By the proposition there is a point-finite, open cover $\mathscr{V}$ of $X\setminus\{a\}$ consisting of compact sets.

By the inductive assumption, every $V\in\mathscr{V}$ has a point-finite family $\mathscr{U}(V)$ of clopen sets, separating points in V.

Put $\mathscr{Z}=\mathscr{V}\cup\bigcup\{\mathscr{U}(V)\colon V\in\mathscr{V}\}$. It is easy to see that $\mathscr{Z}$ is a point-finite family of clopen sets separating points, so the proof of the theorem is finished.

Remark 1. In [2], Example 5–1, it is shown that under some set theoretical assumption, there is a compact space, which is not an E-C, but has a point-countable separating family of open F_σ-sets.

Remark 2. In [2], the proof of Theorem 4.3 and in [6], Proposition 5, was observed, that every strong E-C is scattered, so we obtain the following corollaries:

COROLLARY 1. *A compact subspace of a Σ-product of intervals is a strong* E-C *if and only if it is scattered.*

COROLLARY 2. *An* E-C *is a strong* E-C *if and only if it is scattered.*

COROLLARY 3. *If X is a compact space admiting a point-countable, separating family of open F_σ-sets, then X is a strong* E-C *if and only if it is scattered.*

Remark 3. Let us notice that we can formulate the proposition in the following way:

If X is a scattered compact space and $\mathscr{F}$ is a family of closed subsets of X such that every point belongs to less than $\aleph_\alpha$-many sets in $\mathscr{F}$, where $\aleph_\alpha$ is a regular number, then there are $\{\mathscr{F}_s\colon s\in S\}$ such that $\bigcup\{\mathscr{F}_s\colon s\in S\}=\mathscr{F}$, $|\mathscr{F}_s|<\aleph_\alpha$ for every $s\in S$ and the family $\{\bigcup\mathscr{F}_s\colon s\in S\}$ is point finite.

The author is very grateful to R. Pol for some valuable remarks.

Added in proof. Recently M. Talagrand has shown, without additional set theoretical assumptions, that there is a compact subspace of a Σ-product of the real line which is not Eberlein compact.

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Accepté par la Rédaction le 7. 2. 1977

Finite points of filters in infinite dimensional vector spaces

by

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Abstract. Let E be an infinite dimensional vector space over K, the scalar field. For any subset A of ${}^*E$, the nonstandard extension of E, define $\mathrm{Fin}(A)$ as follows: $z\in\mathrm{Fin}(A)$ if and only if $\lambda z\in A$ for each infinitesimal λ of ${}^*K$, the nonstandard extension of K. A subset A of ${}^*E$ is called Fin-invariant if $\mathrm{Fin}(A)=A$. If A is the monad of a filter $\mathfrak{F}$ in E then $\mathrm{Fin}(A)$ is called the set of finite points of $\mathfrak{F}$.

In this paper, we establish the existence of nontrivial, Fin-invariant sets. Next, sufficient conditions are given, in terms of finite points, for a filter to induce a linear topology on some nontrivial vector subspace of E. Finally, we use Fin-invariant sets of infinitesimals and finite points of filters to produce a class of locally convex, topological vector spaces that are not Schwartz spaces, not metrizable and have invariant nonstandard hulls. In particular, it is shown that the topological vector space induced by the box topology (J. L. Kelley, *General Topology*, p. 107) has invariant nonstandard hulls.

Introduction. In this paper, we study a class of sets first defined in [1], i.e., the set $\mathrm{Fin}(A)$ for any subset A of ${}^*E$, the nonstandard extension of a vector space E (Definition 1.1). The two predominant themes presented here are: when is a subset A of ${}^*E$ Fin-invariant (i.e., $\mathrm{Fin}(A)=A$) and when is A not Fin-invariant. The latter is shown to be of importance in generalizing the concept of finite points in the nonstandard theory of topological vector spaces. Indeed, in Sections 2 and 3, it is shown that the properties of a filter $\mathfrak{F}$ in a vector space E is determined by its finite points (elements of $\mathrm{Fin}(\mu(\mathfrak{F}))$) as well as its monad $\mu(\mathfrak{F})$ whenever $\mathrm{Fin}(\mu(\mathfrak{F}))\neq\emptyset$. In particular, sufficient conditions are given, in terms of finite points, for a filter $\mathfrak{F}$ to induce a linear topology on some non-trivial vector subspace of E (Proposition 3.7).

In Section 1, we consider the basic properties of $\mathrm{Fin}(A)$ for an arbitrary subset A of ${}^*E$. Also the first example of a non trivial, Fin-invariant set is given (Propositions 1.6 and 1.7). It appears that non trivial, Fin-invariant sets are quite abundant; in fact, Fin-invariant subsets of infinitesimals can be generated easily from entities in the "standard" world (Proposition 4.9).

The main results of this paper are found in Sections 4, 5 and 6. In Section 4, we apply the concepts of Fin-invariant sets and finite points of filters to produce a class of function spaces that have, as locally convex spaces, invariant nonstandard hulls (Theorem 4.17). In Section 5, it is shown that these spaces are not metrizable

(Proposition 5.2) and are not Schwartz spaces (Proposition 5.4); thus, the sufficient conditions of [2] are not applicable. Finally, in Section 6, we provide a general description of this class of spaces (Theorem 6.6) and we give a variety of examples to indicate that these spaces are indeed plentiful.

Preliminaries. Throughout this paper, $\boldsymbol{K}$ will denote either the real or the complex numbers and E will symbolize an infinite dimensional vector space over $\boldsymbol{K}$. It is assumed that E and $\boldsymbol{K}$ sit in a full set-theoretical structure

$$B_\Gamma = \{B_\sigma |\ \sigma \in \Gamma\},$$

where Γ is the set of types. The basic framework of nonstandard analysis used here can be found in [6] and [7]; however, we shall only consider a nonstandard structure $^*B_\Gamma$ that is a higher order ultrapower enlargement of B_Γ.

The mapping $x \to {}^*x$ is assumed to be the identity on $\boldsymbol{N}$, the non-negative integers, $\boldsymbol{N}_+$, the positive integers, $\boldsymbol{R}$, the reals and $\boldsymbol{C}$, the complex numbers. Also the extension to $^*\boldsymbol{C}$ of the usual algebraic operations are denoted by the same familiar symbols. The same is true of the extension to $^*\boldsymbol{R}$ of the ordering on $\boldsymbol{R}$.

An element $\lambda \in {}^*\boldsymbol{K}$ is called an *infinitesimal* if and only if $|\lambda| \leqslant \delta$ for each positive δ in $\boldsymbol{R}$ and λ is called *finite* if and only if $|\lambda| \leqslant \delta$ for some positive $\delta \in \boldsymbol{R}$. If $\lambda \in {}^*\boldsymbol{K}$ is not finite then λ is called *infinite*. An element $\lambda \in {}^*\boldsymbol{K}$ is called *near-standard* if there is an element in $\boldsymbol{K}$, denoted by $^0\lambda$, such that $\lambda - {}^0\lambda$ is an infinitesimal. It can be shown that $^0\lambda$ exists and is unique if and only if $\lambda \in {}^*\boldsymbol{K}$ is finite. The set of all infinitesimals in $^*\boldsymbol{K}$ is denoted by $\mu(0)$.

Let X be any entity of B_Γ. We define $^*[X]$ to be the set of standard elements of $^*X$; i.e.,

$$^*[X] = \{^*x |\ x \in X\}.$$

If $\mathfrak{F}$ is a filter on X, then the filter monad of $\mathfrak{F}$, $\mu(\mathfrak{F})$, is defined by

$$\mu(\mathfrak{F}) = \bigcap \{^*F |\ F \in \mathfrak{F}\} = \bigcap {}^*[\mathfrak{F}].$$

Recall that if $^*B_\Gamma$ is an enlargement of B_Γ then for each entity X of B_Γ and each filter $\mathfrak{F}$ on X we have $\mu(\mathfrak{F}) \neq \emptyset$. In fact, there is an element F of $^*\mathfrak{F}$ for which $F \subset \mu(\mathfrak{F})$ ([6], Theorem 2.1.5(a)).

In most cases, E is assumed to be a subset of the base set B_0 of B_Γ. In this instance, E can be regarded as an external subset of $^*E$, i.e., the map $x \to {}^*x$ is the identity on E. Always, however, the extension of vector addition to $^*E$ is denoted by $+$ (as it is in E) and the scalar multiplication operation on $^*\boldsymbol{K} \times {}^*E$ takes (λ, z) to λz.

1. Fin-invariant sets. We begin with the following definition due to Henson and Moore [1].

DEFINITION 1.1. Let E be a vector space over $\boldsymbol{K}$ and let $A \subset {}^*E$. The set $\mathrm{Fin}(A) \subset {}^*E$ is defined by

$$\mathrm{Fin}(A) = \{z |\ \lambda z \in A \text{ for each } \lambda \in \mu(0)\}.$$

Note that $\mathrm{Fin}(A) \neq \emptyset$ if and only if $0 \in A$. Also, it is easily shown that

$$\mathrm{Fin}(A) = \bigcap \{\lambda A |\ \lambda \in {}^*\boldsymbol{K} \text{ is infinite}\} \tag{1.2}$$

whenever $0 \in A$.

PROPOSITION 1.3. *If $A \subset {}^*E$, where E is a vector space over $\boldsymbol{K}$, then $A \subset \mathrm{Fin}(A)$ if and only if $\lambda A \subset A$ for each $\lambda \in \mu(0)$.*

Proof. If $\lambda A \subset A$ for each $\lambda \in \mu(0)$ then $\{0\} = 0A \subset A$ which implies $\mathrm{Fin}(A) \neq \emptyset$. If $a \in A$ then $\lambda a \in \lambda A \subset A$ for $\lambda \in \mu(0)$, i.e., $a \in \mathrm{Fin}(A)$. Conversely, if $A \subset \mathrm{Fin}(A)$ then $\lambda a \in A$ for $a \in A$ and $\lambda \in \mu(0)$, i.e., $\lambda A \subset A$ for each $\lambda \in \mu(0)$. ■

In view of the above proposition, we make the following definition.

DEFINITION 1.4. Let E be a vector space over $\boldsymbol{K}$. A set $A \subset {}^*E$ is *μ-saturated* if and only if $\lambda A \subset A$ for each $\lambda \in \mu(0)$.

Observe that $\mathrm{Fin}(\mu(0)) \subset {}^*\boldsymbol{K}$ is the set of all finite scalars. Consequently $\mathrm{Fin}(\mu(0))$ is a ring with an identity under addition and multiplication in $^*\boldsymbol{K}$ and $\mu(0)$, the set of all infinitesimals, is a maximal ideal in $\mathrm{Fin}(\mu(0))$. This idea can be generalized to vector spaces as follows.

PROPOSITION 1.5. *Let E be a vector space over $\boldsymbol{K}$. If $A \subset {}^*E$ such that $0 \in A$ and $A + A = A$ then $\mathrm{Fin}(A)$ is a module over $\mathrm{Fin}(\mu(0))$.*

Proof. Consider $x, y \in \mathrm{Fin}(A)$. If $\beta \in \mathrm{Fin}(\mu(0))$ then $\lambda(\beta x) = (\lambda\beta)x \in A$ for each $\lambda \in \mu(0)$ since $\lambda \in \mu(0)$ implies $\lambda\beta \in \mu(0)$. Hence $\beta x \in \mathrm{Fin}(A)$. Also $\lambda(x+y) = \lambda x + \lambda y \in A + A = A$ for each $\lambda \in \mu(0)$ which implies $x + y \in \mathrm{Fin}(A)$. ■

It is natural to ask when does $\mathrm{Fin}(A) = A$ for $A \subset {}^*E$. If $A \subset {}^*E$ is a vector space over $^*\boldsymbol{K}$ then $\mathrm{Fin}(A) = A$, a fact easily derived from scalar multiplication over $^*\boldsymbol{K}$. Surprisingly, Definition 1.1 itself generates sets with the above property.

