which has the homotopy type of a 2 dimensional complex with fundamental group isomorphic to G and which does not satisfy FIR. If the group G does not satisfy FIR, then it appears to be much more difficult to construct such a space.

Added in proof. Martha Smith has shown the author a proof by using M. S. Montgomery's *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. 75 (1969), pp. 539–540 that all groups have property C.

References

[1] H. Bass, *Projective modules over free groups are free, notes*, Columbia University, 1964.
[2] K. Borsuk, *Fundamental retracts and extensions of fundamental sequences*, Fund. Math. 64 (1969), pp. 55–85.
[3] — *Theory of Shape*, Polish Scientific Publishers, Warszawa 1975.
[4] D. Edwards and R. Geoghegan, *The stability problem in shape theory and a Whitehead theorem in pro-homotopy*, Trans. Amer. Math. Soc. 214 (1975), pp. 261–278.
[5] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. (3) 4 (1954), pp. 419–436.
[6] J. F. P. Hudson, *Piecewise Linear Topology*, W. A. Benjamin, Inc., New York 1969.
[7] S. Mardešić, *On the Whitehead theorem in shape theory I*, Fund. Math. 91 (1976), pp. 51–64.
[8] M. Moszyńska, *On the homotopy classification of spaces*, Fund. Math. 66 (1969), pp. 65–83.
[9] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York 1966.
[10] J. Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Amer. J. Math. 85 (1963), pp. 541–543.
[11] C. T. C. Wall, *Finiteness conditions for CW-complexes*, Ann. Math. 81 (1965), pp. 56–69.
[12] J. West, *Compact ANR's have finite type*, Bull. Amer. Math. Soc. 81 (1975), pp. 163–165.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
Knoxville, Tennessee
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
Zagreb

Accepté par la Rédaction le 31. 1. 1977

Superextensions and Lefschetz fixed point structures

by

M. van de Vel (Amsterdam)

Abstract. The superextension of a sufficiently connected normal T_1-space admits both a convexoid structure and a semi-complex structure, and it consequently satisfies the Lefschetz fixed point property.

0. Introduction. The concept of supercompactness — a strong compactness property — arose naturally from investigations of de Groot and Aarts on internal characterizations of complete regularity (cf. [1]).

Superextensions are supercompact extensions of topological spaces, constructed in much the same way as Wallman compactifications, but replacing "filters" by "linked systems". This results in a functorial procedure transformating topological spaces into surprisingly nice compact spaces (cf. e. g. A. Verbeek [21], J. van Mill [19]). We first recall some definitions (Verbeek [21], Van Mill [18]):

Let $\mathcal{S}$ be a collection of subsets of a set X. A *linked system in* $\mathcal{S}$ is a collection $\mathcal{M}\subset\mathcal{S}$ such that any two members of $\mathcal{M}$ meet. A *maximal linked system in* $\mathcal{S}$ — briefly: an *mls* — is a linked system not properly contained in another linked system in $\mathcal{S}$.

Let X now be a topological space and let $\mathcal{S}$ be a closed subbase of X. The *superextension of X relative to* $\mathcal{S}$, $\lambda_{\mathcal{S}}(X)$, is the topological space defined on the set of all mls's in $\mathcal{S}$ by the closed subbase

$$\mathcal{S}^+ = \{C^+ |\ C\in\mathcal{S}\},$$

where C^+ denotes the set of all mls's in $\mathcal{S}$ containing C as a member. If $\mathcal{S}$ equals the set of *all* closed subsets of X, then $\lambda_{\mathcal{S}}(X) = \lambda(X)$ is called the *superextension of* X. Each superextension of X is obviously a T_1-space.

A topological space X is *supercompact* provided that there is a closed subbase $\mathcal{S}$ of X such that each linked system $\mathcal{M}$ in $\mathcal{S}$ satisfies

$$\bigcap\{M|\ M\in\mathcal{M}\}\neq\emptyset .$$

In addition, this subbase $\mathcal{S}$ is said to be *binary*. If $\mathcal{S}$ is an arbitrary closed subbase of X, then the corresponding subbase $\mathcal{S}^+$ of $\lambda_{\mathcal{S}}(X)$ is binary, and $\lambda_{\mathcal{S}}(X)$ is supercompact. Observe that a supercompact space is compact by Alexander's lemma.

A subbase $\mathcal{S}$ of X is a T_1*-subbase* if for each $S \in \mathcal{S}$ and for each $x \in X-S$ there exists a $T \in \mathcal{S}$ such that $x \in T$ and $T \cap S = \emptyset$. If $\mathcal{S}$ is a T_1-subbase of X, then the collection $\{S \in \mathcal{S} \mid x \in S\}$ is obviously an mls for each $x \in X$.

A subbase $\mathcal{S}$ of X is called *normal* if for each disjoint pair $S_1, S_2 \in \mathcal{S}$ there exist $S_1', S_2' \in \mathcal{S}$ such that $S_1 \subset X-S_1'$, $S_2 \subset X-S_2'$, and $S_1' \cup S_2' = X$. If $\mathcal{S}$ is a normal T_1-subbase, then $\lambda_{\mathcal{S}}(X)$ is Hausdorff (Van Mill [18]) and $\mathcal{S}^+$ is obviously normal and T_1 again.

We shall need the following two basical results on superextensions.

0.1. THEOREM. *If $\mathcal{S}$ is a T_1-subbase of a T_1-space X, then the mapping $i: X \to \lambda_{\mathcal{S}}(X)$ defined on $x \in X$ by*

$$i(x) = \{S \in \mathcal{S} \mid x \in S\},$$

is an embedding. If $\mathcal{S}$ is binary moreover, then this map is a homeomorphism of X with $\lambda_{\mathcal{S}}(X)$.

(Cf. Verbeek [21] p. 44–46.)

0.2. THEOREM. *Let X and Y be T_1-spaces, let $\mathcal{S}$ be a T_1-subbase of X, and let $\mathcal{T}$ be a normal T_1-subbase of Y. If $f: X \to Y$ is a mapping such that $f^{-1}\mathcal{T} \subset \mathcal{S}$, then f can be extended to a continuous mapping $\lambda(f): \lambda_{\mathcal{S}}(X) \to \lambda_{\mathcal{T}}(Y)$, which is surjective if f is.*

(Cf. Verbeek [21] p. 56.) It follows from 0.1 and 0.2 that a T_1-space with a normal binary T_1-subbase is a retract of its superextension.

Our main result will be the following theorem.

0.3. THEOREM. *Let X be a normal T_1-space with a finite number of components. Then the following assertions are true.*

(i) *$\lambda(X)$ has finitely many components, each of which is acyclic with respect to Čech homology over a field;*

(ii) *$\lambda(X)$ is a Lefschetz space and even a metric* ANR *if X is compact metric;*

(iii) *if X is connected, then $\lambda(X)$ has the fixed point property for continuous mappings.*

A proof is given in Section 3, together with a proof of some related results. Preparatory work is done in Section 1 (order-theoretic structures on superextensions) and in Section 2 (convex sets in superextensions).

The author is indebted to J. van Mill for suggesting some of the problems on which this paper is based.

1. Partial orderings on superextensions.

1.1. DEFINITIONS. If X is a topological space, and if $\mathcal{M}, \mathcal{N} \in \lambda(X)$, then the set

$$I(\mathcal{M}, \mathcal{N}) = \{\mathcal{P} \in \lambda(X) \mid \mathcal{M} \cap \mathcal{N} \subset \mathcal{P}\}$$

is called the *interval between $\mathcal{M}$ and $\mathcal{N}$* (cf. Brouwer and Schrijver [4]).

For each triple $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \lambda(X)$ there exists another mls

$$g(\mathcal{M}, \mathcal{N}, \mathcal{P}) = (\mathcal{M} \cap \mathcal{N}) \cup (\mathcal{N} \cap \mathcal{P}) \cup (\mathcal{P} \cap \mathcal{M})$$

(cf. Verbeek [21]). Observe that $g(\mathcal{M}, \mathcal{N}, \mathcal{P}) \in I(\mathcal{M}, \mathcal{N})$.

