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Some remarks on Eberlein compacts

by

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Abstract. We give a positive answer to a problem of Y. Benyamini, M. E. Rudin, P. Simon and M. Wage by showing that a compact subspace of a Σ-product of intervals is a strong Eberlein compact if and only if it is scattered.

A compact space is called an *Eberlein compact* (E-C), if it is homeomorphic to a weakly compact subset of a Banach space.

The main structure theorem for E-C is due to Amir and J. Lindenstrauss [1]:

A compact space is an E-C *if and only if it is homeomorphic to a compact subset X of the product of intervals, where X has the property, that for each $\varepsilon>0$ and $f\in X$ the set $\{\gamma\in\Gamma\colon f(\gamma)\geqslant\varepsilon\}$ is finite.*

H. P. Rosenthal observed, that X *is* E-C *if and only if it has a σ-point-finite separating family of open F_σ-subsets.*

An E-C X is called *strong* (see [2], [6] and [4]) if it embeds in the Cartesian product of intervals I^Γ in such a way that $x(\gamma)=0$ or $x(\gamma)=1$ for all $x\in X$ and $\gamma\in\Gamma$ and $|\{\gamma\colon x(\gamma)\neq 0\}|<\aleph_0$. Equivalently X is a *strong* E-C if and only if it has a point-finite separating family of clopen sets.

Let us recall that a family $\mathcal{U}$ of subsets of X is called *separating*, if given any $x\neq y$ in X, then there is an $U\in\mathcal{U}$ such that either $x\in U$ and $y\notin U$ or $y\in U$ and $x\notin U$. The family $\mathcal{U}$ is called *point-finite* (*point-countable*), if each x belongs to at most *finitely* (*countably*) many sets in $\mathcal{U}$. It is called *σ-point-finite* if $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$, where each $\mathcal{U}_n$ is point-finite.

A space X is scattered if every closed subset of X has an isolated point.

Let us put $X=X^{(0)}$, $X^{(\alpha+1)}$ is the set of accumulation points of the set X^α and $X^{(\lambda)}=\bigcap_{\beta<\lambda}X^{(\beta)}$ for a limit ordinal λ.

All undefined terms are from [3].

The aim of this note is to prove the following:

THEOREM. *If X is a scattered, compact space, which has a point-countable separating family of open F_σ-sets, it is the same as to say that X is a compact subset of a Σ-product of intervals, then it is a strong* E-C.

This theorem gives a positive answer to the problem formulated by Y. Benyamini, M. E. Rudin, M. Wage ([2], Problem 6) and P. Simon ([6]) (see also [4], p. 109).

The proof of the theorem will be derived from the following:

PROPOSITION. *If $\mathcal{U}$ is a point-countable family of scattered open and compact subsets of a space X then there exists a point-finite family $\mathcal{V}$ of open and compact subsets such, that $\mathcal{V}$ refines $\mathcal{U}$ and $\bigcup\mathcal{U} = \bigcup\mathcal{V}$.*

Proof. We shall prove the proposition by the transfinite induction with respect to the cardinality of $\mathcal{U}$.

If $|\mathcal{U}| = \aleph_0$, ($|\mathcal{U}|$ denotes the cardinality of $\mathcal{U}$), then $\mathcal{U} = \{U_n\colon n = 1, 2, \ldots\}$. It is enough to put

$$V_1 = U_1 \quad \text{and} \quad V_n = U_n \setminus \bigcup_{m=1}^{n-1} U_m . \tag{1}$$

Let us assume that the proposition holds, if $|\mathcal{U}| < \aleph_\alpha$ and suppose, that

$$\mathcal{U} = \{U_\beta\colon \beta < \omega_\alpha\},$$

where ω_α denotes the initial ordinal number of cardinality $\aleph_\alpha$.

If F is a compact scattered set then put

$$Z(F) = \begin{cases} F^{(\beta)}, \text{ where } 1 \leqslant |F^{(\beta)}| < \aleph_0 \text{ if } F \neq \emptyset, \\ \emptyset, \text{ if } F = \emptyset . \end{cases} \tag{2}$$

If $\mathcal{W}$ is a family of compact and scattered sets, then put

$$Z(\mathcal{W}) = \bigcup \{Z(W_1 \cap W_2 \cap \ldots \cap W_n)\colon W_i \in \mathcal{W} \text{ for } i = 1, 2, \ldots, n\}. \tag{3}$$

We define an increasing sequence $\{\mathcal{U}_\beta\colon \beta < \omega_\alpha\}$ of subfamilies of $\mathcal{U}$ by the following formulas:

$$\mathcal{U}_\eta = \{U_\beta\colon \beta \leqslant \eta\} \cup \{U \in \mathcal{U}\colon U \cap Z(\mathcal{U}_\gamma) \neq \emptyset\}, \quad \text{where} \quad \eta = \gamma + 1 \tag{4}$$

and

$$\mathcal{U}_\eta = \bigcup \{\mathcal{U}_\beta\colon \beta < \eta\} \text{ if } \eta \text{ is a limit ordinal number.} \tag{5}$$