PROPOSITION 1.6. *Let E be a complex vector space. If $A \subset {}^*E$ such that $0 \in A$ then $\mathrm{Fin}(\mathrm{Fin}(A)) = \mathrm{Fin}(A)$.*

Proof. Let $z \in \mathrm{Fin}(A)$ and let $\beta \in \mu(0)$. If $\lambda \in \mu(0)$ then $\lambda(\beta z) = (\lambda\beta)z \in A$ since $\lambda\beta \in \mu(0)$. Hence $\beta z \in \mathrm{Fin}(A)$; therefore, $\beta\,\mathrm{Fin}(A) \subset \mathrm{Fin}(A)$ for $\beta \in \mu(0)$. Consequently, $\mathrm{Fin}(A) \subset \mathrm{Fin}(\mathrm{Fin}(A))$ by Proposition 1.3.

Let $z \in \mathrm{Fin}(\mathrm{Fin}(A))$. If $z \notin \mathrm{Fin}(A)$, then there exists $\lambda_0 \in \mu(0)$ such that $\lambda_0 z \notin A$ which implies $z \neq 0$ and $\lambda_0 \neq 0$. Also $\lambda_0 \in \mu(0)$ implies $\sqrt{\lambda_0} \in \mu(0)$. Consequently, $z \in \mathrm{Fin}(\mathrm{Fin}(A))$ implies $\sqrt{\lambda_0}\, z \in \mathrm{Fin}(A)$ which implies $\sqrt{\lambda_0}(\sqrt{\lambda_0}\, z) \in A$ contradicting $\sqrt{\lambda_0}(\sqrt{\lambda_0}\, z) = (\sqrt{\lambda_0}\sqrt{\lambda_0})z = \lambda_0 z \notin A$. Thus $z \in \mathrm{Fin}(A)$. Therefore $\mathrm{Fin}(\mathrm{Fin}(A)) = \mathrm{Fin}(A)$. ■

In the case of real scalars, we must compensate for the lack of square roots of negative numbers. This deficiency imposes a symmetric condition on A, i.e., if we let $A = -A$ then $\lambda z \in A$ for some $\lambda \in {}^*\boldsymbol{R}$ implies $(-\lambda)z \in A$; therefore, it can be assumed that $0 \leqslant \lambda$.

PROPOSITION 1.7. *Let E be a real vector space. If $A \subset {}^*E$ such that $0 \in A$ and $A = -A$ then $\mathrm{Fin}(\mathrm{Fin}(A)) = \mathrm{Fin}(A)$.*

The proof of the above proposition, being analogous to that of Proposition 1.6, is omitted.

DEFINITION 1.9. Let E be a vector space over K. A set $A \subset {}^*E$ is said to be Fin-*invariant* if and only if $\mathrm{Fin}(A) = A$.

Note that Fin-invariant sets are μ-saturated. We shall see, in Section 4, that Fin-invariant sets play a fundamental role in generating topological vector spaces that have invariant nonstandard hulls.

We close this section by making the following observations. If F is a vector subspace of a vector space E over K, then ${}^*F$ is Fin-invariant. If A and B are subsets of ${}^*E$ such that $0 \in A$ and B is Fin-invariant, then

$$\mathrm{Fin}(B \cap A) = B \cap \mathrm{Fin}(A)\,. \tag{1.10}$$

2. Finite points of filters. Now the other side of the question will be considered, i.e., when is $A \subset {}^*E$ not Fin-invariant for some vector space E. In searching for examples, one set seems to clamor for attention, namely, $\mu(0)$, the set of infinitesimals. Clearly, $\mathrm{Fin}(\mu(0)) \neq \mu(0)$ and the fact that $\mu(0)$ is a filter monad appears to imply that any filter monad is not Fin-invariant. The following propositions give some comfort to this belief.

PROPOSITION 2.1. *Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E such that $0 \in \mu(\mathfrak{F})$. If $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ then for $F \in \mathfrak{F}$ there exists $n \in N$ such that $z \in {}^*(nF)$.*

Proof. Let $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ and let $F \in \mathfrak{F}$. Define $A(F) = \{n \in {}^*N \mid z \in n{}^*F\}$. Thus $A(F)$ is an internal subset of ${}^*N$. Also $n \in {}^*N - N$ implies $n^{-1}z \in \mu(\mathfrak{F}) \subset {}^*F$ which implies $z \in n{}^*F$; therefore, ${}^*N - N \subset A(F)$. Now, $A(F)$ being internal implies $A(F)$ has a least element $n_0 \notin {}^*N - N$. Indeed, if $n_0 \in {}^*N - N$ then $n_0 - 1 \in {}^*N - N \subset A(F)$ and $n_0 - 1 < n_0$ contradicting the fact that $n_0 \in A(F)$ is the least element of $A(F)$. Therefore $n_0 \in N$ and $z \in {}^*(n_0 F)$. ■

Recall that a subset B of a vector space E is balanced if and only if $\lambda B \subset B$ for $|\lambda| \leqslant 1$ (see [3], Chapter 2, Section 3, Definition 3).

PROPOSITION 2.2. *Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E that has a filter basis $\mathscr{E} \subset \mathfrak{F}$ of balanced sets. If $z \in {}^*E$ such that $F \in \mathfrak{F}$ implies $z \in {}^*(nF)$ for some $n \in N$ then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$.*

Proof. Since $F \in \mathscr{E}$ is balanced we have $0 \in \mu(\mathfrak{F})$; hence, $\mathrm{Fin}(\mu(\mathfrak{F})) \neq \emptyset$. Let $z \in {}^*E$ such that $F \in \mathfrak{F}$ implies $z \in {}^*(nF)$ for some $n \in N$. If it can be shown that $\lambda \in \mu(0)$ implies $\lambda z \in {}^*F$ for each $F \in \mathscr{E}$ then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$. Indeed, $\lambda z \in {}^*F$ for each $F \in \mathscr{E}$ implies $\lambda z \in \bigcap \{{}^*F \mid F \in \mathscr{E}\} = \mu(\mathfrak{F})$; therefore, $\lambda \in \mu(0)$ implies $\lambda z \in {}^*F$ for each $F \in \mathscr{E}$ which implies $\lambda z \in \mu(\mathfrak{F})$, i.e., $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$.

So let $\lambda \in \mu(0)$ and let $F \in \mathscr{E}$. First, $\lambda = 0$ implies $0z = 0 \in {}^*F$ since F is balanced. Assume $\lambda \neq 0$. There exists $n \in N$ such that $z \in {}^*(nF) = {}^*n{}^*F$. Thus F being balanced implies that the following statement is true for n and F:

"if $x \in E$ such that $x \in nF$ then $x \in \beta F$ for $\beta \in K$ and $n \leqslant |\beta|$".

Hence, the following statement is true for ${}^*n$ and ${}^*F$:

"if $y \in {}^*E$ such that $y \in {}^*n{}^*F$ then $y \in \beta{}^*F$ for $\beta \in {}^*K$ and ${}^*n \leqslant |\beta|$".

Therefore $z \in {}^*n{}^*F$ implies $z \in \lambda^{-1}{}^*F$ since $0 \neq \lambda \in \mu(0)$ implies ${}^*n < |\lambda^{-1}|$. Hence $\lambda z \in {}^*F$. ■

COROLLARY 2.3. *Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E that has a filter basis $\mathscr{E} \subset \mathfrak{F}$ of balanced sets. If $z \in {}^*E$ then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ if and only if for each $F \in \mathfrak{F}$ there exists $n \in N$ such that $z \in {}^*(nF)$.*

Proof. Propositions 2.1 and 2.2. ■

Of course there is a filter in E that has a Fin-invariant monad, e.g., the filter of all subsets of E that contain 0. Hopefully, we will consider somewhat smaller filters. With that thought in mind, we make the following definitions.

DEFINITION 2.4. Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E.

(i) A point $z \in {}^*E$ is called $\mathfrak{F}$-*finite* if and only if $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$.

(ii) A point $x \in E$ is said to be $\mathfrak{F}$-*absorbed* if and only if for $F \in \mathfrak{F}$, there exists $n \in N$ (depending on F) such that $x \in nF$. The set of all $\mathfrak{F}$-absorbed points of E is denoted by $A(\mathfrak{F})$.

(iii) A subset $B \subset E$ is said to be $\mathfrak{F}$-*bounded* if and only if for $F \in \mathfrak{F}$, there exists $n \in N$ (depending on F) such that $B \subset \lambda F$ for $\lambda \in K$ and $n \leqslant |\lambda|$.

Note that $\mathfrak{F}$-bounded sets are subsets of $A(\mathfrak{F})$; however, $A(\mathfrak{F})$ is not necessarily $\mathfrak{F}$-bounded, e.g., consider the neighborhoods of zero in a topological vector space. In fact, $x \in E$ being $\mathfrak{F}$-absorbed does not necessarily imply $\{x\}$ is $\mathfrak{F}$-bounded. Also if $x \in E$, then ${}^*x$ is $\mathfrak{F}$-finite implies x is $\mathfrak{F}$-absorbed (Proposition 2.1) and if $\mathfrak{F}$ has a basis of balanced sets, then x is $\mathfrak{F}$-absorbed if and only if ${}^*x$ is $\mathfrak{F}$-finite (Corollary 2.3). As expected there is a relationship between $\mathfrak{F}$-finite points and $\mathfrak{F}$-bounded subsets of E. However, first we need to discuss $*$-balanced subsets of ${}^*E$ (hypercircled in [1]).

DEFINITION 2.5. Let E be a vector space over K and let B be a subset of ${}^*E$ (internal or external). The $*$-*balanced hull of* B is the set of all elements of ${}^*E$ of the form λz, where $z \in B$, $\lambda \in {}^*K$ and $|\lambda| \leqslant 1$. B is $*$-balanced if it is equal to its $*$-balanced hull.

If $\mathscr{B}$ is the collection of all balanced subsets of E then an internal subset B of ${}^*E$ is $*$-balanced if and only if $B \in {}^*\mathscr{B}$. In particular, if $D \subset E$ is balanced then ${}^*D$ is $*$-balanced. The $*$-balanced hull of an internal set is again internal. In general, the $*$-balanced hull of $B \subset {}^*E$ is the smallest $*$-balanced set containing B. Consequently, the intersection of a collection of $*$-balanced sets is $*$-balanced. In particular, if $\mathfrak{F}$ is a filter on E that has a filter basis of balanced sets then $\mu(\mathfrak{F})$ is $*$-balanced. The following proposition establishes the converse of the previous statement.

PROPOSITION 2.6. *Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E. $\mathfrak{F}$ has a filter basis $\mathscr{E}$ of balanced sets if and only if $\mu(\mathfrak{F})$ is $*$-balanced.*

Proof. As stated above, if $\mathfrak{F}$ has a filter basis of balanced sets then $\mu(\mathfrak{F})$ is $*$-balanced.