We shall also need the concept of a *hyperspace* $H(X)$ of a topological space X. Recall that the points of $H(X)$ are the nonempty closed subsets of X, and that the topology of $H(X)$ is generated by the (open) base consisting of all sets of type

$$\langle 0_1, \ldots, 0_n \rangle = \{A \in H(X) \mid A \subset \bigcup_{i=1}^{n} O_i \text{ and } A \cap O_i \neq \emptyset \text{ for all } i = 1, \ldots, n\},$$

where $O_1, \ldots, O_n$ are open subsets of X (Michael [11]).

1.2. THEOREM. *Let X be a topological space. Then*

(i) *the mapping $g: \lambda(X)^3 \to \lambda(X)$ is continuous;*

(ii) *for each pair $\mathcal{M}, \mathcal{N} \in \lambda(X)$ the restricted mapping*

$$g(\mathcal{M}, \mathcal{N}, -): \lambda(X) \to \lambda(X)$$

is a retraction of $\lambda(X)$ onto $I(\mathcal{M}, \mathcal{N})$;

(iii) *the interval mapping $I: \lambda(X)^2 \to H(\lambda(X))$ is continuous.*

Proof. For each closed set $C \subset X$,

$$g^{-1}(C^+) = (C^+ \times C^+ \times \lambda(X)) \cup (\lambda(X) \times C^+ \times C^+) \cup (C^+ \times \lambda(X) \times C^+),$$

proving (i). If $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$, then $\mathcal{M} \cap \mathcal{N} \subset \mathcal{P}$ and consequently,

$$(\mathcal{M} \cap \mathcal{N}) \cup (\mathcal{N} \cap \mathcal{P}) \cup (\mathcal{P} \cap \mathcal{M}) \subset \mathcal{P}.$$

By maximality of the former, $g(\mathcal{M}, \mathcal{N}, \mathcal{P}) = \mathcal{P}$. In order to see (iii), observe first that for each pair $\mathcal{M}, \mathcal{N} \in \lambda(X)$,

$$\emptyset \neq I(\mathcal{M}, \mathcal{N}) = \bigcap \{T^+ \mid T \in \mathcal{M} \cap \mathcal{N}\},$$

whence $I(\mathcal{M}, \mathcal{N}) \in H(\lambda(X))$. Next observe that $I(\mathcal{M}, \mathcal{N})$ is the image of the map $g(\mathcal{M}, \mathcal{N}, -)$ by (ii). Statement (iii) is then a consequence of the following general result, the terminology of which is adapted from Smithson [12]:

1.3. LEMMA. *Let X, Y, Z be topological spaces of which Y is compact, and let $f: X \times Y \to Z$ be continuous. Then the assignment*

$$F(x) = \text{image of } f(x, -),\ x \in X,$$

yields a continuous multifunction $F: X \to Z$.

Notice that, if the sets $F(x)$ are closed for each $x \in X$, then F becomes a continuous mapping $X \to H(Z)$.

Proof. F is lower semi-continuous: let $O \subset Z$ be an open set meeting the image of $f(x, -): Y \to Z$. Then $f(x, y) \in O$ for some $y \in Y$. By the continuity of f, there exist open sets $O_1 \subset X$, $O_2 \subset Y$ such that

$$x \in O_1, \quad y \in O_2, \quad f(O_1 \times O_2) \subset O.$$

Hence $F(x') \cap O \neq \emptyset$ for $x' \in O_1$.

F is upper semi-continuous. In fact, let $O \subset Z$ be open and let $F(x) \subset O$. For each $y \in Y$ there exist open sets $P_y \subset X$, $Q_y \subset Y$ such that

$$x \in P_y, \quad y \in Q_y, \quad f(P_y \times Q_y) \subset O.$$

By the compactness of Y, there exists a finite covering of Y of type

$$\{Q_{y_1}, \ldots, Q_{y_n}\}, \quad \text{where } y_1, \ldots, y_n \in Y.$$

Let $P = \bigcap_{i=1}^{n} P_{y_i}$. Then $f(P \times Y) \subset O$, whence $F(x') \subset O$ for each $x' \in P$. ∎

1.4. Construction. Let X be a space and let $\mathcal{M} \in \lambda(X)$. A binary relation $\leqslant_{\mathcal{M}}$ (or, briefly: $\leqslant$) is defined of $\lambda(X)$ as follows:

$$\mathcal{P} \leqslant \mathcal{Q} \quad \text{iff} \quad I(\mathcal{M}, \mathcal{P}) \subset I(\mathcal{M}, \mathcal{Q}), \quad \mathcal{P}, \mathcal{Q} \in \lambda(X).$$

This is obviously a quasi ordering. It will turn out to be a partial ordering on $\lambda(X)$, which has first been introduced by J. van Mill (unpublished).

1.5. Theorem. *Let X be a normal T_1-space and let $\mathcal{M} \in \lambda(X)$. Then*

(i) *the associated quasi-order $\leqslant_{\mathcal{M}}$ is a topological partial ordering;*

(ii) *if $\mathcal{C}$ is a nonempty linked collection of closed subsets of X and if*

$$A = \bigcap \{C^+ \mid C \in \mathcal{C}\}$$

is connected, then the ordering $\leqslant_{\mathcal{M}}$ is dense on A;

(iii) *for each pair $\mathcal{N}_1, \mathcal{N}_2 \in \lambda(X)$, the largest mls in $\lambda(X)$ satisfying*

$$\mathcal{P} \leqslant_{\mathcal{M}} \mathcal{N}_1 \quad \text{and} \quad \mathcal{P} \leqslant_{\mathcal{M}} \mathcal{N}_2$$

is $\mathcal{P} = g(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2)$.

Some auxiliary results will be used repeatedly. We therefore number them for later reference.

(1.5; 1) *If X is an arbitrary space and if $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \lambda(X)$ then $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$ iff $I(\mathcal{M}, \mathcal{P}) \subset I(\mathcal{M}, \mathcal{N})$.*

The "if"-part is obvious. Assume that $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$. Then $\mathcal{M} \cap \mathcal{N} \subset \mathcal{P}$ and, for each $\mathcal{Q} \in I(\mathcal{M}, \mathcal{P})$, $\mathcal{M} \cap \mathcal{P} \subset \mathcal{Q}$. Hence, $\mathcal{M} \cap \mathcal{N} \subset \mathcal{M} \cap \mathcal{P} \subset \mathcal{Q}$ proving that $I(\mathcal{M}, \mathcal{P}) \subset I(\mathcal{M}, \mathcal{N})$.

(1.5; 2) *If X is an arbitrary space and if $\mathcal{M}, \mathcal{P}, \mathcal{Q} \in \lambda(X)$, then $\mathcal{P} = \mathcal{Q}$ whenever $I(\mathcal{M}, \mathcal{P}) = I(\mathcal{M}, \mathcal{Q})$.*

If $I(\mathcal{M}, \mathcal{P}) = I(\mathcal{M}, \mathcal{Q})$, then $\mathcal{P} \in I(\mathcal{M}, \mathcal{Q})$ and $\mathcal{Q} \in I(\mathcal{M}, \mathcal{P})$, yielding $\mathcal{M} \cap \mathcal{Q} \subset \mathcal{P}$, $\mathcal{M} \cap \mathcal{P} \subset \mathcal{Q}$. Hence $\mathcal{M} \cap \mathcal{P} = \mathcal{M} \cap \mathcal{Q}$ and it easily follows that $g(\mathcal{M}, \mathcal{P}, \mathcal{Q}) \subset \mathcal{P} \cap \mathcal{Q}$. By maximality of the former linked system, $\mathcal{P} = \mathcal{Q}$.

