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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

The open-cover topology on function spaces

by

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Abstract. A study is made of the convergence of sequences in the open-cover topology on function spaces. Necessary and sufficient conditions are given for a subset of such a function space to be sequentially compact.

1. Introduction. Bessaga and Pełczyński use in [2, p. 121] (see also [4]) a certain natural topology on the space of homeomorphisms from a metric space onto itself. They credit the idea for this topology to a paper by Anderson and Bing [1] in which several conditions are established insuring that a sequence of homeomorphisms of a space converge to a homeomorphism. Open covers are used here to provide a measure of how close a homeomorphism is to the identity. We call this topology the open-cover topology. One advantage that this topology has is that it allows control of the functions throughout the entire domain rather than just a compact set, and does this without the range needing some special structure such as a metric or a uniformity. Our primary concern in this paper will be the investigation of the convergence of sequences in this open-cover topology, and also the characterization of sequentially compact subsets.

If X and Y are topological spaces, the notation $C(X, Y)$ will be used to denote the set of all continuous functions from X into Y. We define the open-cover topology on $C(X, Y)$ as follows. Let $\Gamma(Y)$ denote the set of all open covers of Y. For each $\mathscr{V} \in \Gamma(Y)$ and $f \in C(X, Y)$, let $\mathscr{V}(f) = \{g \in C(X, Y) |$ for every $x \in X$, there exists a $V \in \mathscr{V}$ such that $(f(x), g(x)) \in V \times V\}$. The open-cover topology on $C(X, Y)$ is the topology generated by the subbase

$$\{\mathscr{V}(f) |\ \mathscr{V} \in \Gamma(Y) \text{ and } f \in C(X, Y)\}.$$

This topological space will be denoted by $C_\gamma(X, Y)$. We shall be comparing this topology with two other function space topologies: the compact-open topology, and the topology generated by the supremum metric for some bounded metric ϱ on the range. These topological spaces will be denoted by $C_\varkappa(X, Y)$ and $C_\varrho(X, Y)$, respectively.

We now give a short discussion of several properties enjoyed by the open-cover topology which are either known (see for example [4]) or not too difficult to prove.

For notational convenience, the notation $X \leqslant Y$, for topological spaces X and Y, will mean that X and Y have the same underlying set and that the topology of X is contained in the topology of Y. Also for convenience, all spaces will be assumed to be Hausdorff spaces.

First, for every two topological spaces X and Y, $C_\kappa(X, Y) \leqslant C_\gamma(X, Y)$ and $C_\varrho(X, Y) \leqslant C_\gamma(X, Y)$, when ϱ is any bounded metric for Y. The former inequality is an equality if X is compact, and the latter inequality is an equality if X is pseudocompact. If, in addition, X is completely regular and Y contains a nontrivial path, then compactness (pseudocompactness, respectively) is not only a sufficient condition but a necessary condition for $C_\kappa(X, Y) = C_\gamma(X, Y)$ ($C_\varrho(X, Y) = C_\gamma(X, Y)$, respectively). Finally, if X is normal and Y is the space of real numbers $\boldsymbol{R}$, then $C_\gamma(X, Y)$ is metrizable (also first countable) if and only if X is pseudocompact.

2. Convergence of sequences. In this section we investigate conditions under which a sequence in $C(X, Y)$ converges in the open-cover topology. Such a set of conditions must be stronger than that giving convergence in the compact-open topology. The following results indicate how much stronger it must be. We say that a sequence $\{f_n\}$ in $C(X, Y)$ is *eventually supported on a compact set* if there exists a compact subset K of X and an $m \in N$ (N is the set of natural numbers) such that if $n \in N$ with $n \geqslant m$, then $f_n|_{X \setminus K} = f_m|_{X \setminus K}$.