Conversely, suppose $\mu(\mathfrak{F})$ is $*$-balanced. Let $\mathscr{E}$ be the collection of all balanced hulls of elements of $\mathfrak{F}$. Hence $\mathscr{E}\subset\mathfrak{F}$ which implies $*\mathscr{E}\subset *\mathfrak{F}$. Also if $V\in *\mathfrak{F}$ and if F is the $*$-balanced hull of V, then $F\in *\mathscr{E}$. Let $V_0\in *\mathfrak{F}$ such that $V_0\subset\mu(\mathfrak{F})$ (see [6], Corollary 2.1.6). Now, V_0 being internal and $\mu(\mathfrak{F})$ being $*$-balanced imply F_0, the $*$-balanced hull of V_0, is a subset of $\mu(\mathfrak{F})$ and $F_0\in *\mathscr{E}$. Consequently, if $B\in\mathfrak{F}$, then the following statement is true for $*B$ and $*\mathscr{E}$:

"there exists $F\in *\mathscr{E}$ such that $F\subset *B$".

Indeed, for $B\in\mathfrak{F}$, we have $F_0\subset\mu(\mathfrak{F})\subset *B$. Therefore, the following statement is true for B and $\mathscr{E}$:

"there exists $F\in\mathscr{E}$ such that $F\subset B$".

Thus $\mathscr{E}$ is a filter basis for $\mathfrak{F}$. ■

We can now consider the internal subsets of $\mathrm{Fin}(\mu(\mathfrak{F}))$ for a filter $\mathfrak{F}$ on E that has a $*$-balanced monad. From Proposition 2.2, we infer that if $B\subset E$ is $\mathfrak{F}$-bounded, then $*B\subset\mathrm{Fin}(\mu(\mathfrak{F}))$. Furthermore, the converse is also true. Indeed, the following proposition establishes much more.

PROPOSITION 2.7. *Let E be a vector space over $\boldsymbol{K}$ and let $\mathfrak{F}$ be a filter on E such that $\mu(\mathfrak{F})$ is $*$-balanced. If $B\subset\mathrm{Fin}(\mu(\mathfrak{F}))$ is internal, then for $F\in\mathfrak{F}$, there exists $n\in\boldsymbol{N}$ such that $B\subset *(nF)$.*

Proof. Let $B\subset\mathrm{Fin}(\mu(\mathfrak{F}))$ be internal, i.e., $B\in *\mathscr{P}(E)$. By Proposition 2.6, $\mathfrak{F}$ has a filter basis $\mathscr{E}$ of balanced sets. Now for $F\in\mathscr{E}$ let $\varphi\langle\mathscr{P}(E),\boldsymbol{R},F\rangle$ designate the following sentence:

"if $X\in\mathscr{P}(E)$ such that $X\subset\delta F$ for some positive $\delta\in\boldsymbol{R}$ then either $X\subset\delta F$ for all positive $\delta\in\boldsymbol{R}$ or there exists a positive $\delta_0\in\boldsymbol{R}$ such that $X\subset\delta F$ for $\delta_0<\delta$ and $X\not\subset\delta F$ for $0<\delta<\delta_0$."

Since F is balanced it follows that $\varphi\langle\mathscr{P}(E),\boldsymbol{R},F\rangle$ is true in B_Γ; therefore, $\varphi\langle *\mathscr{P}(E), *\boldsymbol{R}, *F\rangle$ is true in $*B_\Gamma$. Since $B\subset\mathrm{Fin}(\mu(\mathfrak{F}))$, we have $B\subset\lambda *F$ for positive, infinite $\lambda\in *\boldsymbol{R}$ by (1.2). If $B\subset\lambda *F$ for each positive $\lambda\in *\boldsymbol{R}$ then $B\subset *F$. If there exists a positive $\lambda_0\in *\boldsymbol{R}$ such that $B\subset\lambda *F$ for $\lambda_0<\lambda$ and $B\not\subset\lambda *F$ for $0<\lambda<\lambda_0$ then λ_0 is finite. Indeed, if λ_0 were infinite then $2^{-1}\lambda_0$ would be infinite and $2^{-1}\lambda_0<\lambda_0$ which would imply $B\not\subset 2^{-1}\lambda_0 *F$ contradicting $B\subset\mathrm{Fin}(\mu(\mathfrak{F}))\subset 2^{-1}\lambda_0 *F$. In either case there exists $n\in\boldsymbol{N}$ such that $B\subset *(nF)$. ■

In particular, if $B\subset E$ such that $*B$ is a subset of $\mathrm{Fin}(\mu(\mathfrak{F}))$ then B is $\mathfrak{F}$-bounded.

One of the concerns of this paper is when does a filter $\mathfrak{F}$ on a vector space E induce a vector space topology on some non-trivial vector subspace of E. This condition imposes restrictions on $\mathfrak{F}$ that are reflected in the shape of its monad; therefore, the remainder of this section will explore certain properties of filter monads that will be useful in later sections.

First we state a proposition whose proof, being analogous to that of Proposition 2.6, is omitted.

PROPOSITION 2.8. *Let E be a vector space over $\boldsymbol{K}$. If $\mathfrak{F}$ is a filter on E, then $\mu(\mathfrak{F})+\mu(\mathfrak{F})\subset\mu(\mathfrak{F})$ if and only if for $V\in\mathfrak{F}$ there exists $F\in\mathfrak{F}$ such that $F+F\subset V$.*

The next proposition gives conditions for a filter $\mathfrak{F}$ on E so that the monad of $\mathfrak{F}$ has a linear structure.

PROPOSITION 2.9. *Let E be a vector space over $\boldsymbol{K}$. If $\mathfrak{F}$ is a filter on E such that $\mu(\mathfrak{F})$ is $*$-balanced and $\mu(\mathfrak{F})+\mu(\mathfrak{F})=\mu(\mathfrak{F})$ then $\mu(\mathfrak{F})$ is a module over $\mathrm{Fin}(\mu(0))$.*

Proof. By Proposition 2.6, $\mathfrak{F}$ has a filter basis $\mathscr{E}$ of balanced sets. Let $z\in\mu(\mathfrak{F})$ and $\lambda\in\mathrm{Fin}(\mu(0))$ which implies $z\in *F$ for each $F\in\mathscr{E}$ and there exists $n_0\in\boldsymbol{N}$ such that $|\lambda|\leqslant n_0$. If it can be shown that $n_0 z\in *F$ for each $F\in\mathscr{E}$ then $\lambda z\in\mu(\mathfrak{F})$. Indeed, $n_0 z\in *F$ for each $F\in\mathscr{E}$ implies $\lambda z=(\lambda n_0^{-1})n_0 z\in(\lambda n_0^{-1})*F\subset *F$ for each $F\in\mathscr{E}$ since $*F$ is $*$-balanced for each $F\in\mathscr{E}$.

Now, let $F\in\mathscr{E}$. We can infer from Proposition 2.8 that there exists $F_0\in\mathscr{E}$ such that $F_1+\ldots+F_{n_0}\subset F$ where $F_i=F_0$ for each $i\in\{1,\ldots,n_0\}$. So $z\in\mu(\mathfrak{F})$ implies $z\in *F_0$ which implies $n_0 z\in *F$. Therefore $\lambda z\in\mu(\mathfrak{F})$ for $z\in\mu(\mathfrak{F})$ and $\lambda\in\mathrm{Fin}(\mu(0))$.

If $z_1, z_2\in\mu(\mathfrak{F})$ then $z_1+z_2\in\mu(\mathfrak{F})+\mu(\mathfrak{F})=\mu(\mathfrak{F})$. Consequently, $\mu(\mathfrak{F})$ is a module over $\mathrm{Fin}(\mu(0))$. ■

COROLLARY 2.10. *Let E be a vector space over $\boldsymbol{K}$. If $\mathfrak{F}$ is a filter such that $\mu(\mathfrak{F})$ is $*$-balanced and $\mu(\mathfrak{F})+\mu(\mathfrak{F})=\mu(\mathfrak{F})$ then $\lambda\mu(\mathfrak{F})=\mu(\mathfrak{F})$ for $\lambda\in\mathrm{Fin}(\mu(0))$ such that $\lambda\notin\mu(0)$.*

Proof. If $\lambda\in\mathrm{Fin}(\mu(0))$ such that $\lambda\notin\mu(0)$, then $\lambda^{-1}\in\mathrm{Fin}(\mu(0))$. Consequently, $\lambda\mu(\mathfrak{F})\subset\mu(\mathfrak{F})$ and $\lambda^{-1}\mu(\mathfrak{F})\subset\mu(\mathfrak{F})$ by Proposition 2.9. Hence, $\lambda^{-1}\mu(\mathfrak{F})\subset\mu(\mathfrak{F})$ implies $\mu(\mathfrak{F})\subset\lambda\mu(\mathfrak{F})$. Therefore, $\lambda\mu(\mathfrak{F})=\mu(\mathfrak{F})$. ■

COROLLARY 2.11. *Let E be a vector space over $\boldsymbol{K}$ and let $\mathfrak{F}$ be a filter on E such that $\mu(\mathfrak{F})$ is $*$-balanced and $\mu(\mathfrak{F})+\mu(\mathfrak{F})=\mu(\mathfrak{F})$. If $B\subset E$ such that $nB\in\mathfrak{F}$ for some $n\in\boldsymbol{N}$ then $B\in\mathfrak{F}$.*

Proof. If $nB\in\mathfrak{F}$ for some $n\in\boldsymbol{N}$ then $\mu(\mathfrak{F})\subset *(nB)=n*B$. By Corollary 2.10, we have $\mu(\mathfrak{F})=n^{-1}\mu(\mathfrak{F})\subset n^{-1}(n*B)=*B$.

Finally, in the next two propositions we consider two filters on a vector space. The objective is to use finite points in determining when one filter is a subcollection of the other.

PROPOSITION 2.13. *Let E be a vector space over $\boldsymbol{K}$ and let $\mathfrak{F}_1$ be a filter on E such that $\mu(\mathfrak{F}_1)$ is $*$-balanced. If $\mathfrak{F}_2$ is another filter on E then $\mu(\mathfrak{F}_2)\subset\mathrm{Fin}(\mu(\mathfrak{F}_1))$ if and only if for each $F\in\mathfrak{F}_1$, there exists $n\in\boldsymbol{N}$ such that $nF\in\mathfrak{F}_2$.*

Proof. Assume that for each $F\in\mathfrak{F}_1$ there exists $n\in\boldsymbol{N}$ such that $nF\in\mathfrak{F}_2$. If $z\in\mu(\mathfrak{F}_2)$ and $F\in\mathfrak{F}_1$ then $z\in *(nF)$ for some $n\in\boldsymbol{N}$ which implies $z\in\mathrm{Fin}(\mu(\mathfrak{F}_1))$ by Propositions 2.6 and 2.2. Therefore $\mu(\mathfrak{F}_2)\subset\mathrm{Fin}(\mu(\mathfrak{F}_1))$.