Using (1.5; 2) the quasi-ordering $\leqslant_{\mathcal{M}}$ is easily seen to be a partial ordering. By (1.5; 1); the set of predecessors of $\mathcal{N} \in \lambda(X)$ equals $I(\mathcal{M}, \mathcal{N})$. Hence the graph of the (inverse) relation $\geqslant_{\mathcal{M}}$ equals the graph (in $\lambda(X)^2$) of the continuous multifunction $I(\mathcal{M}, -)$. The space X being normal T_1, $\lambda(X)$ is a regular space, and hence the graph of $I(\mathcal{M}, -)$ is closed (Smithson [12] p. 35). It easily follows that the graph of the partial ordering $\leqslant_{\mathcal{M}}$ is closed in $\lambda(X) \times \lambda(X)$, and hence that $\leqslant_{\mathcal{M}}$ is a topological ordering on $\lambda(X)$ (Ward [22] p. 92). This proves the first statement of Theorem 1.5.

For a proof of (ii) we need the following facts:

(1.5; 3) *If X is an arbitrary space and if $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$, then*

$$I(\mathcal{M}, \mathcal{P}) \cap I(\mathcal{P}, \mathcal{N}) = \{\mathcal{P}\}.$$

In fact, let $\mathcal{R} \in I(\mathcal{M}, \mathcal{P}) \cap I(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$. Then

$$\mathcal{M} \cap \mathcal{N} \subset \mathcal{P}, \quad \mathcal{M} \cap \mathcal{P} \subset \mathcal{R}, \quad \mathcal{P} \cap \mathcal{N} \subset \mathcal{R}.$$

In particular, $\mathcal{M} \cap \mathcal{N} \subset \mathcal{M} \cap \mathcal{P} \subset \mathcal{R}$, and using the above inclusions,

$$g(\mathcal{M}, \mathcal{N}, \mathcal{P}) = \mathcal{R}.$$

By the retraction property of $g(\mathcal{M}, \mathcal{N}, -)$, $g(\mathcal{M}, \mathcal{N}, \mathcal{P}) = \mathcal{P}$ and hence $\mathcal{R} = \mathcal{P}$.

(1.5; 4) *Let X be an arbitrary space. If $\mathcal{P}_1, \mathcal{P}_2 \in I(\mathcal{M}, \mathcal{N})$, then*

$$I(\mathcal{M}, \mathcal{P}_1) \cap I(\mathcal{P}_2, \mathcal{N}) \neq \emptyset \quad \textit{iff} \quad \mathcal{P}_2 \in I(\mathcal{M}, \mathcal{P}_1) \textit{ and } \mathcal{P}_1 \in I(\mathcal{P}_2, \mathcal{N}).$$

The "if"-part is obvious. Assume that there is an mls

$$\mathcal{R} \in I(\mathcal{M}, \mathcal{P}_1) \cap I(\mathcal{P}_2, \mathcal{N}).$$

Then $\mathcal{M} \cap \mathcal{P}_1 \subset \mathcal{R}$ and $\mathcal{P}_2 \cap \mathcal{N} \subset \mathcal{R}$, and consequently the collection

$$(\mathcal{M} \cap \mathcal{P}_1) \cup (\mathcal{P}_2 \cap \mathcal{N})$$

is linked. We then obtain two extended linked systems

$$\mathcal{S}' = (\mathcal{M} \cap \mathcal{P}_1) \cup (\mathcal{P}_2 \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{P}_2),$$
$$\mathcal{T}' = (\mathcal{M} \cap \mathcal{P}_1) \cup (\mathcal{P}_2 \cap \mathcal{N}) \cup (\mathcal{P}_1 \cap \mathcal{N}).$$

Let $\mathcal{S} \supset \mathcal{S}'$ and $\mathcal{T} \supset \mathcal{T}'$ be mls's. Then

$$\mathcal{S} \in I(\mathcal{M}, \mathcal{P}_2) \cap I(\mathcal{P}_2, \mathcal{N}), \quad \mathcal{P}_2 \in I(\mathcal{M}, \mathcal{N}),$$
$$\mathcal{T} \in I(\mathcal{M}, \mathcal{P}_1) \cap I(\mathcal{P}_1, \mathcal{N}), \quad \mathcal{P}_1 \in I(\mathcal{M}, \mathcal{N}),$$

and applying (1.5; 3), $\mathcal{S} = \mathcal{P}_2$ and $\mathcal{T} = \mathcal{P}_1$. In particular,

$$\mathcal{M} \cap \mathcal{P}_1 \subset \mathcal{P}_2, \quad \mathcal{P}_2 \cap \mathcal{N} \subset \mathcal{P}_1,$$

showing that $\mathcal{P}_2 \in I(\mathcal{M}, \mathcal{P}_1)$ and $\mathcal{P}_1 \in I(\mathcal{P}_2, \mathcal{N})$.

(ii) can now be seen as follows. Let $\mathscr{C}\subset H(X)$ be a nonempty linked collection such that the space $A=\bigcap\{C^+|\ C\in\mathscr{C}\}$ is connected. Let $\mathscr{P},\mathscr{Q}\in A$. Then $\mathscr{C}\subset\mathscr{P}\cap\mathscr{Q}$ and consequently $I(\mathscr{P},\mathscr{Q})\subset A$. By Theorem 1.2 (ii), $I(\mathscr{P},\mathscr{Q})$ is a retract of $\lambda(X)$ and hence of A. It follows that $I(\mathscr{P},\mathscr{Q})$ is a connected T_1-space. Assume now that $\mathscr{P}\leqslant_{\mathscr{M}}\mathscr{Q}$ and $\mathscr{P}\neq\mathscr{Q}$. Then there exists a point $\mathscr{R}\in I(\mathscr{P},\mathscr{Q})-\{\mathscr{P},\mathscr{Q}\}$. Applying (1.5; 1), $I(\mathscr{P},\mathscr{Q})\subset I(\mathscr{M},\mathscr{Q})$, whence $\mathscr{R}$ and $\mathscr{P}$ are in $I(\mathscr{M},\mathscr{Q})$. Also, $\mathscr{R}\in I(\mathscr{M},\mathscr{R})\cap I(\mathscr{P},\mathscr{Q})$, and applying (1.5; 4), $\mathscr{P}\in I(\mathscr{M},\mathscr{R})$. This shows that $\mathscr{P}<_{\mathscr{M}}\mathscr{R}<_{\mathscr{M}}\mathscr{Q}$.

We finally prove the following result which includes statement (iii) of the theorem, in view of (1.5; 1):

(1.5; 5) *Let X be an arbitrary space. Then for each triple $\mathscr{M},\mathscr{N}_1,\mathscr{N}_2\in\lambda(X)$,*

$$I(\mathscr{M},\mathscr{N}_1)\cap I(\mathscr{M},\mathscr{N}_2)=I(\mathscr{M},g(\mathscr{M},\mathscr{N}_1,\mathscr{N}_2)).$$

We put

$$\mathscr{N}_0=g(\mathscr{M},\mathscr{N}_1,\mathscr{N}_2),\qquad A=I(\mathscr{M},\mathscr{N}_1)\cap I(\mathscr{M},\mathscr{N}_2).$$

For each $\mathscr{P}\in A$ we have $\mathscr{M}\cap\mathscr{N}_1\subset\mathscr{P}$ and $\mathscr{M}\cap\mathscr{N}_2\subset\mathscr{P}$, and in particular,

$$\mathscr{M}\cap\mathscr{N}_1\subset\mathscr{P}\cap\mathscr{N}_1\subset g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1),$$

showing that $\mathscr{P}$ and $g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)$ are in $I(\mathscr{M},\mathscr{N}_1)$. Moreover, the set

$$I(\mathscr{M},g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1))\cap I(\mathscr{P},\mathscr{N}_1)$$

is nonempty since it contains $g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)$. Applying (1.5; 4) yields that

$$\mathscr{P}\in I(\mathscr{M},g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)).$$

We now show that $g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)=\mathscr{N}_0$, yielding one half of (1.5; 5). In fact, the sets $\mathscr{P}\cap\mathscr{N}_0$ and $\mathscr{N}_0\cap\mathscr{N}_1$ are contained in $g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)$. Now,

$$\mathscr{P}\cap\mathscr{N}_0=(\mathscr{P}\cap\mathscr{M}\cap\mathscr{N}_1)\cup(\mathscr{P}\cap\mathscr{M}\cap\mathscr{N}_2)\cup(\mathscr{P}\cap\mathscr{N}_1\cap\mathscr{N}_2)$$
$$\supset(\mathscr{M}\cap\mathscr{N}_1)\cup(\mathscr{M}\cap\mathscr{N}_2)$$

since $\mathscr{M}\cap\mathscr{N}_1\subset\mathscr{P}$ and $\mathscr{M}\cap\mathscr{N}_2\subset\mathscr{P}$. Also, $\mathscr{N}_0\cap\mathscr{N}_1\supset\mathscr{N}_1\cap\mathscr{N}_2$ by the construction of $\mathscr{N}_0$. Hence

$$g(\mathscr{P},\mathscr{N}_0,\mathscr{N}_1)\supset(\mathscr{M}\cap\mathscr{N}_1)\cup(\mathscr{M}\cap\mathscr{N}_2)\cup(\mathscr{N}_1\cap\mathscr{N}_2)=\mathscr{N}_0,$$

yielding the desired result.