2.1. Lemma. *If $\{f_n\}$ is a sequence in $C(X, Y)$ which converges to f in $C_\kappa(X, Y)$ and is eventually supported on a compact set, then $\{f_n\}$ converges to f in $C_\gamma(X, Y)$.*

Proof. Let $\mathscr{V}(g)$ be a subbasic open subset of $C_\gamma(X, Y)$ containing f. There exists a compact subset K of X and an $m_1 \in N$ such that if $n \geqslant m_1$, then $f_n|_{X\setminus K} = f_{m_1}|_{X \setminus K}$. Since $C_\gamma(K, Y) = C_\kappa(K, Y)$, then $\mathscr{V}(g|_K)$ is an open neighborhood of $f|_K$ in $C_\kappa(K, Y)$, so that there exists an $m_2 \in N$ with $m_2 \geqslant m_1$ such that if $n \geqslant m_2$, then $f_n|_K \in \mathscr{V}(g|_K)$. Then if $n \geqslant m_2$, $f_n \in \mathscr{V}(g)$. ∎

Perhaps a more useful thing to know is the extent to which the converse of Lemma 2.1 is true. First, it is easy to see that it is not true in general; for example, take the domain to be $\boldsymbol{R}$ and the range to be the rational numbers. However, if we require that the range contain a nontrivial path, then we get a partial converse of Lemma 2.1. This is given by Theorem 2.3 which follows the next lemma.

2.2. Lemma. *Let X be a completely regular space, and let Y be a regular space containing a nontrivial path. Let $\{f_n\}$ be a sequence in $C(X, Y)$, and let $f \in C(X, Y)$. If there exists a sequence $\{x_n\}$ in X having no cluster point in X such that $f_n(x_n) \neq f(x_n)$ for every $n \in N$, then no subsequence of $\{f_n\}$ converges to f in $C_\gamma(X, Y)$.*

Proof. Let N_1 be any cofinal subset of N, and let φ be a homeomorphism from the closed unit interval, $[0, 1]$, into Y. Choose disjoint open subsets W_0 and W_1 of Y so that $\varphi(0) \in W_0$ and $\varphi(1) \in W_1$, and so that there is a cofinal subset N_2 of N_1 with either $\{f_n(x_n)|\ n \in N_2\} \subseteq W_0$ or $\{f_n(x_n)|\ n \in N_2\} \subseteq W_1$ — say the latter. Now there is a $t \in (0, 1)$ such that $\varphi([0, t)) \subseteq W_0$. Let $\{t_n\}$ be a strictly decreasing sequence in $(0, t)$ converging to 0. Let $y_0 = \varphi(t)$, and for each $n \in N$, let $y_n = \varphi(t_n)$.

There exists a cofinal subset N_3 of N_2 such that for each $n \in N_3$,

$$f(x_n) \notin \{f_m(x_m)|\ m \in N_3 \text{ and } m \leqslant n\}.$$

Since f is continuous and since $\{x_n\}$ has no cluster point in X, there exists a discrete collection $\{U_n|\ n \in N_3\}$ of open subsets of X such that for each $n \in N_3$, $x_n \in U_n$ and

$$f(U_n) \subseteq Y \setminus \{f_m(x_m)|\ m \in N_3 \text{ and } m \leqslant n\}.$$

Then for each $n \in N_3$, define $g_n\colon \{x_n\} \cup (\overline{U}_n \setminus U_n) \to Y$ by $g_n(x_n) = y_n$ and $g_n(x) = y_0$ if $x \in \overline{U}_n \setminus U_n$. Since X is completely regular, each g_n has a continuous extension $\bar{g}_n\colon \overline{U}_n \to Y$ such that $\bar{g}_n(\overline{U}_n) \subseteq \varphi([t_n, t])$. Finally define $g \in C(X, Y)$ by $g(x) = \bar{g}_n(x)$ if $x \in U_n$ and $g(x) = y_0$ if $x \in X \setminus \bigcup\{U_n|\ n \in N_3\}$. Let $Y_0 = \{\varphi(0)\} \cup \{y_n|\ n \in N_3\}$, and for each $n \in N_3$, let $Y_n = Y_0 \setminus \{y_n\}$. Also for each $n \in N_3$, define

$$V_n = Y \setminus (\{f_n(x_n)\} \cup Y_n),$$

and define

$$\mathscr{V} = \{V_n|\ n \in N_3\} \cup \{Y \setminus \overline{W}_1,\ Y \setminus Y_0\}.$$