Conversely, assume that $\mu(\mathfrak{F}_2)\subset \mathrm{Fin}(\mu(\mathfrak{F}_1))$. There exists $W\in {}^*\mathfrak{F}_2$ such that $W\subset\mu(\mathfrak{F}_2)$ ([6], Corollary 2.1.6); therefore, $W\subset \mathrm{Fin}(\mu(\mathfrak{F}_1))$. Proposition 2.7 implies that for $F\in\mathfrak{F}_1$ there exists $n\in N$ such that $W\subset {}^*(nF)$ which implies ${}^*(nF)\in {}^*\mathfrak{F}_2$ which implies $nF\in\mathfrak{F}_2$. ∎

PROPOSITION 2.14. *Let E be a vector space over K and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters on E such that $\mu(\mathfrak{F}_i)$ is $*$-balanced for $i = 1, 2$. If $\mu(\mathfrak{F}_1)+\mu(\mathfrak{F}_1) = \mu(\mathfrak{F}_1)$ then $\mathrm{Fin}(\mu(\mathfrak{F}_1))\subset\mathrm{Fin}(\mu(\mathfrak{F}_2))$ if and only if $\mu(\mathfrak{F}_1)\subset\mu(\mathfrak{F}_2)$.*

Proof. Clearly $\mu(\mathfrak{F}_1)\subset\mu(\mathfrak{F}_2)$ implies $\mathrm{Fin}(\mu(\mathfrak{F}_1))\subset\mathrm{Fin}(\mu(\mathfrak{F}_2))$. If $\mathrm{Fin}(\mu(\mathfrak{F}_1))\subset\mathrm{Fin}(\mu(\mathfrak{F}_2))$ then $\mu(\mathfrak{F}_1)\subset\mathrm{Fin}(\mu(\mathfrak{F}_2))$ by Proposition 1.3 since $\mu(\mathfrak{F}_1)$ is $*$-balanced. Proposition 2.13 implies that for $F\in\mathfrak{F}_2$ there exists $n\in N$ such that $nF\in\mathfrak{F}_1$; therefore, $F\in\mathfrak{F}_1$ by Corollary 2.11. Consequently $\mathfrak{F}_2\subset\mathfrak{F}_1$ which implies $\mu(\mathfrak{F}_1)\subset\mu(\mathfrak{F}_2)$. ∎

3. Subadditive filters. In this section we consider filters on a vector space E that generate topologies on E. Primarily we seek sufficient conditions, in terms of finite points, so that a filter on E will generate a linear topology on some non-trivial vector subspace of E. This requirement limits the kinds of filters we can examine. In particular, the filter we consider must have a translation-invariant property, i.e., neighborhoods of points are obtained by translations of elements of the filter. Also we would like any generated topology to be Hausdorff. To be more specific, let E be a vector space over K and let $\mathscr{U}(0)$ denote the ultra filter of all subsets of E that contain 0.

DEFINITION 3.1. Let E be a vector space over K and let $\mathfrak{F}$ be a filter on E. $\mathfrak{F}$ is said to be *subadditive* if and only if $\mathfrak{F}\subset\mathscr{U}(0)$ and the following conditions hold:

1) If $x\in E$ such that $x\neq 0$ then there exists $F\in\mathfrak{F}$ such that $x\notin F$.

2) For $F\in\mathfrak{F}$, there exists $F_1\in\mathfrak{F}$ such that $F_1+F_1\subset F$.

Proposition 2.8 implies that a filter $\mathfrak{F}$ on a vector space E is subadditive if and only if $0\in\mu(\mathfrak{F})$, $\mu(\mathfrak{F})+\mu(\mathfrak{F}) = \mu(\mathfrak{F})$ and $x\in E$ such that ${}^*x\in\mu(\mathfrak{F})$ implies $x = 0$. Also, it is easily shown that if $\mathfrak{F}$ is a subadditive filter on E then there exists an unique Hausdorff topology $\hat\theta$ on E such that $\mathfrak{F} = \eta_{\hat\theta}(0)$, the $\hat\theta$-neighborhoods of 0, and vector addition is $\hat\theta$-continuous. We will refer to $\hat\theta$ as the topology generated by $\mathfrak{F}$. The topology $\hat\theta$ is not necessarily a group topology on E since the map $x\to -x$ may not be $\hat\theta$-continuous. However, if $\mu(\mathfrak{F}) = -\mu(\mathfrak{F})$ then $(E,\hat\theta)$ is a Hausdorff topological group.

PROPOSITION 3.2. *Let E be a vector space over K, let $\mathfrak{F}$ be a subadditive filter on E and let $\hat\theta$ be the topology on E generated by $\mathfrak{F}$. $(E,\hat\theta)$ is a topological vector space if and only if $\mu(\mathfrak{F})$ is $*$-balanced and ${}^*x\in\mathrm{Fin}(\mu(\mathfrak{F}))$ for each $x\in E$.*

Proof. $\hat\theta$ is a vector space topology if and only if $\mathfrak{F}$ has a filter basis $\mathscr{E}$ of balanced, absorbing sets so that for $F\in\mathscr{E}$ there exists $F_1\in\mathscr{E}$ such that $F_1+F_1\subset F$ ([3], Theorem 1, p. 81). Therefore the proposition follows from Proposition 2.6, Corollary 2.3 and Definition 3.1. ∎

Using Proposition 1.5 together with Proposition 2.6, Corollary 2.3 and Definition 3.1, we obtain an immediate generalization of Proposition 3.2 which is stated without proof.

PROPOSITION 3.3. *Let E be a vector space over K, let $\mathfrak{F}$ be a subadditive filter on E and let $\hat\theta$ be the topology on E generated by $\mathfrak{F}$. If $\mu(\mathfrak{F})$ is $*$-balanced, then $A(\mathfrak{F})$ (Definition 2.1) is a vector subspace of E and $\hat\theta$ induces a linear topology θ on $A(\mathfrak{F})$ such that $\mathfrak{F}_A$, the trace of $\mathfrak{F}$ on $A(\mathfrak{F})$, is the collection of θ-neighborhoods of 0 in $A(\mathfrak{F})$.*

Now consider an arbitrary subadditive filter $\mathfrak{F}$ on a vector space E. For $F\in\mathfrak{F}$, let F_C denote the balanced core of F, i.e., F_C is the largest (with respect to set inclusion) balanced subset of F. Note that F_C is not empty, for $F\in\mathfrak{F}$, since $0\in F_C$ and F_C is said to be non-trivial if and only if there exists $x\in F_C$ such that $x\neq 0$. Clearly the collection $\{F_C|\ F\in\mathfrak{F}\}$ generates a filter, denoted by $\mathfrak{F}_C$, that contains $\mathfrak{F}$. Also $\mathfrak{F}_C\neq\mathscr{U}(0)$ if and only if each F_C is non-trivial for $F\in\mathfrak{F}$.

Using Proposition 2.1 and 2.2 we obtain the following inclusions:

$$A(\mathfrak{F}_C)\subset FP(\mathfrak{F})\subset A(\mathfrak{F}), \tag{3.4}$$

where $A(\mathfrak{F}_C)$ and $A(\mathfrak{F})$ are the absorbed points of $\mathfrak{F}_C$ and $\mathfrak{F}$ respectively and $FP(\mathfrak{F})$ is the set of all standard $\mathfrak{F}$-finite points of E, i.e., $x\in FP(\mathfrak{F})$ if and only if ${}^*x\in\mathrm{Fin}(\mu(\mathfrak{F}))$ (see Definition 2.4). Observe that $FP(\mathfrak{F})$ is a *vector subspace* of E by Propositions 2.8 and 1.5 since $\mathfrak{F}$ is subadditive.

PROPOSITION 3.5. *Let E be a vector space over K. If $\mathfrak{F}$ is a subadditive filter on E then $A(\mathfrak{F}_C) = FP(\mathfrak{F})$.*

Proof. Let $x\in E$ such that ${}^*x\in\mathrm{Fin}(\mu(\mathfrak{F}))$ and let $F\in\mathfrak{F}$. If $\beta\in\mu(0)$ such that $0<\beta$ then $\lambda{}^*x\in\mu(\mathfrak{F})\subset{}^*F$ for $\lambda\in{}^*K$ such that $|\lambda|\leqslant\beta$. Consequently, the following statement is true for ${}^*x$, ${}^*F$ and ${}^*K$:

"there exists $\beta\in{}^*K$ such that $0<\beta$ and $\lambda{}^*x\in{}^*F$ for $\lambda\in{}^*K$ such that $|\lambda|\leqslant\beta$"

Passing this condition back to x, F and K we infer there exists $\beta\in K$ such that $0<\beta$ and $\lambda x\in F$ for $|\lambda|\leqslant\beta$. Therefore, if $n\in N$ such that $n^{-1}\leqslant\beta$, then $n^{-1}x\in F_C$ which implies $x\in nF_C$. Consequently, $x\in A(\mathfrak{F}_C)$ since $F\in\mathfrak{F}$ was arbitrary.

The proposition follows from (3.4) and the above argument. ∎

If a subadditive filter $\mathfrak{F}$, on a vector space E, is to induce a linear topology on some non-trivial vector subspace E_0 of E then $E_0\subset FP(\mathfrak{F})$ which implies $\mathrm{Fin}(\mu(\mathfrak{F}))$ must have a non zero standard point. In general, $\mathrm{Fin}(\mu(\mathfrak{F}))$ having a non zero standard point does not necessarily imply $\mathfrak{F}$ induces a linear topology on some non-trivial subspace of $FP(\mathfrak{F})$. However, the picture changes if we assume $\mu(\mathfrak{F})$ is μ-saturated (Definition 1.4).

LEMMA 3.6. *Let E be a vector space over K and let $\mathfrak{F}$ be a subadditive filter on E such that $\mu(\mathfrak{F})$ is μ-saturated. If $x\in E$ such that $x\neq 0$ and ${}^*x\in\mathrm{Fin}(\mu(\mathfrak{F}))$, then $\lambda{}^*x\in\mu(\mathfrak{F})$ if and only if $\lambda\in\mu(0)$.*

Proof. If $\lambda \in \mu(0)$ then $\lambda{}^*x \in \mu(\mathfrak{F})$ since ${}^*x \in \mathrm{Fin}(\mu(\mathfrak{F}))$. Conversely, let $\lambda \in {}^*\boldsymbol{K}$ such that $\lambda{}^*x \in \mu(\mathfrak{F})$. Thus, λ is not infinite. Indeed, if λ were infinite then $\lambda{}^*x \in \mu(\mathfrak{F})$ would imply ${}^*x = \lambda^{-1}(\lambda{}^*x) \in \mu(\mathfrak{F})$, since λ^{-1} would be infinitesimal and $\mu(\mathfrak{F})$ is μ-saturated, contradicting ${}^*x \notin \mu(\mathfrak{F})$ since $\mathfrak{F}$ is subadditive and $x \neq 0$. Therefore λ is finite which implies ${}^0\lambda \in \boldsymbol{K}$ exists. Since ${}^0\lambda - \lambda \in \mu(0)$ we infer

$$ {}^0\lambda{}^*x = \lambda{}^*x + ({}^0\lambda - \lambda){}^*x \in \mu(\mathfrak{F}) + \mu(\mathfrak{F}) = \mu(\mathfrak{F}) $$

which implies ${}^0\lambda x = 0$. Consequently, ${}^0\lambda = 0$, since $x \neq 0$, which implies $\lambda \in \mu(0)$. ■

PROPOSITION 3.7. *Let E be a vector space over $\boldsymbol{K}$ and let $\mathfrak{F}$ be a subadditive filter on E. If $\mu(\mathfrak{F})$ is μ-saturated then $\mathfrak{F}$ induces a linear topology on some non-trivial vector subspace of E if and only if there exists a non zero $x \in E$ such that ${}^*x \in \mathrm{Fin}(\mu(\mathfrak{F}))$.*

Proof. If $\mathfrak{F}$ induces a linear topology on a non-trivial vector subspace E_0 of E then ${}^*x \in \mathrm{Fin}(\mu(\mathfrak{F}))$ for any $x \in E_0$. Conversely, assume that there exists a non zero $x \in E$ such that ${}^*x \in \mathrm{Fin}(\mu(\mathfrak{F}))$. Let $E_0 = \{\lambda x |\ \lambda \in \boldsymbol{K}\}$ and let θ be the topology on E_0 induced by $\mathfrak{F}$. From Lemma 3.6 we infer the map $\lambda \to \lambda x$ of $\boldsymbol{K}$ onto E_0 is a linear homeomorphism with respect to θ (see [7], Theorem 4.2.7). Therefore θ is a linear topology on E_0. ■

4. Invariant nonstandard hulls and Property 1. We now apply the concepts of the previous sections to produce a class of locally convex spaces that have invariant nonstandard hulls. In view of the work of Henson and Moore [2], it is shown, under an appropriate hypothesis, that these spaces are not metrizable and are not Schwarz spaces (Propositions 5.2 and 5.4). Since function spaces are used to define this class of spaces, we begin by establishing the necessary notation.