It is easy to see that

$$\bigcup \{\mathcal{U}_\beta\colon \beta < \omega_\alpha\} = \mathcal{U} . \tag{6}$$

Notice, that $|Z(\mathcal{U}_\eta)| \leqslant |\mathcal{U}_\eta| \leqslant \bar{\eta}\cdot \aleph_0$, where $\bar{\eta}$ denotes the cardinality of η, so

$$|U \in \mathcal{U}\colon U \cap Z(\mathcal{U}_\eta) \neq \emptyset| \leqslant \bar{\eta}\cdot\aleph_0 ,$$

because $\mathcal{U}$ is point-countable. From the last fact it follows that

$$|\mathcal{U}_\beta| \leqslant \bar{\beta}\cdot\aleph_0 \quad \text{for} \quad \beta < \omega_\alpha . \tag{7}$$

Now let us put

$$\mathcal{W}_\beta = \mathcal{U}_{\beta+1} \setminus \mathcal{U}_\beta \quad \text{for} \quad \beta < \omega_\alpha . \tag{8}$$

By (6), (5) and (8) we infer that

$$\bigcup \{\mathcal{W}_\beta\colon \beta < \omega_\alpha\} = \mathcal{U} \tag{9}$$

and by (7) and (8) that

$$|\mathcal{W}_\beta| < \aleph_\alpha \quad \text{for} \quad \beta < \omega_\alpha . \tag{10}$$

Applying the inductive assumption to $\mathcal{W}_\beta$, we can find a point-finite family $\mathcal{V}_\beta$ of open and compact sets, which refines $\mathcal{W}_\beta$ such that

$$\bigcup \mathcal{V}_\beta = \bigcup \mathcal{W}_\beta . \tag{11}$$

Put

$$\mathcal{V} = \bigcup \{\mathcal{V}_\beta\colon \beta < \omega_\alpha\} . \tag{12}$$

It is easy to see that $\mathcal{V}$ refines $\mathcal{U}$. By (9) and (11) we infer that $\bigcup \mathcal{U} = \bigcup \mathcal{V}$. In order to prove that $\mathcal{V}$ is point-finite, it is enough to show that, if

$$W_n \in \mathcal{W}_{\beta_n}, \quad \text{where} \quad \beta_1 < \beta_2 < \ldots < \beta_n < \ldots \tag{13}$$

then $\bigcap_{n=1}^{\infty} W_n = \emptyset$.

Suppose to contrary that there is $x \in \bigcap_{n=1}^{\infty} W_n$. Put

$$C_n = \bigcap_{i=1}^{n} W_i . \tag{14}$$

If α_n is such that $Z(C_n) = C_n^{(\alpha_n)}$, then there is m such that for every $n \geqslant m$ $\alpha_n = \alpha_{n+1}$. From the definition of m it follows that

$$W_{m+2} \cap Z(W_1 \cap \ldots \cap W_m) \supset Z(W_1 \cap \ldots \cap W_{m+2}) \neq \emptyset ,$$

so $W_{m+2} \in \mathcal{U}_{\beta_m+2}$, contradicting $W_{m+2} \in \mathcal{W}_{\beta_{m+2}}$, where $\beta_{m+2} > \beta_{m+1} \geqslant \beta_m + 1$.

Proof of the theorem. We shall prove the theorem by the transfinite induction with respect to the ordinal number α defined by

$$Z(X) = X^{(\alpha)}. \tag{15}$$

If $\alpha = 0$, then X is finite, so Theorem holds. Let us suppose, that the theorem holds for every $\beta < \alpha$ and that $Z(X) = X^{(\alpha)}$.

It is easy to see, that we can assume without loss of generality, that $|Z(X)| = 1$. Put $Z(X) = \{a\}$.

Let $\mathcal{H}$ be a point-countable separating family of open and F_σ-sets in X.

Since X is zero-dimensional (see [5], Th. 8.5.4), so we can assume, replacing eventually $\mathcal{H}$ by another family, that $\mathcal{H}$ is a family of clopen sets.

If $\{H \in \mathcal{H}\colon a \in H\} = \{H_n\colon n = 1, 2, \ldots\}$ then $\mathcal{U} = \{H \in \mathcal{H}\colon a \notin H\} \cup \cup \{X \setminus H_n\colon n = 1, 2, \ldots\}$ is an open, point-countable cover of $X \setminus \{a\}$ consisting of compact sets.

By the proposition there is a point-finite, open cover $\mathscr{V}$ of $X\setminus\{a\}$ consisting of compact sets.