To prove the other half of (1.5; 5), observe that $\mathscr{N}_0=g(\mathscr{M},\mathscr{N}_1,\mathscr{N}_2)\in A$, and applying (1.5; 2),

$$I(\mathscr{M},\mathscr{N}_0)\subset I(\mathscr{M},\mathscr{N}_1),\qquad I(\mathscr{M},\mathscr{N}_0)\subset I(\mathscr{M},\mathscr{N}_2).\ \blacksquare$$

We shall need one other result on the order-theoretic structure of $\lambda(X)$.

1.6. Lemma. *Let X be an arbitrary space and let $\mathscr{M},\mathscr{N},\mathscr{P}\in\lambda(X)$. Then*

$$\mathscr{M}\cap\mathscr{P}\subset\mathscr{N}\quad\textit{iff}\quad\mathscr{N}\subset\mathscr{M}\cup\mathscr{P}.$$

Proof. Assume first that $\mathscr{M}\cap\mathscr{P}\subset\mathscr{N}$ and let $N\in\mathscr{N}$. If $N\notin\mathscr{M}$ and $N\notin\mathscr{P}$, then by maximality of $\mathscr{M}$ and $\mathscr{P}$ there exist $M\in\mathscr{M}$ and $P\in\mathscr{P}$ such that

$$M\cap N=\emptyset\qquad P\cap N=\emptyset.$$

Hence $(M\cup P)\cap N=\emptyset$. But $M\cup P\in\mathscr{M}\cap\mathscr{P}\subset\mathscr{N}$, contradicting that $\mathscr{N}$ is linked.

Assume next that $\mathscr{N}\subset\mathscr{M}\cup\mathscr{P}$, and let $T\in\mathscr{N}\cap\mathscr{P}$. For each $N\in\mathscr{N}$, either $N\cap M\neq\emptyset$ for all $M\in\mathscr{M}$, or, $N\cap P\neq\emptyset$ for all $P\in\mathscr{P}$. Hence $N\cap T\neq\emptyset$, and by the maximality of $\mathscr{N}$, $T\in\mathscr{N}$. $\blacksquare$

1.7. Corollary. *Let X be an arbitrary space, and let $\mathscr{M},\mathscr{N}\in\lambda(X)$. Then*

$$\{\mathscr{Q}|\ \mathscr{N}\leqslant_{\mathscr{M}}\mathscr{Q}\}=\bigcap\{N^+|\ N\in\mathscr{N}-\mathscr{M}\}.$$

Proof. $\mathscr{N}\leqslant_{\mathscr{M}}\mathscr{Q}$ means, by definition, $\mathscr{M}\cap\mathscr{Q}\subset\mathscr{N}$, or, $\mathscr{N}\subset\mathscr{M}\cup\mathscr{Q}$ by Lemma 1.6. The latter statement is equivalent to $\mathscr{N}-\mathscr{M}\subset\mathscr{Q}$. Hence $\mathscr{N}\leqslant_{\mathscr{M}}\mathscr{Q}$ iff $\mathscr{Q}\in\bigcap\{\mathscr{N}^+|\ N\in\mathscr{N}-\mathscr{M}\}$. $\blacksquare$

2. Convex subsets of superextensions.

2.1. Definition. Let X be a topological space. A nonempty closed subset Y of $\lambda(X)$ is said to be *convex* if there is a nonempty (linked) collection $\mathscr{C}$ of closed subsets of X, such that $Y=\bigcap\{C^+|\ C\in\mathscr{C}\}$.

Several examples of convex sets have already been met with, e.g. an interval $I(\mathscr{M},\mathscr{N})$ (= set of $\leqslant_{\mathscr{M}}$-predecessors of $\mathscr{N}$), and the set of $\leqslant_{\mathscr{M}}$-successors of $\mathscr{N}$ (cf. Corollary 1.7). Observe that for each $\mathscr{P}\in\lambda(X)$,

$$\{\mathscr{P}\}=\bigcap\{P^+|\ P\in\mathscr{P}\}$$

is convex, and that $\lambda(X)=X^+$ is convex. We let $K(\lambda(X))$ denote the space of all convex subsets of $\lambda(X)$ endowed with the subspace topology of the hyperspace $H(\lambda(X))$.

2.2. Lemma. *Let X be a normal T_1-space and let Y be a nonempty closed subset of $\lambda(X)$. Then the following assertions are equivalent.*

(i) *Y is a convex set,*

(ii) *for each pair of mls's $\mathscr{P}_1,\mathscr{P}_2\in Y$, $I(\mathscr{P}_1,\mathscr{P}_2)\subset Y$,*

(iii) *for each $\mathscr{M}\in\lambda(X)$ with associated ordering $\leqslant_{\mathscr{M}}$ on $\lambda(X)$, the set Y has a smallest element with respect to $\leqslant_{\mathscr{M}}$.*

Proof of (i)⇒(ii). Let $Y\subset\lambda(X)$ be convex, say, $Y=\bigcap\{C^+|\ C\in\mathscr{C}\}$, where $\mathscr{C}\subset H(X)$ is linked, and let $\mathscr{P}_1,\mathscr{P}_2\in Y$. Then $\mathscr{C}\subset\mathscr{P}_1\cap\mathscr{P}_2$, and hence $I(\mathscr{P}_1,\mathscr{P}_2)\subset Y$.

Proof of (ii)⇒(iii). Let $\mathscr{M}\in\lambda(X)$ with an associated ordering $\leqslant_{\mathscr{M}}$ on $\lambda(X)$, and let $\mathscr{P}_1,\mathscr{P}_2\in Y$. Then

$$\mathscr{P}_0=g(\mathscr{M},\mathscr{P}_1,\mathscr{P}_2)\in I(\mathscr{P}_1,\mathscr{P}_2)\subset Y,$$

whence $\mathscr{P}_0\leqslant_{\mathscr{M}}\mathscr{P}_1$ and $\mathscr{P}_0\leqslant_{\mathscr{M}}\mathscr{P}_2$ in Y, proving that $\leqslant_{\mathscr{M}}$ is filtered below on Y. Since Y is compact and since $\leqslant_{\mathscr{M}}$ is a topological ordering (cf. Theorem 1.5 (i)), Y has at least one minimal element, which must therefore be the smallest in Y.

Proof of (iii)⇒(i). Assume that Y has a unique minimal element with respect to the ordering $\leqslant_{\mathscr{M}}$ on $\lambda(X)$, associated to each $\mathscr{M}\in\lambda(X)$. This minimal element is denoted by $\langle\mathscr{M}, Y\rangle$, $\mathscr{M}\in\lambda(X)$. If $Y=\lambda(X)$, then Y is convex. Let $Y\neq\lambda(X)$, and let $\mathscr{M}\in\lambda(X)-Y$. For each $\mathscr{P}\in Y$ we have $\langle\mathscr{M}, Y\rangle\leqslant_{\mathscr{M}}\mathscr{P}$, whence by Corollary 1.7, $Y\subset\bigcap\{N^+|\ N\in\langle\mathscr{M}, Y\rangle-\mathscr{M}\}$, whereas $\mathscr{M}$ is not in the latter set. Applying this for each $\mathscr{M}\in\lambda(X)-Y$ yields

$$Y=\bigcap\{N^+|\ N\in\langle\mathscr{M}, Y\rangle-\mathscr{M},\ \mathscr{M}\in\lambda(X)-Y\}\in K(\lambda(X)). \ \blacksquare$$

Interpreting intervals in a superextension as line segments in a topological vector space, statement (ii) of the above lemma yields a motivation for the term "convex set".