Unless stated otherwise, I will denote an infinite set and $\Delta(I)$ is the collection of all finite subsets of I (we assume $\emptyset \in \Delta(I)$ also). In this section E is $\boldsymbol{K}^I$, the vector space of all $\boldsymbol{K}$-valued functions on I. For $x \in E$ define

$$ s(x) = \{i \in I |\ x(i) \neq 0\} \tag{4.1} $$

and let $E_0 = \boldsymbol{K}^{(I)}$, the set of all $x \in E$ for which $s(x)$ is finite, i.e., $s(x) \in \Delta(I)$. Hence, $\emptyset \in \Delta(I)$ implies E_0 is a proper vector subspace of E.

The ith projection is symbolized by π_i, i.e., $\pi_i\colon E \to \boldsymbol{K}$ for which $\pi_i(x) = x(i)$ for $x \in E$ and $i \in I$. Also, for $i \in I$, define $e_i \in E_0$ as follows: $e_i(i) = 1$ and $e_i(j) = 0$ for $j \in I$ such that $j \neq i$. Clearly, $\{e_i |\ i \in I\}$ is a *Hamel* basis for E_0. Furthermore, if $z \in E_0$, then

$$ z = \sum_{i \in s(z)} \pi_i(z) e_i \,. \tag{4.2} $$

Let e symbolize the function defined as follows: $e(i) = 1$ for each $i \in I$. Let $\mathfrak{A}(E)$ be the set of all $x \in E$ for which $0 < \pi_i(x)$ for each $i \in I$ and let $\mathfrak{A}(E_0)$ denote the set of all $z \in E_0$ such that $0 < \pi_i(z)$ for $i \in s(z)$. Note that $\mathfrak{A}(E)$ and E_0 are disjoint sets.

For this and the remaining sections the set-theoretical structure B_Γ will have $B_0 = I \cup C$ as its base set (see preliminaries). Thus E and E_0 must not be considered as external subsets of ${}^*E$ since the entities $\boldsymbol{K}^I$ and $\boldsymbol{K}^I$ are elements of $B_{((0,0))}$. However, we will identify I with the external subset ${}^*[I] = \{{}^*i |\ i \in I\}$ of ${}^*I$; therefore, the notation ${}^*I\text{-}I$ represents the elements of ${}^*I$ that are not in ${}^*[I]$. In general, the index set I will not have an algebraic structure. So, if X_1 and X_2 are subsets of I, then

$$ X_1 - X_2 = \{i \in I |\ i \in X_1 \text{ and } i \notin X_2\} \,. $$

DEFINITION 4.3. Let I be an infinite set and let $E = \boldsymbol{K}^I$. For $x \in E$ define $[x] \subset E$ as follows: $z \in [x]$ if and only if $|\pi_i(z)| \leqslant |\pi_i(x)|$ for each $i \in I$. If $A \subset E$ is nonempty then let

$$ \mathscr{E}(A) = \{[x] |\ x \in A\} \,. $$

Now we are ready to define a new class of linear spaces that have invariant non-standard hulls. The idea is to give conditions on a non-empty set $A \subset E$ so that $\mathscr{E}(A)$ will induce a linear topology on E_0 that has the desired property.

DEFINITION 4.4. Let I be an infinite set and let $E = \boldsymbol{K}^I$. A set $A \subset E$ is said to *satisfy Property* 1 if and only if $A \neq \emptyset$, $A \subset \mathfrak{A}(E)$ and the following conditions hold:

1. For $x, y \in A$, there exists $z \in A$ such that $\pi_i(z) \leqslant \min(\pi_i(x), \pi_i(y))$ for each $i \in I$.

2. For $x \in A$, there exists $y, z \in A$ such that $\pi_i(2y) \leqslant \pi_i(x)$ and $\pi_i(z) \leqslant (\pi_i(x))^2$ for each $i \in I$.

3. For $\delta > 0$ and $i \in I$ there exists $x \in A$ for which $\pi_i(x) < \delta$.

4. If $z \in E$ such that for each $x \in A$, there exists $\lambda \in \boldsymbol{K}$ (depending on x) for which $|\pi_i(z)| \leqslant \lambda \pi_i(x)$ for each $i \in I$, then $s(z)$ is finite, i.e., $s(z) \in \Delta(I)$.

5. If $\imath \in {}^*I - I$, then there exists $x \in A$ such that $\pi_\imath({}^*x) \in \mu(0)$.

PROPOSITION 4.5. *Let I be an infinite set, let $E = \boldsymbol{K}^I$ and let $E_0 = \boldsymbol{K}^{(I)}$. If $A \subset E$ satisfies Property 1 then $\mathscr{E}(A)$ is a filter basis for a subadditive filter $\mathfrak{F}$ on E such that $\mu(\mathfrak{F})$ is $*$-balanced and $z \in E_0$ if and only if ${}^*z \in \mathrm{Fin}(\mu(\mathfrak{F}))$.*

Proof. Conditions 1, 2 and 3 of Definition 4.4 imply $\mathscr{E}(A)$ is a filter basis for a subadditive filter $\mathfrak{F}$ on E. Also $\mu(\mathfrak{F})$ is $*$-balanced by Proposition 2.6 since $\mathscr{E}(A)$ is a collection of balanced convex sets. Finally, it follows from Corollary 2.3, $A \subset \mathfrak{A}(E)$ and Condition 4 of Definition 4.4 that $z \in E_0$ if and only if ${}^*z \in \mathrm{Fin}(\mu(\mathfrak{F}))$. ■

PROPOSITION 4.6. *Let I be an infinite set, let $E = \boldsymbol{K}^I$ and let $E_0 = \boldsymbol{K}^{(I)}$. If $A \subset E$ satisfies Property 1, then the subadditive filter $\mathfrak{F}$ on E generated by $\mathscr{E}(A)$ induces a Hausdorff, locally convex linear topology θ on E_0 such that $\mathfrak{F}_{E_0}$, the trace of $\mathfrak{F}$ on E_0, is the filter of θ-neighborhoods of 0 in E_0.*

Proof. Propositions 4.5, 3.3 and the fact that $\mathscr{E}(A)$ is a collection of balanced convex sets. ■

Remark. The topology θ of Proposition 4.6 will sometimes be called the linear topology induced on E_0 by A, whenever $A \subset E$ satisfies Property 1.

It will be shown that if $A \subset E$ satisfies Property 1, then (E_0, θ), where θ is the linear topology induced on E_0 by A, has an invariant nonstandard hull (Theorem 4.17). However, before considering the nonstandard hull of (E_0, θ), we must examine the finite points of $\mathfrak{F}$, the filter on E generated by $\mathscr{E}(A)$. To aid in this examination we now introduce an interesting collection of ${}^*\boldsymbol{K}$-subsets.

Definition 4.7. Let I be an infinite set, let $E = \boldsymbol{K}^I$ and let $B \subset E$ be nonempty. For $\iota \in {}^*I$, define $\nu_\iota(B) \subset {}^*\boldsymbol{K}$ as follows: $\lambda \in \nu_\iota(B)$ if and only if $|\lambda| \leqslant |\pi_\iota({}^*x)|$ for each $x \in B$.

For $A \subset E$ satisfying Property 1 and $\mathfrak{F}$, the filter on E generated by $\mathscr{E}(A)$, the following propositions show that the monad of $\mathfrak{F}$ and the finite points of $\mathfrak{F}$ are completely determined by the collection $\{\nu_\iota(A) |\ \iota \in {}^*I\}$. It is also shown that $\nu_\iota(A)$ has the peculiar property of being Fin-invariant when $\iota \in {}^*I - I$ and not Fin-invariant when $\iota \in I$. Finally, observe that $\nu_\iota(A)$ is $*$-balanced for each $\iota \in {}^*I$.

Proposition 4.8. *Let I be an infinite set, let $E = \boldsymbol{K}^I$ and let $A \subset E$ satisfy Property* 1. *If $i \in I$, then $\nu_i(A) = \mu(0)$.*

Proof. For $i \in I$, let $z \in \nu_i(A)$ and let $\delta \in \boldsymbol{K}$ such that $0 < \delta$. By Condition 3 of Property 1, there exists $x \in A$ for which $\pi_i(x) < \delta$. Hence ${}^*\pi_i({}^*x) < \delta$ which implies $|z| < \delta$ since $|z| \leqslant {}^*\pi_i({}^*x)$; therefore, $z \in \mu(0)$ since $\delta > 0$ was arbitrary. Consequently $\nu_i(A) \subset \mu(0)$.

Conversely, if $z \in \mu(0)$ then $|z| < {}^*\pi_i({}^*x)$ for each $x \in A$ since $\pi_i(x) > 0$ for each $x \in A$. Hence $z \in \nu_i(A)$ by Definition 4.7. Thus $\nu_i(A) = \mu(0)$. ∎

Proposition 4.9. *Let I be an infinite set, let $E = \boldsymbol{K}^I$ and let $A \subset E$ satisfy Property* 1. *If $\iota \in {}^*I - I$ then $\nu_\iota(A) \subset \mu(0)$ and $\nu_\iota(A)$ is Fin-invariant, i.e., $\mathrm{Fin}(\nu_\iota(A)) = \nu_\iota(A)$.*

Proof. Let $\iota \in {}^*I - I$. Condition 5 of Definition 4.4 implies there exists $x_0 \in A$ such that $\pi_\iota({}^*x_0) \in \mu(0)$. So $\lambda \in \nu_\iota(A)$ implies $|\lambda| \leqslant \pi_\iota({}^*x_0)$, which implies $\lambda \in \mu(0)$. Hence $\nu_\iota(A) \subset \mu(0)$. Also $\nu_\iota(A) \subset \mathrm{Fin}(\nu_\iota(A))$ since $\nu_\iota(A)$ is $*$-balanced (Proposition 1.3).

Let $\lambda \in \mathrm{Fin}(\nu_\iota(A))$ and let $x \in A$ which implies $0 < \pi_\iota({}^*x)$.

Assume $\pi_\iota({}^*x) \in \mu(0)$; therefore,

$$\pi_\iota({}^*x)\lambda \in \nu_\iota(A) \tag{4.10}$$

by Definition 1.1. By Condition 2 of Property 1, there exists $z \in A$ such that $\pi_i(z) \leqslant (\pi_i(x))^2$ for each $i \in I$ which implies $\pi_\varkappa({}^*z) \leqslant (\pi_\varkappa({}^*x))^2$ for each $\varkappa \in {}^*I$. In particular, $\pi_\iota({}^*z) \leqslant (\pi_\iota({}^*x))^2$. From (4.10) we infer

$$|\pi_\iota({}^*x)\lambda| \leqslant \pi_\iota({}^*z) \leqslant (\pi_\iota({}^*x))^2$$

which implies

$$|\lambda| \leqslant (\pi_\iota({}^*x))^{-1}(\pi_\iota({}^*x))^2 = \pi_\iota({}^*x)\,.$$

Therefore, if $x \in A$ for which $\pi_\iota({}^*x) \in \mu(0)$, then $|\lambda| \leqslant \pi_\iota({}^*x)$.

If $x \in A$ such that $\pi_\iota({}^*x) \notin \mu(0)$ then there exists $\delta \in \boldsymbol{K}$ such that $0 < \delta \leqslant \pi_\iota({}^*x)$ which implies

$$|\lambda| \leqslant \pi_\iota({}^*x_0) < \delta \leqslant \pi_\iota({}^*x)$$

since $x_0 \in A$ for which $\pi_\iota({}^*x_0) \in \mu(0)$.