To see that $f_n \notin \mathscr{V}(g)$ for each $n \in N_3$, note that $g(x) = y_n$. The only members of $\mathscr{V}$ containing y_n are V_n and $Y \setminus \overline{W}_1$; but $f_n(x_n) \notin V_n \cup (Y \setminus \overline{W}_1)$. Finally, to see that $f \in \mathscr{V}(g)$, let $x \in X$. First note that $g(X) \subseteq \varphi((0, t]) \subseteq Y \setminus \overline{V}_1$. If

$$x \in X \setminus \bigcup \{U_n|\ n \in N_3\},$$

then $g(x) = y_0$, so that $g(x) \in (Y \setminus \overline{W}_1) \cap (Y \setminus Y_0)$. Since $(Y \setminus \overline{W}_1) \cup (Y \setminus Y_0) = Y$, then $g(x)$ and $f(x)$ are both contained in the same member of $\mathscr{V}$. On the other hand, if $x \in U_n$ for some $n \in N_3$, then $f(x) \notin \{f_m(x_m)|\ m \in N_3 \text{ and } m \leqslant n\}$. If there is an $m \in N_3$ such that $g(x) = y_m$, then $g(x) = y_m$, then $g(x) \in V_m$; also since $g(x) \in \varphi([t_n, t])$, then $m \leqslant n$. But either $f(x) \in Y \setminus \overline{W}_1$, or, since $f(x) \neq f_m(x_m)$, $f(x) \in V_m$. If there is no $m \in N_3$ with $g(x) = y_m$, then $g(x) \in (Y \setminus \overline{W}_1) \cap (Y \setminus Y_0)$. Thus in either case, $f(x)$ and $g(x)$ are both contained in the same member of $\mathscr{V}$. Therefore $f \in \mathscr{V}(g)$, so that $\{f_n|\ n \in N_1\}$ does not converge to f in $C_\gamma(X, Y)$. ∎

2.3. Theorem. *Let X be a paracompact locally compact space, and let Y be a regular space containing a nontrivial path. Then the sequence $\{f_n\}$ in $C(X, Y)$ converges to f in $C_\gamma(X, Y)$ if and only if $\{f_n\}$ converges to f in $C_\kappa(X, Y)$ and is eventually supported on a compact set.*

Proof. To prove the necessity, recall that since X is a paracompact locally compact space, it is the free union of σ-compact spaces $\{X_\alpha|\ \alpha \in A\}$. Then for each $\alpha \in A$, $X_\alpha = \bigcup_{n=1}^{\infty} K_\alpha^n$, where each K_α^n is compact and contained in the interior of K_α^{n+1}. Suppose that $\{f_n\}$ is not eventually supported on a compact set. Then by induction, sequences $\{n_i\}$, $\{\alpha_i\}$, and $\{x_i\}$ can be constructed so that for each $i \in N$, $x_i \in X_{\alpha_i} \setminus \bigcup_{j=0}^{i-1} K_{\alpha_j}^i$ (take $K_{\alpha_0}^i = \emptyset$) and $f_{n_i}(x_i) \neq f(x_i)$. By construction, $\{x_i\}$ has no cluster point in X. Therefore by Lemma 2.2, $\{f_{n_i}\}$ does not converge to f in $C_\gamma(X, Y)$, and hence $\{f_n\}$ does not converge to f in $C_\gamma(X, Y)$. ∎

For an example, let Ω denote the ordinal numbers less than the first uncountable ordinal with the order topology. Since Ω is pseudocompact, $C_\gamma(\Omega, \mathbf{R}) = C_\varrho(\Omega, \mathbf{R})$, where ϱ is a bounded metric on $\mathbf{R}$. Thus a sequence of distinct constant functions can be found in $C_\gamma(\Omega, \mathbf{R})$ converging to some constant function. Such a sequence is not eventually supported on a compact set. This shows that the paracompactness of X cannot be omitted from the hypotheses of Theorem 2.3.

3. Sequentially compact subsets. In this final section we establish a version of Ascoli's Theorem which characterizes the sequentially compact subsets of $C_\gamma(X, Y)$.

Recall that a subset $F \subseteq C(X, Y)$ is *evenly continuous* if for every $x \in X$, $y \in Y$, and neighborhood V of y in Y, there exist neighborhoods U of x and W of Y such that for every $f \in F$ with $f(x) \in W$, $f(U) \subseteq V$. Ascoli's Theorem says that for a locally compact space X, closed subset F is compact in $C_\varkappa(X, Y)$ if and only if F is evenly continuous and the closed orbit $\overline{F[x]}$ is compact for every $x \in X$ (see [3]).

We need to introduce an additional property in order to deal with subsets of $C_\gamma(X, Y)$. If $F \subseteq C(X, Y)$ and $S \subseteq X$, we say that S is a *supporting set* of F if there exists a finite subset F_0 of F such that for every $f \in F$, there exists an $f_0 \in F_0$ with $f|_{X \setminus S} = f_0|_{X \setminus S}$.