Lemma 2.2 is used to prove the following theorem, which will be our main tool in deriving several new and known properties of superextensions. We use the notation, introduced in the proof of 2.2.

2.3. THEOREM. *Let X be a normal T_1-space. Then the mapping*

$$p\colon \lambda(X)\times K(\lambda(X))\to\lambda(X),$$

defined by $p(\mathscr{M}, A)=\langle\mathscr{M}, A\rangle$, is continuous.

Proof. Let $A\in K(\lambda(X))$, say $A=\bigcap\{C^+|\ C\in\mathscr{C}\}$ for some nonempty linked system $\mathscr{C}\subset H(X)$, and let $\mathscr{M}\in\lambda(X)$. Then the following collection is easily seen to be linked:

$$p'(\mathscr{M}, A)=\mathscr{C}\cup\{M\in\mathscr{M}|\ M\cap C\neq\emptyset \text{ for all } C\in\mathscr{C}\}.$$

In [20], J. van Mill has proved that this collection is a *pre-mls*, i.e. it is contained in exactly one mls, which we denote by $p''(\mathscr{M}, A)$. Since $\mathscr{C}\subset p'(\mathscr{M}, A)\subset p''(\mathscr{M}, A)$, we find that $p''(\mathscr{M}, A)\in A$ and hence that $\langle\mathscr{M}, A\rangle\leqslant_{\mathscr{M}}p''(\mathscr{M}, A)$. On the other hand, we have for each $\mathscr{P}\in A$ that $\mathscr{C}\subset\mathscr{P}$, whence

$$\mathscr{M}\cap\mathscr{P}\subset\{M\in\mathscr{M}|\ M\cap C\neq\emptyset \text{ for all } C\in\mathscr{C}\}\subset p''(\mathscr{M}, A),$$

whence $p''(\mathscr{M}, A)\leqslant_{\mathscr{M}}\mathscr{P}$. Applying this for $\mathscr{P}=\langle\mathscr{M}, A\rangle$ yields that $p''(\mathscr{M}, A)=\langle\mathscr{M}, A\rangle=p(\mathscr{M}, A)$.

Using this explicit formula for $\langle\mathscr{M}, A\rangle$, we can now prove that p is continuous. Let $A\in K(\lambda(X))$ and $\mathscr{M}\in\lambda(X)$ again, and let $G\subset X$ be a closed set. Then $p(\mathscr{M}, A)\in G^+$, iff $G\in p(\mathscr{M}, A)$, iff G meets all members of the pre-mls $p'(\mathscr{M}, A)$. This is in turn equivalent to the following statement: $\{G\}\cup\mathscr{C}$ is linked (i.e. $G^+\cap A\neq\emptyset$) and for each $M\in\mathscr{M}$ with $M^+\cap A\neq\emptyset$, $M\cap G\neq\emptyset$. Hence, $p(\mathscr{M}, A)\notin G^+$ iff $G^+\cap A=\emptyset$, or, there exists an $M\in\mathscr{M}$ with $M^+\cap A\neq\emptyset$ and $M\cap G=\emptyset$.

We claim that the latter alternative is equivalent to the statement that $A\not\subset G^+$ and $\mathscr{M}\notin G^+$.

In fact, assume first that for some $M\in\mathscr{M}$, $M^+\cap A\neq\emptyset$ and $M\cap G=\emptyset$. If $A\subset G^+$, then $M^+\cap G^+\neq\emptyset$ and hence $M\cap G\neq\emptyset$, contradiction. Also, $\mathscr{M}\notin G^+$ since G does not meet $M\in\mathscr{M}$.

Assume next that $A\not\subset G^+$ and that $\mathscr{M}\notin G^+$. Then there is an mls $\mathscr{N}\in A-G^+$ and an $M\in\mathscr{M}$ not meeting G. Choose $N\in\mathscr{N}$ such that $N\cap G=\emptyset$. Since $\mathscr{N}\in A$, $\mathscr{C}\cup\{N\}$ is linked and, *a fortiori*, $\mathscr{C}\cup\{M\cup N\}$ is linked. Moreover, $M\cup N\in\mathscr{M}$ and $G\cap(M\cup N)=\emptyset$, proving the desired equivalence.

We have now obtained the following formula:

$$p^{-1}(\lambda(X)-G^+)=\lambda(X)\times(\langle\lambda(X)-G^+\rangle\cap K(\lambda(X))\cup$$
$$\cup(\lambda(X)-G^+)\times(\langle\lambda(X)-G^+, \lambda(X)\rangle\cap K(\lambda(X))),$$

which is an open subset of $\lambda(X)\times K(\lambda(X))$. Hence, p is continuous. ■

We shall need two more results on $K(\lambda(X))$.

2.4. THEOREM. *Let X be a normal T_1-space. Then the set $K(\lambda(X))$ is closed in the hyperspace of $\lambda(X)$.*

Proof. Let $Y\in H(\lambda(X))-K(\lambda(X))$. By compactness of Y, each topological ordering on Y admits at least one minimal element. Since Y is nonconvex, and applying Lemma 2.2(iii), there is an $\mathscr{M}\in\lambda(X)$ such that the corresponding partial ordering $\leqslant_{\mathscr{M}}$ admits two different minimal elements in Y, say $\mathscr{P}_1$, $\mathscr{P}_2$. Applying (1.5; 5) and (1.5; 1), the set

$$I(\mathscr{M}, g(\mathscr{M}, \mathscr{P}_1, \mathscr{P}_2))=I(\mathscr{M}, \mathscr{P}_1)\cap I(\mathscr{M}, \mathscr{P}_2)$$
$$=\{\mathscr{Q}\in\lambda(X)|\ \mathscr{Q}\leqslant_{\mathscr{M}}\mathscr{P}_1, \mathscr{Q}\leqslant_{\mathscr{M}}\mathscr{P}_2\}$$

does not meet Y. Since X is normal T_1, $\lambda(X)$ is Hausdorff and hence normal. We can then find disjoint open sets $O, P\subset\lambda(X)$ such that

$$I(\mathscr{M}, g(\mathscr{M}, \mathscr{P}_1, \mathscr{P}_2))\subset P, \qquad Y\subset 0.$$

By the continuity of the composed mapping (cf. Theorem 1.2(i) and (iii))

$$\lambda(X)^2\xrightarrow{g(\mathscr{M},-,-)}\lambda(X)\xrightarrow{I(\mathscr{M},-)}H(\lambda)X)),$$

there exist (disjoint) open neighborhoods $O_1, O_2\subset O$ of $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively, such that for all $\mathscr{P}_1'\in O_1$, $\mathscr{P}_2'\in O_2$, $I(\mathscr{M}, g(\mathscr{M}, \mathscr{P}_1', \mathscr{P}_2'))\subset P$. We now show that the neighborhood $\langle O, O_1, O_2\rangle$ of Y does not meet $K(\lambda(X))$. In fact, let $Y'\in\langle O, O_1, O_2\rangle$. Then $Y'\subset O\cup O_1\cup O_2=O$ and there exist (different) mls's

$$\mathscr{P}_1'\in Y'\cap O_1, \qquad \mathscr{P}_2'\in Y'\cap O_2.$$

Hence $I(\mathscr{M}, g(\mathscr{M}, \mathscr{P}_1', \mathscr{P}_2'))$ is contained in P and it therefore does not meet Y'. Applying Theorem 1.5(iii), there is no element in Y' which is smaller than both $\mathscr{P}_1'$ and $\mathscr{P}_2'$ under the relation $\leqslant_{\mathscr{M}}$. It follows that Y' is not convex. ■