In either case, $x \in A$ implies $|\lambda| \leqslant \pi_\iota({}^*x)$ which implies $\lambda \in \nu_\iota(A)$. Consequently $\mathrm{Fin}(\nu_\iota(A)) = \nu_\iota(A)$ since $\lambda \in \mathrm{Fin}(\nu_\iota(A))$ was arbitrary. ∎

It can be shown that if A satisfies Property 1 then for any $\iota \in {}^*I$ there exists $\lambda_\iota \in {}^*\boldsymbol{K}$ for which $0 < \lambda_\iota$ and $\alpha \in \nu_\iota(A)$ for $|\alpha| \leqslant \lambda_\iota$. This fact is a consequence of $\{{}^*x |\ x \in A\}$ being an external subset of some $*$-finite subset of ${}^*A$.

Proposition 4.11. *Let I be an infinite set and let $E = \boldsymbol{K}^I$. For $A \subset E$ satisfying Property* 1, *let $\mathfrak{F}$ be the filter on E generated by $\mathscr{E}(A)$. If $z \in {}^*E$ then $z \in \mu(\mathfrak{F})$ if and only if $\pi_\iota(z) \in \nu_\iota(A)$ for each $\iota \in {}^*I$.*

Proof. If $z \in {}^*E$ then $z \in {}^*[x]$ for each $x \in A$ if and only if $|\pi_\iota(z)| \leqslant \pi_\iota({}^*x)$ for each $\iota \in {}^*I$ and each $x \in A$. ∎

Proposition 4.12. *Let I be an infinite set and let $E = \boldsymbol{K}^I$. For $A \subset E$ satisfying Property* 1, *let $\mathfrak{F}$ be the filter on E generated by $\mathscr{E}(A)$. If $z \in {}^*E$ then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ if and only if $\pi_\iota(z) \in \mathrm{Fin}(\nu_\iota(A))$ for each $\iota \in {}^*I$.*

Proof. If $z \in {}^*E$, then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ if and only if $\lambda z \in \mu(\mathfrak{F})$ for each $\lambda \in \mu(0)$ if and only if $\lambda\pi_\iota(z) = \pi_\iota(\lambda z) \in \nu_\iota(A)$ for each $\lambda \in \mu(0)$ and each $\iota \in {}^*I$ (Proposition 4.11). Consequently $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ if and only if $\pi_\iota(z) \in \mathrm{Fin}(\nu_\iota(A))$ for each $\iota \in {}^*I$. ∎

Proposition 4.12 was predictable; however, the cartesian product of the external family $\{\mathrm{Fin}(\nu_\iota(A)) |\ \iota \in {}^*I\}$, where $A \subset E$ satisfies Property 1, contains too many external ${}^*\boldsymbol{K}$-valued functions on ${}^*I$. The following proposition offers a deeper insight into the nature of the finite points generated by $\mathscr{E}(A)$.

Proposition 4.13. *Let I be an infinite set and let $E = \boldsymbol{K}^I$. For $A \subset E$ satisfying Property* 1, *let $\mathfrak{F}$ be the filter on E generated by $\mathscr{E}(A)$. If $z \in {}^*E$ then $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ if and only if there exists a finite set $S \in \Delta(I)$ such that ${}^*\pi_i(z) \in \mathrm{Fin}(\nu_i(A))$ for $i \in S$ and $\pi_\iota(z) \in \nu_\iota(A)$ for $\iota \in {}^*I - S$.*

Proof. Let $z \in {}^*E$. If there exists $S \in \Delta(I)$ such that ${}^*\pi_i(z) \in \mathrm{Fin}(\nu_\iota(A))$ for $i \in S$ and $\pi_\iota(z) \in \nu_\iota(A)$ for $\iota \in {}^*I - S$ then Proposition 4.12 implies $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ since $\nu_\iota(A)$ is $*$-balanced for each $\iota \in {}^*I$ (see Proposition 1.3).

Conversely, assume $z \in \mathrm{Fin}(\mu(\mathfrak{F}))$ which implies $\pi_\iota(z) \in \mathrm{Fin}(\nu_\iota(A))$ for each $\iota \in {}^*I$ by Proposition 4.12. Hence $\pi_\iota(z) \in \mathrm{Fin}(\mu(0))$ for each $\iota \in {}^*I$ since $\nu_\iota(A) \subset \mu(0)$ for each $\iota \in {}^*I$ (Propositions 4.8 and 4.9). Therefore, ${}^0(\pi_\iota(z)) \in \boldsymbol{K}$ exists for each $\iota \in {}^*I$.

Define $y \in E$ as follows: $\pi_i(y) = {}^0({}^*\pi_i(z))$ for each $i \in I$.

Let $x \in A$. There exists $n \in \boldsymbol{N}$ for which $z \in {}^*(n[x]) = {}^*[nx]$ by Proposition 2.1. So,

$$|\pi_\iota(z)| \leqslant \pi_\iota({}^*(nx)) = n\pi_\iota({}^*x) \quad \text{for each } \iota \in {}^*I$$

which implies

$$|{}^*\pi_i(z)| \leqslant {}^*(n\pi_i(x)) < {}^*((n+1)\pi_i(x)) \quad \text{for each } i \in I.$$

Thus

$$|\pi_i(y)| = |{}^0({}^*\pi_i(z))| < (n+1)\pi_i(x) \quad \text{for each } i \in I.$$

Consequently $s(y)$ is finite, i.e., $s(y) \in \Delta(I)$ by Condition 4 of Property 1 since $x \in A$ was arbitrary.

We infer, from Proposition 4.8, that ${}^*\pi_i(z) \in \mu(0) = \nu_i(A)$ for $i \in I - s(y)$ since $0 = \pi_i(y) = {}^0({}^*\pi_i(z))$ for $i \in I - s(y)$. Also Proposition 4.9 implies

$$\pi_\iota(z) \in \operatorname{Fin}(\nu_\iota(A)) = \nu_\iota(A)$$

for $\iota \in {}^*I - I$. Therefore if we let $S = s(y)$ then $S \in \Delta(I)$, ${}^*\pi_i(z) \in \operatorname{Fin}(\nu_i(A))$ for $\iota \in S$ and $\pi_\iota(z) \in \nu_\iota(A)$ for $\iota \in {}^*I - S$ since ${}^*I - S = ({}^*I - I) \cup (I - S)$. ■

PROPOSITION 4.14. *Let I be an infinite set, let $E = K^I$ and let $E_0 = K^{(I)}$. For $A \subset E$ satisfying Property 1 let $\mathfrak{F}$ be the filter on E generated by $\mathscr{E}(A)$. If $z \in \operatorname{Fin}(\mu(\mathfrak{F}))$ then there exists $x \in E_0$ such that $z - {}^*x \in \mu(\mathfrak{F})$.*

Proof. Let $z \in \operatorname{Fin}(\mu(\mathfrak{F}))$; therefore, there exists a finite set $S \in \Delta(I)$ such that ${}^*\pi_i(z) \in \operatorname{Fin}(\nu_i(A))$ for $i \in S$ and $\pi_\iota(z) \in \nu_\iota(A)$ for $\iota \in {}^*I - S$ by Proposition 4.13. Now Proposition 4.8 implies $\operatorname{Fin}(\nu_i(A)) = \operatorname{Fin}(\mu(0))$ for $i \in S$; hence, ${}^0({}^*\pi_i(z))$ exists for $i \in S$.

Define $x \in E_0$ as follows: $\pi_i(x) = {}^0({}^*\pi_i(z))$ for $i \in S$ and $\pi_i(x) = 0$ for $i \in I - S$. Consequently $\pi_\iota({}^*x) = {}^0(\pi_\iota(z))$ for $\iota \in {}^*S = S$ and $\pi_\iota({}^*x) = 0$ for $\iota \in {}^*I - S$.

Proposition 4.8 implies

$$\pi_\iota(z - {}^*x) = \pi_\iota(z) - \pi_\iota({}^*z) \in \mu(0) = \nu_\iota(A) \quad \text{for } \iota \in S.$$

Also for $\iota \in {}^*I - S$ we have

$$\pi_\iota(z - {}^*x) = \pi_\iota(z) - \pi_\iota({}^*x) = \pi_\iota(z) \in \nu_\iota(A).$$

Therefore $\pi_\iota(z - {}^*x) \in \nu_\iota(A)$ for each $i \in {}^*I$ which implies $z - {}^*x \in \mu(\mathfrak{F})$ by Proposition 4.11. ■

Let $A \subset E$ satisfy Property 1. We now prove the nonstandard hulls of (E_0, θ), where θ is the linear topology induced on E_0 by A, are invariant. The basic theory of nonstandard hulls of locally convex spaces is developed in [2]; a few definitions and details will be reproduced here for convenience. Although the following discussion is valid for any Hausdorff linear space, we will confine our attentions to (E_0, θ).

The monad of the filter $\eta_\theta(0)$ of θ-neighborhoods of 0 in E_0 is denoted by $\mu_\theta(0)$. An element p of ${}^*E_0$ is called *θ-near-standard* if there exists $x \in E_0$ such that $p - {}^*x \in \mu_\theta(0)$, p is called θ-pre-near standard if for each $V \in \eta_\theta(0)$ there exists $x \in E_0$ for which $p \in {}^*x + {}^*V$ and p is called *θ-finite* if $p \in \operatorname{Fin}(\mu_\theta(0))$. The sets of θ-near-standard and θ-pre-near-standard points of ${}^*E_0$ are symbolized by $\operatorname{ns}_\theta({}^*E_0)$ and $\operatorname{pns}_\theta({}^*E_0)$ respectively. These subsets of ${}^*E_0$ are related by

$$\mu_\theta(0) \subset \operatorname{ns}_\theta({}^*E_0) \subset \operatorname{pns}_\theta({}^*E_0) \subset \operatorname{Fin}(\mu_\theta(0)). \tag{4.15}$$

The collection $\{{}^*V \cap \operatorname{Fin}(\mu_\theta(0)) \mid V \in \eta_\theta(0)\}$ is a filter base for a locally convex linear topology $\hat{\theta}$ on $\operatorname{Fin}(\mu_\theta(0))$ under which $\operatorname{pns}_\theta({}^*E_0)$ and $\mu_\theta(0)$ are closed sets. The nonstandard hull of (E_0, θ) with respect to the enlargement ${}^*B_\Gamma$ is the Hausdorff quotient space

$$(\hat{E}_0, \hat{\theta}) = (\operatorname{Fin}(\mu_\theta(0)), \hat{\theta})/\mu_\theta(0). \tag{4.16}$$

Let $\varphi\colon \operatorname{Fin}(\mu_\theta(0)) \to \hat{E}_0$ be the natural quotient map. The map taking x to $\varphi({}^*x)$ is a topological vector space isomorphism of (E_0, θ) into $(\hat{E}_0, \hat{\theta})$.

The nonstandard hull $(\hat{E}_0, \hat{\theta})$ of (E_0, θ) contains the completion of (E_0, θ) as the image under φ of the set $\operatorname{pns}_\theta({}^*E_0)$. When the nonstandard hull $(\hat{E}_0, \hat{\theta})$ is equal to the image of $\operatorname{pns}_\theta({}^*E_0)$ under φ, we say that *the nonstandard hulls of (E_0, θ) are invariant.*

THEOREM 4.17. *Let I be an infinite set, let $E = K^I$ and let $E_0 = K^{(I)}$. If $A \subset E$ satisfies Property 1 then the nonstandard hulls of (E_0, θ), where θ is the linear topology induced on E_0 by A, are invariant.*

Proof. If $\mathfrak{F}$ is the subadditive filter on E generated by A then $\mathfrak{F}_{E_0}$, the trace of $\mathfrak{F}$ on E_0, is the filter of θ-neighborhoods of 0 in E_0 which implies

$$\mu_\theta(0) = {}^*E_0 \cap \mu(\mathfrak{F})$$

and

$$\operatorname{Fin}(\mu_\theta(0)) = {}^*E_0 \cap \operatorname{Fin}(\mu(\mathfrak{F}))$$

since ${}^*E_0$ is Fin-invariant.