3.1. Lemma. *Let X be a paracompact locally compact space, and let Y be a regular space containing a nontrivial path. If F is a sequentially compact subset of $C_\gamma(X, Y)$, then F has a compact supporting set.*

Proof. Suppose F does not have a compact supporting set. Using the same technique as that in the proof of Theorem 2.3, it is possible to find sequences $\{x_n\}$ in X and $\{f_n\}$ in F such that $\{x_n\}$ has no cluster point in X and, for each $n \in N$, $f_{n+1}(x_n) \neq f_i(x_n)$ for $1 \leqslant i \leqslant n$. But then for every $f \in C(X, Y)$, there exists a $k \in N$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that for every $i \geqslant k$, $f_i(x_{n_i}) \neq f(x_{n_i})$. Therefore by Lemma 2.2, no subsequence of $\{f_n\}$ can converge in $C_\gamma(X, Y)$, so that F is not sequentially compact. ∎

In order to extend Lemma 3.1 to a characterization of sequential compactness, we define one additional concept. If $F \subseteq C(X, Y)$, we define $\mathscr{K}(F)$ to be the set of equivalence classes of the equivalence relation on F defined by: f is equivalent to g if there exists a compact subset K of X such that $f|_{X \setminus K} = g|_{X \setminus K}$.

3.2. Theorem. *Let X be a paracompact locally compact space, let Y be a metric space containing a nontrivial path, and let F be a subset of $C(X, Y)$. Then the following are equivalent.*

(i) *F is sequentially compact in $C_\gamma(X, Y)$.*

(ii) *F is closed in $C_\varkappa(X, Y)$, F is evenly continuous, $\overline{F[x]}$ is compact for every $x \in X$, and F has a compact supporting set.*

(iii) *F is compact in $C_\varkappa(X, Y)$, and F has a compact supporting set.*

(iv) *$\mathscr{K}(F)$ is finite, and each member of $\mathscr{K}(F)$ is sequentially compact as a subset of $C_\gamma(X, Y)$.*

Proof. Clearly (iv) implies (i). To see that (i) implies (ii), note that if F is sequentially compact in $C_\gamma(X, Y)$, then by Lemma 3.1, F has a compact supporting set. Also F will be sequentially compact in $C_\varrho(X, Y)$, where ϱ is a metric on Y. Therefore F will be compact in $C_\varrho(X, Y)$, and hence F is compact in $C_\varkappa(X, Y)$. Then (ii) follows from Ascoli's Theorem. Also the fact that (ii) implies (iii) follows from Ascoli's Theorem.

It remains then to establish that (iii) implies (iv). Let F be a compact subset of $C_\varkappa(X, Y)$ having a compact supporting set K. Then there exists a finite subset F_0 of F such that for every $f \in F$, there exists an $f_0 \in F_0$ with $f|_{X \setminus K} = f_0|_{X \setminus K}$. Then certainly $\mathscr{K}(F)$ has no more elements than F_0, and is hence finite. Also if $F_1 \in \mathscr{K}(F)$, then the open-cover topology on F_1 is equal to the compact-open topology on F_1 since all functions in F_1 agree outside of the compact set K. But since the topology generated by the supremum metric is sandwiched between the compact-open topology, and the open-cover topology, then F_1 is metrizable as a subspace of $C_\varkappa(X, Y)$. It is not difficult to see that F_1 is closed in F relative to the compact-open topology, so that F_1 is compact as a subset of $C_\varkappa(X, Y)$. Then since F_1 is metrizable, it is sequentially compact as a subset of $C_\varkappa(X, Y)$, and hence sequentially compact as a subset of $C_\gamma(X, Y)$. ∎

3.3. Corollary. *Let X be a paracompact locally compact space, and let Y be a metric space containing a nontrivial path. If F is a sequentially compact subset of $C_\gamma(X, Y)$, then F is compact in $C_\gamma(X, Y)$.*

Proof. By the proof of Theorem 3.2, if F is sequentially compact in $C_\gamma(X, Y)$, then $\mathscr{K}(F)$ is finite and each element of $\mathscr{K}(F)$ is compact in $C_\gamma(X, Y)$. Therefore F is compact in $C_\gamma(X, Y)$. ∎

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