. Suppose $A \subset B$ where B is closed. Then $\xi(A^*) \subset \xi(B^*) = B$. Hence $\xi(A^*)$ is the smallest closed set containing A.

We let $m(p)$ be the monad of p in the topology. We will show eventually that $m(p) = \xi^{-1}\{p\}$, i.e., ξ is the standard point map. We first show one direction (the easier one).

LEMMA 3. $\xi^{-1}\{P\} \subset m(P)$.

Proof. Let $p \in U$ where U is open. Then $\varepsilon(U'^*) \subset U'$ by the way the topology is defined. Since U'^* is the complement of U^* it follows that $\xi(x) \in U \to x \in U^*$. Hence $\xi^{-1}\{p\} \subset U^*$. Since U is arbitrary open containing p, $\xi^{-1}\{p\} \subset m(p)$.

LEMMA 4. *S is compact.*

Proof. Every point $x \in S^*$ is contained in $\xi^{-1}\xi(x)$. Hence x is nearstandard by Lemma 3, in fact, $x \in m\xi(x)$.

Remark. It is curious that compactness is much easier to prove than Hausdorffness in our situation. This is unusual in topology.

LEMMA 5. $m(P) \subset \xi^{-1}\{P\}$.

Note. It follows immediately from this lemma that monads are disjoint, hence the space is Hausdorff. Thus this lemma is enough to complete the proof of the theorem.

Remark. This lemma uses the full force of second enlargements with condition (3). It is important to note that the proof of this lemma which is the deepest part of the theorem is not a translation of a proof found in [5] but a proof discovered directly by nonstandard methods which takes advantage of the convenient nonstandard characterizations in Theorems 2 and 5.

Proof. Suppose $q \in m(p)$. Then for every open set U such that $p \in U$ it follows that $q \in U^*$. Hence $q \notin U^* \to p \notin U$, i.e., $q \in U'^* \to p \in U'$. This may be stated as follows: for any closed set C, $q \in C^* \to p \in C$. Now let $q \in A^*$ where A is standard (not assumed to be closed). Then $\xi(A^*)$ is closed by Lemma 1. Also $A \subset \xi(A^*)$. Hence $q \in A^* \subset [\xi(A^*)]^*$. Therefore $p \in \xi(A^*)$. Let $A_1, A_2, \ldots, A_n$ be a finite collection of sets such that $A_1^*, A_2^*, \ldots, A_n^*$ contain q. Then $q \in A_1^* \cap A_2^* \cap \ldots \cap A_n^* = (A_1 \cap A_2 \cap \ldots \cap A_n)^*$. Hence $p \in \xi(A_1 \cap A_2 \cap \ldots \cap A_n)^* = \xi(A_1^* \cap A_2^* \cap \ldots \cap A_n^*)$. Therefore the following relation in X^* is concurrent: $R[A, s]$ where A is a subset of S, $q \in A^*$, $s \in A^*$ and $\varepsilon(s) = p$. Since S^{**} is an enlargement of S^* there is an element t such that, for all $A \subset S$ $q \in A^* \to t \in A^{**}$ and $\xi^*(t) = p$. Then $q \sim \mu(t)$. Hence $\xi q = \xi\mu(t) = \xi\nu\xi^*(t)$ by (3). Further this equals $\xi\nu(P)$ by the above which is ξP since ν is, of course, the identity on S. Finally $\xi P = P$. Hence $q \in \xi^{-1}\{P\}$. This completes the proof of the lemma and the theorem.

6. Further remarks. The main theorem shows that compact Hausdorff spaces may be regarded as maps $S^* \to S$ satisfying certain axioms. This does not quite give the main result mentioned in section 4 since the theorem refers to objects only. However; that is the hardest part. By the way the category D was defined, the morphisms correspond to maps which commute with the standard point map. It is well-known that these are exactly the continuous maps. Furthermore it is easy to see that the pair of adjoint functors referred to in the beginning of Section 4 (not defined there but found in [1] or [2]) are essentially the same as β and the forgetful. This is left to the interested reader.

These results can be extended to the category of compact Hausdorff algebras of a given type. For example, instead of the Stone-Čech compactification of a set we have the Bohr compactification of a discrete group. Indescirnability takes on a more complicated form in that case. Essentially, it is necessary to extend $\sim$ to a congruence relation. The analogue of the main theorem works because the requirement that the standard point mapping be a homomorphism is equivalent to the requirement that an algebra be a topological algebra.

In conclusion it appears that the blend of category theory and nonstandard analysis leads to interesting possibilities and is worth further exploration.

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