Let $z \in \operatorname{Fin}(\mu_\theta(0))$. There exists $x \in E_0$ for which $z - {}^*x \in \mu(\mathfrak{F})$ by Proposition 4.14. Hence $z - {}^*x \in {}^*E_0$ implies $z - {}^*x \in \mu_\theta(0)$. Therefore

$$\operatorname{ns}_\theta({}^*E_0) = \operatorname{Fin}(\mu_\theta(0))$$

which implies the nonstandard hulls of (E_0, θ) are invariant by (4.15), Corollary 2.3 and Lemma 1(iii) of [2]. ■

Observe that in the proof of Theorem 4.17 it is shown

$$\operatorname{ns}_\theta({}^*E_0) = \operatorname{pns}_\theta({}^*E_0); \tag{4.18}$$

consequently, *(E_0, θ) is complete* ([6], Theorem 3.14.1).

5. Property 1 and uncountable index sets. We now show that if the index set I is uncountable then (E_0, θ), where θ is a linear topology induced on E_0 by some Property 1 subset of E, is not metrizable and is not a Schwartz space. Thus (E_0, θ), for I uncountable, does not satisfy the sufficient conditions of Theorem 1 and Theorem 4 in [2]. The following proposition is the corner stone of this section.

PROPOSITION 5.1. *Let I be an uncountable set, let $E = K^I$ and let $E_0 = K^{(I)}$. If $A \subset E$ satisfies Property 1, then $B \subset E_0$ is θ-bounded, where θ is the linear topology induced on E_0 by A, if and only if $B \subset [y]$ for some $y \in \mathfrak{A}(E_0)$.*

Proof. Let $B \subset E_0$ and assume $B \subset [y]$ for some $y \in \mathfrak{A}(E_0)$. Now $y \in \mathfrak{A}(E_0)$ implies $s(y)$ is finite and $0 < \pi_i(y)$ for each $i \in s(y)$. Therefore, if $x \in A$, then $B \subset \beta[x]$, where $\beta \in N$ such that

$$\max\{\pi_i(y)|\ i \in s(y)\} < \beta \min\{\pi_i(x)|\ i \in s(y)\}.$$

Consequently B is θ-bounded.

Conversely, assume that B is θ-bounded. Let

$$s(B) = \{i \in I|\ \pi_i(z) \neq 0 \text{ for some } z \in B\}$$

and define $y \in E$ as follows: $\pi_i(y) = 0$ for $i \notin s(B)$ and $\pi_i(y) = \sup_{z \in B} |\pi_i(z)|$ for $i \in s(B)$. The function y is well defined since B is θ-bounded. In fact, if $x \in A$, then there exists $n \in N$ such that $|\pi_i(z)| \leqslant n\pi_i(x)$ for each $i \in I$ and each $z \in B$; therefore, $\pi_i(y) \leqslant n\pi_i(x)$ for each $i \in I$. From Condition 4 of Property 1 we infer $s(y)$ is finite. Furthermore, $y \in \mathfrak{A}(E_0)$ and $B \subset [y]$ by definition of y. ■

Proposition 5.2. *Let I be an uncountable set, let $E = K^I$ and let $E_0 = K^{(I)}$. If $A \subset E$ satisfies Property 1 then (E_0, θ) is not metrizable, where θ is the linear topology induced on E_0 by A.*

Proof. Let S be a countable subset of I and consider $\mathscr{C} = \{\{e_i\}|\ i \in S\}$ which is a sequence of θ-bounded subsets of E_0. If $B \subset E_0$ is θ-bounded then $\mathrm{sp}(B)$, the linear subspace of E_0 generated by B, is finite dimensional by Proposition 5.1; therefore, $\bigcup \mathscr{C}$ is not a subset of $\mathrm{sp}(B)$ since $\bigcup \mathscr{C}$ is an infinite set of linearly independent elements. Consequently, (E_0, θ) does not satisfy Mackey's countability condition ([3], Chapter 2, Section 7, Proposition 3) which implies (E_0, θ) is not metrizable. ■

Before proving (E_0, θ) is not a Schwartz space, we need a technical lemma. Recall that e is the function on I which has a constant value of 1.

Lemma 5.3. *Let I be an uncountable set, let $E = K^I$ and let $E_0 = K^{(I)}$. If $y \in \mathfrak{A}(E)$, then there exists $\lambda_0 > 0$ such that*

$$[y] \cap E_0 \not\subset \lambda_0[e] + [p] \quad \text{for any } p \in \mathfrak{A}(E_0).$$

Proof. For $n \in N_+$ let

$$S_n = \{i \in I|\ n^{-1} < \pi_i(y)\}.$$

Since I is uncountable and $y \in \mathfrak{A}(E)$ implies $I = \bigcup_{n=1}^{\infty} S_n$, there exists $m \in N$ for which S_m is infinite. Let $\lambda_0 = m^{-1}$.

If $p \in \mathfrak{A}(E_0)$, then $s(p)$ is finite which implies there exists $k \in S_m$ such that $k \notin s(p)$. Therefore, $\pi_k(y)e_k \in [y] \cap E_0$ and

$$\lambda_0|\pi_k(e)| + |\pi_k(p)| = m^{-1} < \pi_k(y)$$

which implies $\pi_k(y)e_k \notin \lambda_0[e] + [p]$. ■

Proposition 5.4 *Let I be an uncountable set, let $E = K^I$ and let $E_0 = K^{(I)}$. If $A \subset E$ satisfies Property 1 then (E_0, θ) is not a Schwartz space, where θ is the linear topology induced on E_0 by A.*

Proof. By Proposition 5.1, it is sufficient to show that for $x, y \in A$ there exists $\lambda_0 > 0$ such that

$$[y] \cap E_0 \not\subset \lambda_0[x] + [q] \quad \text{for any } q \in \mathfrak{A}(E_0)$$

(see [3], Proposition 4(ii), p. 276).

Let $x, y \in A$. Define $y_0 \in \mathfrak{A}(E)$ as follows: $\pi_i(y_0) = (\pi_i(x))^{-1}\pi_i(y)$ for each $i \in I$. By Lemma 5.3, there exists $\lambda_0 > 0$ such that $[y_0] \cap E_0 \not\subset \lambda_0[e] + [p]$ for any $p \in \mathfrak{A}(E_0)$.

Let $q \in \mathfrak{A}(E_0)$. Define $p \in E$ as follows: $\pi_i(p) = (\pi_i(x))^{-1}\pi_i(q)$ for each $i \in I$. Hence $p \in \mathfrak{A}(E_0)$ which implies there exists $z \in [y_0] \cap E_0$ and $k \in I$ such that

$$\lambda_0 + \pi_k(p) < |\pi_k(z)|;$$

therefore,

$$\lambda_0 \pi_k(x) + \pi_k(q) < |\pi_k(x)\pi_k(z)| \tag{5.5}$$

since $\pi_k(x)\pi_k(p) = \pi_k(q)$.

If we define $z_0 \in E$ as follows: $\pi_i(z_0) = \pi_i(x)\pi_i(z)$ for each $i \in I$, then $z_0 \in [y] \cap E_0$. Indeed, $s(z_0) = s(z)$ and

$$|\pi_i(z)| \leqslant \pi_i(y_0) = (\pi_i(x))^{-1}\pi_i(y) \quad \text{for } i \in s(z)$$

implies

$$|\pi_i(z_0)| = |\pi_i(x)\pi_i(z)| \leqslant \pi_i(y) \quad \text{for } i \in s(z_0).$$

However, $z_0 \notin \lambda_0[x] + [q]$ by (5.5). Consequently

$$[y] \cap E_0 \not\subset \lambda_0[x] + [q] \quad \text{for any } q \in \mathfrak{A}(E_0).\ ■$$

6. Examples. In this section, it is shown that Property 1 is not as restrictive as it appears. Indeed, if the index set I is uncountable and admits a first countable, compact Hausdorff topology, then there is a plethora of sets of positive functions satisfying Property 1. However, before giving a general description of such sets, it is necessary to have a few facts about first countable, compact Hausdorff topologies which we state without proof.

Proposition 6.1. *Let I be an infinite set and let τ be a first countable, compact Hausdorff topology on I. If $i \in I$ and if V is a τ-closed neighborhood of i for which $V \neq I$ then there exists a τ-continuous function x mapping I into the closed interval $[0, 1]$ such that $x(i) = 0$, $x(j) > 0$ for $j \in I - \{i\}$ and $x(j) = 1$ for $j \in I - V$.*

Proposition 6.2. *Let I be an infinite set. If τ is a first countable, compact Hausdorff topology on I then there exist $i \in I$, $\{i_n\}_{n=1}^{\infty} \subset I$ and a countable infinite collection $\{V_n\}_{n=1}^{\infty}$ of τ-closed sets such that $\{V_n\}_{n=1}^{\infty}$ is a basis for the filter of τ-neighborhoods of i, $V_{n+1} \subset V_n$, where $\mathring{V}_n$ is the τ-interior of V_n, and $i_n \in \mathring{V}_n - V_{n+1}$ for each $n \in N_+$.*

PROPOSITION 6.3. *Let I be an infinite set and let τ be a first countable, compact Hausdorff topology on I. Let $i \in I$ and let $\{V_n\}_{n=1}^{\infty}$ be a countable filter basis for the τ-neighborhoods of i such that V_n is τ-closed and $V_{n+1} \subset \mathring{V}_n$ for each $n \in N_+$. If $\{i_n\}_{n=1}^{\infty} \subset I$ such that $i_n \in \mathring{V}_n - V_{n+1}$ for each $n \in N_+$, then there exists a τ-continuous function $x\colon I \to [0, 1]$ for which $x(j) = 0$ for $j \in F = \{i_n\}_{n=1}^{\infty} \cup \{i\}$ and $0 < x(j)$ for $j \in I - F$.*

PROPOSITION 6.4. *Let I be an infinite set and let τ be a first countable, compact Hausdorff topology on I. Let $y\colon I \to [0, \infty)$. If $i \in I$ and $\{i_n\}_{n=1}^{\infty} \subset I$ for which $i \notin \{i_n\}_{n=1}^{\infty}$, $i_n \to i$ with respect to τ, $0 < y(i_n)$ for $n \in N_+$ and $y(i_n) \to 0$ then there exist an infinite subsequence $\{j_k\}_{k=1}^{\infty}$ of $\{i_n\}_{n=1}^{\infty}$ and a τ-continuous function $x\colon I \to [0, 1]$ such that $x(i) = 0$, $x(j) > 0$ for $j \in I - \{i\}$ and $x(j_k) = y(j_k)$ for each $k \in N_+$.*

Satisfying the first three conditions of Property 1 (Definition 4.4) is not difficult. However, fulfilling Condition 4 and especially condition 5 requires $\mathfrak{A}(K^I)$ to have an abundance of a certain type of functions which we now define (recall that $\mathfrak{A}(E)$ is the set of all positive valued functions on I, where $E = K^I$).