2.5. THEOREM. *Let X be normal and T_1, and let $Y\subset\lambda(X)$ be convex and connected. Then the subspace $K(\lambda(X))\cap H(Y)$ of $H(\lambda(X))$ is densely ordered by inclusion.*

Proof. Let $A\subset B$ be two different convex subsets of $Y\subset\lambda(X)$. Choose $\mathscr{M}\in B-A$, and let $\mathscr{N}\in A$ be the smallest element w.r.t. $\leqslant_{\mathscr{M}}$. In particular, $\mathscr{M}\neq\mathscr{N}$. Applying Theorem 1.5(ii), the ordering $\leqslant_{\mathscr{M}}$ is dense on Y, and hence there is a $\mathscr{P} \in Y$ such that $\mathscr{M} <_{\mathscr{M}} \mathscr{P} <_{\mathscr{M}} \mathscr{N}$. Observe that $\mathscr{P} \notin A$, but $\mathscr{P} \in I(\mathscr{M}, \mathscr{N}) \subset B$ by Lemma 2.2(ii). Using Corollary 1.7, the set $D = \{\mathscr{Q} \in \lambda(X) |\ \mathscr{P} \leqslant_{\mathscr{M}} \mathscr{Q}\}$ is convex. We then have that $B \cap D \in K(\lambda(X))$ and $A \subset B \cap D \subset B$.

Moreover, $A \neq B \cap D$ since $\mathscr{P} \in (B \cap D) - A$, and $B \cap D \neq B$ since $\mathscr{M} \in B - (B \cap D)$. ■

3. The fixed point property and related results. Our first result is on the acyclicity of the components of a superextension. Acyclicity is always meant with respect to the *Čech homology functor*, $H = (H_n)_{n \in N}$, *over a field.* Recall that H is defined on a compact space X by $H(X) = \varprojlim H(\mathscr{U})$, where $\mathscr{U}$ ranges over the finite open coverings of X, and H on the right stands for simplicial homology over the given field.

3.1. THEOREM. *Let X be a normal T_1-space with a finite number of components. Then $\lambda(X)$ has a finite number of components, each of which is acyclic. If X is connected moreover, then $\lambda(X)$ is acyclic. If X is a compact locally connected metric space, then the components of $\lambda(X)$ are contractible.*

Proof. Let X be normal T_1 with a finite number of components, k say. As was shown by Verbeek in [21] p. 109, $\lambda(X)$ has exactly $\lambda(k)$ components, where $\lambda(k)$ is the (finite) number of mls's existing on the discrete space $\{1, \ldots, k\}$. Actually, Verbeek's proof shows even more. Let π: $X \to \{1, \ldots, k\}$ be the decomposition map of X. Then there is an induced map (cf. Theorem 0.2)

$$\lambda(\pi) = \bar{\pi}\colon \lambda(X) \to \lambda\{1, \ldots, k\} \approx \{1, \ldots, \lambda(k)\}$$

of which the following properties are derived in [21]:

(i) $\bar{\pi}$ is the decomposition map of $\lambda(X)$ (whence $\bar{\pi}^{-1}(1), \ldots, \bar{\pi}^{-1}(\lambda(k))$ are the components of $\lambda(X)$);

(ii) for each mls $\mathscr{K} \in \lambda\{1, \ldots, k\}$, $\bar{\pi}^{-1}(\mathscr{K}) = \bigcap \{\pi^{-1}(S)^+ |\ S \in \mathscr{K}\}$ (whence each component of $\lambda(X)$ is a convex set).

Let $C \subset \lambda(X)$ be a component. Since $H(\lambda(X))$ and $H(C)$ are compact (Michael [11]), and since C is convex and connected, it follows from Theorems 2.4 and 2.5 that $K(\lambda(X)) \cap H(C)$ is a compact space which is densely ordered by inclusion. Hence (Ward [22]), each maximal linearly ordered set in $K(\lambda(X)) \cap H(C)$ is compact and connected. Let J be such a set. As we observed in 2.1, $K(\lambda(X)) \cap H(C)$ contains C as well as all singletons in C. By the maximality of J, there exists a $\mathscr{P} \in C$ with $\{\mathscr{P}\} \in J$, and $C \in J$. Finally, each finite open covering of a linearly ordered continuum can be refined by a finite covering of open intervals which admits no proper subcovering. Such a cover is easily seen to be a chain, which is acyclic under simplicial homology. It follows that J is acyclic.

Let p^*: $C \times J \to \lambda(X)$ be the restriction of the "nearest point map" p, introduced in Theorem 2.3. Since each member of J is a convex subset of C, it follows that p^* maps into C. Also, by the construction of p, we have that

$$p^*(-, C) = \text{identity map of } C,$$
$$p^*(-, \{\mathscr{P}\}) = \text{constant map onto } \mathscr{P} \in C.$$

Dealing with compact T_2 spaces C, J, the projection q: $C \times J \to C$ onto first coordinates is a closed mapping. Its fiber, J, is acyclic, and hence q is a Vietoris mapping, inducing an isomorphism $H(q)$ of $H(C \times J)$ with $H(C)$ (cf. e.g. Begle [2]).

Next, consider the mappings h_0, h_1: $C \to C \times J$, which are defined by $h_0(\mathscr{M}) = (\mathscr{M}, C)$ and $h_1(\mathscr{M}) = (\mathscr{M}, \{\mathscr{P}\})$. Then $qh_0 = qh_1$, and since $H(q)$ is an isomorphism, $H(h_0) = H(h_1)$. Consequently,

$$H(p^* h_0) = H(p^*) H(h_0) = H(p^*) H(h_1) = H(p^* h_1).$$

But $H(p^* h_0) = H(p^*(-, C))$ is the identity homomorphism of $H(C)$, whereas $H(p^* h_1) = H(p^*(-, \{\mathscr{P}\}))$ factors through the Čech homology of a one-point space. In these circumstances, C must be acyclic.

If X is normal, connected and T_1, then $\lambda(X)$ is connected since $\lambda(1) = 1$. Applying the above argument on $C = \lambda(X)$ yields that $\lambda(X)$ is acyclic.

Finally, assume that X is a compact, locally connected, metric space. Then $\lambda(X)$ is metrizable (Verbeek [21]) and X is normal T_1 with a finite number of components. Hence the components of $\lambda(X)$ are convex by the argument at the beginning of this proof. Let C be a component of $\lambda(X)$, let $J \subset K(\lambda(X)) \cap H(C)$ be a maximal linearly ordered subspace, and let p^*: $C \times J \to C$ be the mapping considered previously.

$\lambda(X)$ being metrizable, $H(\lambda(X))$ can be given the Hausdorff metric, and in particular, J is metrizable. Hence J is homeomorphic to the unit interval (Ward [22]) and the mapping p^* becomes an ordinary contraction of C onto a point of C. ■

3.2. GENERALITIES. We shall now work towards the Lefschetz fixed point property of superextensions. For this purpose, some explanation of terminology is in order.

Let X be a compact space. A continuous selfmap f: $X \to X$ induces a sequence of endomorphisms $H_n(f)$: $H_n(X) \to H_n(X)$, $n \in N$. If $H(X)$ is *finitely generated*, that is, if all the vectorspaces $H_n(X)$ are zero except for a finite number of them, which are finite-dimensional, then the *trace* of f is defined to be

$$\operatorname{tr}(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr} H_n(f)$$

where $\operatorname{tr} H_n(f)$ denotes the algebraic trace of the endomorphism $H_n(f)$. The space X is called a *Lefschetz space* (Browder [6]) if each selfmap of X with non-zero trace has a fixed point. This property is called the *Lefschetz fixed point property*.

The following types of Lefschetz spaces will be met with hereafter: *convexoid spaces* (cf. Leray [10]) and, more generally, *retracts of convexoid spaces* (cf. Deleanu [7]), which include the compact metric ANR's; spaces with an SC-*structure* (Browder [5]) which include the compact ANR's, and spaces with a WSC-*structure* (Thompson [15]).