By the inductive assumption, every $V\in\mathscr{V}$ has a point-finite family $\mathscr{U}(V)$ of clopen sets, separating points in V.

Put $\mathscr{Z}=\mathscr{V}\cup\bigcup\{\mathscr{U}(V)\colon V\in\mathscr{V}\}$. It is easy to see that $\mathscr{Z}$ is a point-finite family of clopen sets separating points, so the proof of the theorem is finished.

Remark 1. In [2], Example 5–1, it is shown that under some set theoretical assumption, there is a compact space, which is not an E-C, but has a point-countable separating family of open F_σ-sets.

Remark 2. In [2], the proof of Theorem 4.3 and in [6], Proposition 5, was observed, that every strong E-C is scattered, so we obtain the following corollaries:

COROLLARY 1. *A compact subspace of a Σ-product of intervals is a strong* E-C *if and only if it is scattered.*

COROLLARY 2. *An* E-C *is a strong* E-C *if and only if it is scattered.*

COROLLARY 3. *If X is a compact space admiting a point-countable, separating family of open F_σ-sets, then X is a strong* E-C *if and only if it is scattered.*

Remark 3. Let us notice that we can formulate the proposition in the following way:

If X is a scattered compact space and $\mathscr{F}$ is a family of closed subsets of X such that every point belongs to less than $\aleph_\alpha$-many sets in $\mathscr{F}$, where $\aleph_\alpha$ is a regular number, then there are $\{\mathscr{F}_s\colon s\in S\}$ such that $\bigcup\{\mathscr{F}_s\colon s\in S\}=\mathscr{F}$, $|\mathscr{F}_s|<\aleph_\alpha$ for every $s\in S$ and the family $\{\bigcup\mathscr{F}_s\colon s\in S\}$ is point finite.

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Added in proof. Recently M. Talagrand has shown, without additional set theoretical assumptions, that there is a compact subspace of a Σ-product of the real line which is not Eberlein compact.

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Finite points of filters in infinite dimensional vector spaces

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Abstract. Let E be an infinite dimensional vector space over K, the scalar field. For any subset A of $^*E$, the nonstandard extension of E, define $\mathrm{Fin}(A)$ as follows: $z\in\mathrm{Fin}(A)$ if and only if $\lambda z\in A$ for each infinitesimal λ of $^*K$, the nonstandard extension of K. A subset A of $^*E$ is called Fin-invariant if $\mathrm{Fin}(A)=A$. If A is the monad of a filter $\mathfrak{F}$ in E then $\mathrm{Fin}(A)$ is called the set of finite points of $\mathfrak{F}$.

In this paper, we establish the existence of nontrivial, Fin-invariant sets. Next, sufficient conditions are given, in terms of finite points, for a filter to induce a linear topology on some nontrivial vector subspace of E. Finally, we use Fin-invariant sets of infinitesimals and finite points of filters to produce a class of locally convex, topological vector spaces that are not Schwartz spaces, not metrizable and have invariant nonstandard hulls. In particular, it is shown that the topological vector space induced by the box topology (J. L. Kelley, *General Topology*, p. 107) has invariant nonstandard hulls.

Introduction. In this paper, we study a class of sets first defined in [1], i.e., the set $\mathrm{Fin}(A)$ for any subset A of $^*E$, the nonstandard extension of a vector space E (Definition 1.1). The two predominant themes presented here are: when is a subset A of $^*E$ Fin-invariant (i.e., $\mathrm{Fin}(A)=A$) and when is A not Fin-invariant. The latter is shown to be of importance in generalizing the concept of finite points in the nonstandard theory of topological vector spaces. Indeed, in Sections 2 and 3, it is shown that the properties of a filter $\mathfrak{F}$ in a vector space E is determined by its finite points (elements of $\mathrm{Fin}(\mu(\mathfrak{F}))$) as well as its monad $\mu(\mathfrak{F})$ whenever $\mathrm{Fin}(\mu(\mathfrak{F}))\neq\emptyset$. In particular, sufficient conditions are given, in terms of finite points, for a filter $\mathfrak{F}$ to induce a linear topology on some non-trivial vector subspace of E (Proposition 3.7).

In Section 1, we consider the basic properties of $\mathrm{Fin}(A)$ for an arbitrary subset A of $^*E$. Also the first example of a non trivial, Fin-invariant set is given (Propositions 1.6 and 1.7). It appears that non trivial, Fin-invariant sets are quite abundant; in fact, Fin-invariant subsets of infinitesimals can be generated easily from entities in the "standard" world (Proposition 4.9).

The main results of this paper are found in Sections 4, 5 and 6. In Section 4, we apply the concepts of Fin-invariant sets and finite points of filters to produce a class of function spaces that have, as locally convex spaces, invariant nonstandard hulls (Theorem 4.17). In Section 5, it is shown that these spaces are not metrizable