DEFINITION 6.5. Let I be an infinite set, let $E = K^I$ and let τ be a topology on I. For X, a non empty subset of I, we say that $x \in E$ is an *X-funnel, with respect to* τ, if and only if $x \in \mathfrak{A}(E)$ and the function $\Psi(x) \in E$, defined by: $\pi_i(\Psi(x)) = \pi_i(x)$ for $i \in I - X$ and $\pi_i(\Psi(x)) = 0$ for $i \in X$, is τ-continuous. If $X = \{i\}$, for $i \in I$, then an X-funnel is called an *i-funnel* (or a *funnel at i*).

The set of all functions in E that are *S-funnels, with respect to* τ, for some non empty finite set $S \subset I$ is denoted by $[\Delta(I); \tau]$.

Remark. The existence of an X-funnel depends on the nature of X and the topology τ on I.

For $x, y \in \mathfrak{A}(E)$, where $E = K^I$ for some infinite set I, we will let $x \wedge y$ denote the function defined by:

$$\pi_i(x \wedge y) = \min\{\pi_i(x), \pi_i(y)\} \quad \text{for each } i \in I.$$

Similarly, for $x \in \mathfrak{A}(E)$ let x^2 denote the function defined by:

$$\pi_i(x^2) = (\pi_i(x))^2 \quad \text{for each } i \in I.$$

THEOREM 6.6. *Let I be an infinite set, let $E = K^I$ and let τ be a first countable, compact Hausdorff topology on I. If $A \subset \mathfrak{A}(E)$ for which $x \wedge y$, x^2, $\frac{1}{2}x \in A$ whenever $x, y \in A$ and $[\Delta(I); \tau] \subset A$ then A satisfies Property 1.*

Proof. First, $A \neq \emptyset$ by Proposition 6.1 and Conditions 1 and 2 of Property 1 are satisfied by hypothesis. Also Condition 3 can be fulfilled by taking an i-funnel with a sufficiently small value at i (see Proposition 6.1).

Let $z \in E$ and define $z_0 \in E$ by: $\pi_i(z_0) = |\pi_i(z)|$ for each $i \in I$. Hence

$$s(z) = s(z_0)$$

(recall that for $y \in E$, $s(y) \subset I$ for which $i \in s(y)$ if and only if $\pi_i(y) \neq 0$). Assume $s(z_0)$ is infinite. By the hypothesis for τ, there exist $i \in I$ and an infinite sequence $\{i_n\}_{n=1}^{\infty} \subset s(z_0)$ of distinct points for which $i \notin \{i_n\}_{n=1}^{\infty}$ and $i_n \to i$, with respect to τ. It can be assumed that $\pi_{i_n}(z_0) \leqslant 1$ for each $n \in N_+$ and $\pi_{i_n}(z_0) \to 0$ as $n \to \infty$. From Proposition 6.4 we infer there exist a funnel x at i and an infinite subsequence $\{j_k\}_{k=1}^{\infty}$ of $\{i_n\}_{n=1}^{\infty}$ such that $\pi_j(x) \leqslant 1$ for $j \in I$ and $\pi_{j_k}(x) = \pi_{j_k}(z_0)$ for each $k \in N_+$.

Consider $[x^2]$ and $\lambda > 0$. There exists $m \in N_+$ for which $\pi_{j_k}(x) < \lambda^{-1}$ for $m \leqslant k$ since $\pi_{j_k}(z_0) \to 0$. Therefore

$$(\pi_{j_k}(x))^2 < \lambda^{-1}\pi_{j_k}(x) = \lambda^{-1}\pi_{j_k}(z_0) \quad \text{for } m \leqslant k$$

since $\pi_j(x) \leqslant 1$ for $j \in I$ implies $(\pi_j(x))^2 \leqslant \pi_j(x)$ for $j \in I$. So,

$$\lambda\pi_{j_k}(x^2) = \lambda(\pi_{j_k}(x))^2 < \pi_{j_k}(z_0) \quad \text{for } m \leqslant k$$

which implies $z_0 \notin \lambda[x^2]$. Consequently Condition 4 of Property 1 is satisfied.

Now, consider $^*I$. Since τ is compact we have that each point of $^*I$ is τ-near-standard ([7], Theorem 4.1.13), i.e., for $\iota \in {}^*I$ there exists $i \in I$ for which $\iota \in \mu_\tau(i)$, where $\mu_\tau(i)$ is the monad of the filter of τ-neighborhoods of i. Let $\iota \in {}^*I - I$ and let $i \in I$ such that $\iota \in \mu_\tau(i)$. By Proposition 6.1, there exists a τ-continuous function $x\colon I \to [0, 1]$ that takes the value 0 only at i. Hence

$$^*x[\mu_\tau(i)] \subset \mu(0) \cap {}^*[0, 1]$$

which implies

$$0 < {}^*x(\iota) = \pi_\iota({}^*x) \in \mu(0)$$

(see [7], Theorem 4.2.7).

We fulfill Condition 5 of Property 1 by defining $x_0 \in [\Delta(I); \tau]$ as follows: $\pi_j(x_0) = \pi_j(x)$ for $j \in I - \{i\}$ and $\pi_i(x_0) = 1$. ■

For the remainder of this section, we will assume that I is an uncountable infinite set and τ is a first countable, compact Hausdorff topology on I with no isolated points. Also, E and E_0 will retain their usual meaning, i.e., $E = K^I$ and $E_0 = K^{(I)}$.

EXAMPLE 1. Let $A = \mathfrak{A}(E)$, the set of all positive functions on I. Clearly $\mathfrak{A}(E)$ fulfills the hypothesis of Theorem 6.6, i.e., $\mathfrak{A}(E)$ satisfies Property 1. Also it is well known that since I is uncountable, the linear topology induced on E_0 by $\mathfrak{A}(E)$ is strictly weaker than the strongest possible locally convex linear topology on E_0. Since $\mathscr{E}(A)$ generates the "box" topology on E (see [4], problem V, p. 107), we shall denote its induced topology on E_0 by θ^b.

EXAMPLE 2. Let $A = [\Delta(I); \tau]$; therefore, A satisfies Property 1 by Theorem 6.6. We shall see, in subsequent examples, that the linear topology induced on E_0 by $[\Delta(I); \tau]$ is strictly weaker than θ^b of Example 1. We symbolize this topology by θ^f.

EXAMPLE 3. By Proposition 6.2, there exists a collection

$$[i, \{i_n\}_{n=1}^{\infty}, \{V_n\}_{n=1}^{\infty}]$$

such that $i \in I$, $\{i_n\}_{n=1}^{\infty} \subset I$ and $\{V_n\}_{n=1}^{\infty}$ is a basis for the filter of τ-neighborhoods of i for which V_n is τ-closed, $V_{n+1} \subset \mathring{V}_n$ and $i_n \in \mathring{V}_n - V_{n+1}$ for each $n \in N_+$, where $\mathring{V}_n$ is the τ-interior of V_n. Let

$$X = \{i_n\}_{n=1}^{\infty} \cup \{i\}$$

and define $A[X] \subset \mathfrak{A}(E)$ as follows: $x \in A[X]$ if and only if x is a Q-funnel with respect to τ, where $Q \subset X \cup S$ for some finite subset S of I. It is easily shown that $A[X]$ fulfills the hypothesis of Theorem 6.6; therefore, $A[X]$ satisfies Property 1. Proposition 6.3 implies the existence of a τ-continuous function $x: I \to [0, 1]$ that assumes the value of 0 only on the set X. Therefore the function x_0, defined by: $\pi_j(x_0) = \pi_j(x)$ for $j \in I - X$, $\pi_i(x_0) = 1$ and $\pi_{i_n}(x_0) = 1$ for $n \in N_+$, is an X-funnel which implies $x_0 \in A(X]$.

If y is an S-funnel for some finite subset S of I then there exists $j_0 \in X$ for which $j_0 \notin S$. Thus by choosing a sufficient τ-neighborhood of j_0, we can produce an index $j \in I$ for which $\pi_j(x_0) < \pi_j(y)$. We therefore conclude that the linear topology θ induced on E_0 by $A[X]$ is strictly stronger than θ^f of Example 2.

On the other hand, there exists another collection $[k, \{k_n\}_{n=1}^{\infty}, \{U_n\}_{n=1}^{\infty}]$ satisfying the conclusion of Proposition 6.2 for which $k \neq i$ (I is uncountable). Consequently if $F = \{k_n\}_{n+1}^{\infty} \cup \{k\}$ then each F-funnel is not an element of $A[X]$ and each X-funnel is not an element of $A[F]$ since $U_n \cap X$ and $V_n \cap F$ are finite subsets of I for some $n \in N_+$. Thus we infer, (1) if θ_1 is the linear topology induced on E_0 by $A[F]$ then θ and θ_1 are not compatible, (2) θ is strictly weaker than θ^b of Example 1.

Example 4. In this example, it is shown that there exists a sequence $\{A_m\}_{m=1}^{\infty}$ of sets satisfying Property 1 for which θ_{m+1} is strictly weaker than θ_m, where θ_m is the linear topology induced on E_0 by A_m for $m \in N_+$.

Let $[i, \{i_n\}_{n=1}^{\infty}, \{V_n\}_{n=1}^{\infty}]$ satisfy the conclusion of Proposition 6.2 (see Example 3). Define the collection

$$\{\{i_{(m,n)}\}_{n=1}^{\infty} |\ m \in N_+\}$$

of subsequences of $\{i_n\}_{n=1}^{\infty}$ inductively as follows: let $i_{(1,n)} = i_n$ for $n \in N_+$ and for $m \in N_+$, let $i_{(m+1,n)} = i_{(m,2n)}$ for $n \in N_+$. Observe that $\{i_{(m+1,n)}\}_{n=1}^{\infty}$ is a subsequence of $\{i_{(m,n)}\}_{n=1}^{\infty}$ for which

$$\{i_{(m,n)}\}_{n=1}^{\infty} - \{i_{(m+1,n)}\}_{n=1}^{\infty} \tag{6.7}$$

is an infinite subset of I since the elements of $\{i_n\}_{n=1}^{\infty}$ are distinct.

For $m \in N_+$ let

$$X_m = \{i_{(m,n)}\}_{n=1}^{\infty} \cup \{i\}$$

and consider $A[X_m] \subset \mathfrak{A}(E)$ defined as follows: $x \in A[X_m]$ if and only if x is a Q-funnel with respect to τ, where $Q \subset X_m \cup S$ for some finite set $S \subset I$. As shown in Example 3, each $A[X_m]$ satisfies Property 1 for $m \in N_+$. Also $X_{m+1} \subset X_m$ implies $A[X_{m+1}] \subset A[X_m]$ for each $m \in N_+$. However, the X_m-funnels, which exist by Proposition 6.3, are not elements of $A[X_{m+1}]$ for each $m \in N_+$ since $X_m - X_{m+1}$ is an infinite subset of I by (6.7) for $m \in N_+$.

Consequently, we have that θ_{m+1} is strictly weaker than θ_m for each $m \in N_+$, where θ_m is the linear topology induced on E_0 by $A[X_m]$. Also note that θ^f is strictly weaker than θ_m and θ_m is strictly weaker than θ^b for each $m \in N_+$, where θ^b and θ^f are the linear topologies of Examples 1 and 2 respectively.

Remark. If τ is a metrizable, compact Hausdorff topology on I then it is possible to use the metric ϱ on I to generate a non negative, τ-continuous function on I that vanishes only on F when F is a proper τ-closed subset of I; e.g., let $x(i) = \varrho(i, F)$, the distance from i to F, for each $i \in I$. Therefore, it is possible to obtain funnels on some pretty bizarre sets (for example, let $I = [0, 1]$ and consider the Cantor set).

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Accepté par la Redaction le 14. 2. 1977