Defining these concepts explicitely would lead us too far, and we refer the interested reader to the bibliography. We like to draw attention onto the following facts:

(3.2; 1) The only relationship between the above classes of spaces (except for the one mentioned already) seems to be the implication SC⇒WSC. (cf. Thompson [15]).

(3.2; 2) Retracts of spaces with a WSC-structure again carry a WSC-structure (cf. Thompson [16], or Van de Vel [17]). A similar result on SC-structures seems to hold for metric spaces only (cf. Thompson [16]).

3.3. LEMMA. *Let X be a normal T_1-space. Then*

(i) *each convex subset of $\lambda(X)$ is a retract of $\lambda(X)$,*

(ii) *$\lambda(X)$ is locally convex, i.e. each point has a neighborhood base consisting of convex sets.*

Statement (i) is a reformulation of a result in J. van Mill [20]. Actually, we have the necessary tools available for a short proof:

Proof of (i). Define a map $g\colon \lambda(X)\to\lambda(X)$ by $g(\mathscr{M}) = p(\mathscr{M}, A)$, where A is a fixed convex set and p is the mapping introduced in Theorem 2.3. By the construction of p, g is a retraction of $\lambda(X)$ onto A.

Proof of (ii). The collection of all sets of type

$$O^+ = \{\mathscr{M} \mid M\subset O \text{ for some } M\in\mathscr{M}\}, \quad O\subset X \text{ open},$$

yields an open subbase for $\lambda(X)$ (Verbeek [21]). If $\mathscr{M}\in\lambda(X)$ and if O is a neighborhood of $\mathscr{M}$, then there exist open sets $O_1, \ldots, O_n\subset X$ such that

$$\mathscr{M}\in\bigcap_{i=1}^{n} O_i^+\subset O .$$

For each $i = 1, .., n$, there exists an $M_i\in\mathscr{M}$ with $M_i\subset O_i$. By normality of X there exist closed sets P_i, $Q_i\subset X$ such that

$$M_i\subset X-P_i\subset Q_i\subset O_i .$$

Hence $\bigcap_{i=1}^{n} Q_i^+$ is a convex neighborhood of $\mathscr{M}$ contained in O. ■

We can now obtain our main result.

3.4. THEOREM. *Let X be a normal T_1-space with a finite number of components. Then $\lambda(X)$ is a Lefschetz space. In fact, $\lambda(X)$ is convexoid, and it carries an SC-structure. If X is connected moreover, then $\lambda(X)$ has the fixed point property.*

Proof. $\lambda(X)$ has a finite number of (acyclic) components by Theorem 3.1. In particular, these components are open.

Let $A\subset\lambda(X)$ be a convex set contained in some component C of $\lambda(X)$. By Lemma 3.3 (i), A is a retract of $\lambda(X)$ and hence of C. Therefore, A is acyclic. Let $\mathscr{C}$ denote the collection of all convex sets contained in some component of $\lambda(X)$. All members of $\mathscr{C}$ are acyclic by the above argument, and $\mathscr{C}$ is closed under the formation of finite nonempty intersections. By Lemma 3.3 (ii), using the fact that the $\lambda(X)$-components are open, each point of $\lambda(X)$ has a base of neighborhoods which are a member of the collection $\mathscr{C}$. It easily follows that $\lambda(X)$ is convexoid.

The above argument also shows that the interiors of members of $\mathscr{C}$ form an open base on $\lambda(X)$. Hence, each finite open covering of $\lambda(X)$ can be refined by a finite covering consisting of members of $\mathscr{C}$, whose interiors still cover $\lambda(X)$. Using the techniques exposed in Van de Vel [17], it can be seen that $\lambda(X)$ is an lc-space (Begle [3]), and it therefore carries an SC-structure (Browder [5]).

If X is connected, normal and T_1, then $\lambda(X)$ is an acyclic Lefschetz space, and hence it has the fixed point property. ■

3.5. THEOREM. *Let X be a compact locally connected metric space. Then $\lambda(X)$ is a metric ANR. If X is connected, then $\lambda(X)$ is an AR.*

Proof. This is a modification of the above proof. By Theorem 3.1, $\lambda(X)$ has a finite number of components, each of which is contractible. These components are open. Let $\mathscr{C}$ again denote the set of all convex sets contained in some component of $\lambda(X)$. Then all members of $\mathscr{C}$ are contractible, and the interiors of members of $\mathscr{C}$ form an open base. Let $\mathscr{U}$ be an open covering of $\lambda(X)$. Then $\mathscr{U}$ can be refined by a closed cover $\mathscr{D}\subset\mathscr{C}$ such that

$$\operatorname{int}(\mathscr{D}) = \{\operatorname{int}(D) \mid D\in\mathscr{D}\}$$

is still a covering of $\lambda(X)$. Using the fact that finite nonempty intersections in $\mathscr{D}$ are contractible, it easily follows that each partial realization of a polyhedron into $\operatorname{int}(\mathscr{D})$ can be extended to a full realization into $\mathscr{D}$ (and hence into $\mathscr{U}$). Noticing that $\lambda(X)$ is metrizable (Verbeek [21]), it follows that $\lambda(X)$ is a metric ANR (Dugundji [8]).

If X is connected moreover, then $\lambda(X)$ is contractible, and hence it is a metric AR. ■

A different proof for the second part of Theorem 3.5 can be found in Van Mill [19].

We finally derive some interesting consequences of the above results.

3.6. COROLLARY. *Let X be a normal T_1-space. Then $\lambda(X)$ has trivial n-th homotopy groups for each $n>0$.*

Proof. Let $n>0$. The n-sphere S^n is a compact connected metric space and hence $\lambda(S^n)$ is contractible. Let $f\colon S^n\to\lambda(X)$ be continuous. Since $\lambda(X)$ is normal T_1, f extends to a continuous mapping

$$\lambda(f)\colon \lambda(S^n)\to\lambda(\lambda(X))$$

(cf. Theorem 0.2). The extension is with respect to the embeddings

$$i\colon \lambda(X)\to\lambda(\lambda(X)), \quad j\colon S^n\to\lambda(S^n)$$

(cf. Theorem 0.1). Since $\lambda(S^n)$ is contractible, j extends to a continuous mapping $j'\colon E^{n+1}\to\lambda(S^n)$, where E^{n+1} denotes the unit $(n+1)$-cell. Finally, the canonical closed subbase of $\lambda(X)$ is binary, normal and T_1, and hence there is a retraction $r\colon \lambda(\lambda(X)) \to \lambda(X)$. It follows from the commutativity of the diagram

$$\begin{array}{ccccc} E^{n+1} \supset & S^n & \xrightarrow{f} & \lambda(X) & \\ \searrow^{j'} & \downarrow j & & \downarrow i & \searrow^{\mathrm{id}} \\ & \lambda(S^n) & \xrightarrow{\lambda(f)} & \lambda(\lambda(X)) & \xrightarrow{r} \lambda(X) \end{array}$$

that $r \circ \lambda(f) \circ j'\colon E^{n+1} \to \lambda(X)$ extends f. Hence (Spanier [13] p. 43), the nth homotopy group $\pi_n(\lambda(X)) = 0$. ■

3.7. Corollary. *Let X be a compact Hausdorff space with a finite number of components, carrying a normal binary T_1-subbase. Then X is a Lefschetz space. In fact, X is a retract of a convexoid space, and it carries a* WSC-*structure. Moreover, the components of X are acyclic.*

Proof. As we noticed in the introduction, X is a retract of $\lambda(X)$. In view of (3.2; 2), X also carries a WSC-structure. If $r\colon \lambda(X) \to X$ is a retraction and if $C \subset X$ is a component, then there is a component $D \subset \lambda(X)$ with the property that $r(D) = C$, and hence C is acyclic. ■

3.8. Corollary. *Let X be a completely regular T_1-space with a finite number of components. The collection $\mathcal{S}$ of all zerosets in X is a normal T_1-(sub-) base of X and $\lambda_{\mathcal{S}}(X)$ is a Lefschetz space with finitely many components, each of which is acyclic.*

Proof. The first part of the corollary can be found in Gillman and Jerison [9], p. 38. Notice that $\mathcal{S}^+$ is a normal binary T_1-subbase of $\lambda_{\mathcal{S}}(X)$, whence $\lambda_{\mathcal{S}}(X)$ is a retract of $\lambda(X)$. In particular, $\lambda_{\mathcal{S}}(X)$ has a finite number of components, and applying Corollary 3.7 yields the desired result. ■

3.9. Corollary. *Let $f\colon X \to Y$ be a continuous closed surjection between normal connected T_1-spaces. Then the induced mapping $\lambda(f)\colon \lambda(X) \to \lambda(Y)$ is a Vietoris mapping.*

Proof. In [21] p. 56, the extension $\lambda(f)$ is constructed as follows. If $\mathcal{M} \in \lambda(X)$, then $\lambda(f)(\mathcal{M})$ is the unique mls of Y containing the linked system (pre-mls)

$$\{N \mid N \subset Y \text{ is closed and } f^{-1}(N) \in \mathcal{M}\}.$$

Using the fact that f is surjective and closed, it can easily been proved that for each $\mathcal{N} \in \lambda(Y)$,

$$\lambda(f)^{-1}(\mathcal{N}) = \bigcap \{f^{-1}(N)^+ \mid N \in \mathcal{N}\},$$

which is a convex set in $\lambda(X)$. Since X is connected, $\lambda(f)^{-1}(\mathcal{N})$ is acyclic, being a retract of the acyclic $\lambda(X)$. Moreover, $\lambda(f)$ is onto (cf. Theorem 0.2) and closed, i.e. $\lambda(f)$ is a Vietoris mapping (Begle [2]). ■

4. Concluding remarks. The acyclity of superextensions — Theorem 3.1 — and the ANR property of superextensions of metric continua — Theorem 3.5, due originally to J. van Mill [19] — yield a solution to different problems posed in

Verbeek's dissertation ([21] p. 142). We also like to draw attention onto Corollary 3.7, giving a surprising connection between purely analytic conditions (the existence of a certain closed subbase e.g.) and highly sophisticated structures of an algebraic nature involving Čech homology.

Proceeding as in the proof of Corollary 3.7, it can be deduced from Theorem 3.5 that a metric continuum carrying a normal binary T_1-subbase is a metric AR. It is then natural to ask which class of compact metric AR admit normal binary T_1-subbases. At present state this is only known for compact trees, [4], and their products, [21]. It may be of interest to notice that compact metric spaces have binary subbases, a result which has recently been proved by Strok and Szymański, [14].

Finally, it appears that superextensions can take the role of the Hilbert cube, or of topological vector spaces (or rather of its compact convex sets). As an example, it can easily been derived from Theorem 3.5 that a compact locally connected metric space is an ANR iff it is a neighborhood retract of its superextension.

In this respect, certain similarities between superextensions and vectorspaces, which appear in this paper, are not pure coincidence.

References

[1] J. M. Aarts and J. de Groot, *Complete regularity as a separation axiom*, Canad. J. Math. 21 (1969), pp. 96–105.
[2] E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. of Math. 51 (3) (1950), pp. 534–543.
[3] — *Locally connected spaces and generalized manifolds*, Amer. J. Math. 64 (1942), pp. 553–574.
[4] A. E. Brouwer and A. Schrijver, *A characterization of supercompactness with an application to tree-like spaces*, Report Mathematical Centre ZW 34/74, Amsterdam 1974.
[5] F. E. Browder, *On the fixed point index for continuous mappings on locally connected spaces*, Summa Bras. Math. 4 (7) (1960), pp. 253–293.
[6] — *Fixed point theorems on infinite dimensional manifolds*, Trans. Amer. Math Soc. 119 (1965), pp. 179–194.
[7] A. Deleanu, *Théorie des points fixes: sur les rétractes de voisinage des espaces convexoides*, Bull. Soc. Math. France 87 (1959), pp. 235–243.
[8] J. Dugundji, *Absolute neighborhood retracts and local connectedness in arbitrary metric spaces*, Comp. Math. 13 (1958), pp. 229–246.
[9] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J. 1960.
[10] J. Leray, *Sur les équations et les transformations*, J. Math. Pures et Appl. 9-e série, t. 24 (1945), pp. 201–248.
[11] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1) (1951), pp. 152–182.
[12] R. E. Smithson, *Multifunctions*, Nieuw Arch. vr. Wiskunde 20 (3) (1972), pp. 31–53.
[13] E. Spanier, *Algebraic Topology*, Mc. Graw-Hill, New York 1966.
[14] M. Strok and A. Szymański, *Compact metric spaces have binary bases*, Fund. Math. 89 (1975), pp. 81–91.
[15] R. B. Thompson, *A unified approach to local and global fixed point indices*, Adv. Math. 3 (1969), pp. 1–71.
[16] — *Retracts of semi-complexes*, Ill. J. Math. 15 (1971), pp. 258–272.

[17] M. van de Vel, *The intersection property: a contribution to almost fixed point theory*, Dissertation, University of Antwerp, Belgium 1975.

[18] J. van Mill, *On supercompactness and superextensions*, Report 37, Free University of Amsterdam, 1975.

[19] — *The superextension of the closed unit interval is homeomorphic to the Hilbert cube*, Fund. Math. 103 (1979), pp. 151–175.

[20] — *A pseudo interior of λI*, Comp. Math. 36 (1) (1978), pp. 75–82.

[21] A. Verbeek, *Superextensions of topological spaces*, Math. Centre tracts 41, Mathematisch Centrum Amsterdam, 1972.

[22] L. E. Ward, Jr., *Topology*, Marcel Dekker, New York 1972.

WISKUNDIG SEMINARIUM
FREE UNIVERSITY OF AMSTERDAM

Accepté par la Rédaction le 31. 1. 1977

Some remarks on Eberlein compacts

by

K. Alster (Warszawa)

Abstract. We give a positive answer to a problem of Y. Benyamini, M. E. Rudin, P. Simon and M. Wage by showing that a compact subspace of a Σ-product of intervals is a strong Eberlein compact if and only if it is scattered.

A compact space is called an *Eberlein compact* (E-C), if it is homeomorphic to a weakly compact subset of a Banach space.

The main structure theorem for E-C is due to Amir and J. Lindenstrauss [1]:

A compact space is an E-C *if and only if it is homeomorphic to a compact subset X of the product of intervals, where X has the property, that for each $\varepsilon>0$ and $f\in X$ the set $\{\gamma\in\Gamma\colon f(\gamma)\geqslant\varepsilon\}$ is finite.*

H. P. Rosenthal observed, that X *is* E-C *if and only if it has a σ-point-finite separating family of open F_σ-subsets.*

An E-C X is called *strong* (see [2], [6] and [4]) if it embeds in the Cartesian product of intervals I^Γ in such a way that $x(\gamma)=0$ or $x(\gamma)=1$ for all $x\in X$ and $\gamma\in\Gamma$ and $|\{\gamma\colon x(\gamma)\neq 0\}|<\aleph_0$. Equivalently X is a *strong* E-C if and only if it has a point-finite separating family of clopen sets.

Let us recall that a family $\mathcal{U}$ of subsets of X is called *separating*, if given any $x\neq y$ in X, then there is an $U\in\mathcal{U}$ such that either $x\in U$ and $y\notin U$ or $y\in U$ and $x\notin U$. The family $\mathcal{U}$ is called *point-finite* (*point-countable*), if each x belongs to at most *finitely* (*countably*) many sets in $\mathcal{U}$. It is called *σ-point-finite* if $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$, where each $\mathcal{U}_n$ is point-finite.

A space X is scattered if every closed subset of X has an isolated point.

Let us put $X=X^{(0)}$, $X^{(\alpha+1)}$ is the set of accumulation points of the set X^α and $X^{(\lambda)}=\bigcap_{\beta<\lambda}X^{(\beta)}$ for a limit ordinal λ.

All undefined terms are from [3].

The aim of this note is to prove the following:

THEOREM. *If X is a scattered, compact space, which has a point-countable separating family of open F_σ-sets, it is the same as to say that X is a compact subset of a Σ-product of intervals, then it is a strong* E-C.