Metrization, paracompactness, and real-valued functions II

by

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Abstract. Regarding collections of real-valued functions, the following hierarchy is shown: partition of unity → relatively complete → equicontinuous → evenly continuous, and conditions are given to show when the arrows are reversible. Some of the results are used to obtain the following theorem: A topological space X is pseudometrizable iff X has the weak topology induced by a σ-equicontinuous family of real-valued functions. Characterizations of paracompact spaces are also given.

In classical analysis, as well as in general topology, the concept of an equicontinuous family of real-valued functions has been studied extensively and several interesting results have been proved. No doubt, this has served to stimulate investigations concerning collections of real functions affixed with various properties, and notable consequences have followed. For example, Michael [6] characterized paracompact spaces by appealing to the notion of a partition of unity. And in [5], Kelley generalized equicontinuity by introducing the concept of an evenly continuous family and obtained several results pertaining to function spaces.

It is the purpose of this paper to investigate the relationships among these concepts and how they, in turn, relate to relatively complete collections of real functions, a property that was studied in [3] and [8]. It will be shown that the implication diagram partition of unity → relatively complete → equicontinuous → evenly continuous holds, and examples and conditions will be given to show under what circumstances the arrows are reversible. Furthermore, a functional characterization of pseudometrizable spaces will be given, and this result will be used to determine when open covers are precisely normal covers, yielding equivalent conditions for paracompactness weaker than those involving partitions of unity.

In the following definitions and theorems, $\mathscr{F} = \{f_\alpha\colon \alpha \in A\}$ will denote a family of continuous real-valued functions defined on a topological space X. Then

(a) $\mathscr{F}$ is a *partition of unity* if each $f_\alpha\colon X \to [0, 1]$, for each $x \in X$ $f_\alpha(x) \neq 0$ for only finitely many $\alpha \in A$, and for each $x \in X$, $\sum\{f_\alpha(x)\colon \alpha \in A\} = 1$.

(b) $\mathscr{F}$ is *relatively complete* if for each $B \subset A$, the real functions $\inf\{f_\beta(x)\colon \beta \in B\}$ and $\sup\{f_\beta(x)\colon \beta \in B\}$ are continuous.

(c) $\mathscr{F}$ is *equicontinuous at a point* $x \in X$ iff given $\varepsilon>0$, there is a nbd U of x such that $|f_\alpha(y)-f_\alpha(x)|<\varepsilon$ for every $y \in U$ and for every $\alpha \in A$.

(d) $\mathscr{F}$ is *evenly continuous at a point* $x \in X$ iff for each real number y and each $\varepsilon>0$, there is a nbd V of x and a $\delta>0$ such that $|f_\alpha(a)-y|<\varepsilon$ whenever $|f_\alpha(x)-y|<\delta$, for every $a \in V$ and for every $\alpha \in A$.

(e) $\mathscr{F}$ is *pointwise bounded* iff for each $x \in X$, there is a real number M such that $|f_\alpha(x)|<M$ for every $\alpha \in A$.

Our original proofs of Theorem 1 and Theorem 4 were by means of nets. We are indebted to Professor R. Pol for suggesting the simpler method of proof used here.

For brevity, let us say that $\mathscr{F}$ is equicontinuous (evenly continuous) iff $\mathscr{F}$ is equicontinuous (evenly continuous) at each $x \in X$.

THEOREM 1. *A collection $\mathscr{F}$ of real-valued functions is relatively complete iff $\mathscr{F}$ is equicontinuous and pointwise bounded.*

Proof. Assume that $\mathscr{F}$ is relatively complete. It is clear that $\mathscr{F}$ is bounded at each x since $\inf\{f_\alpha(x)\colon \alpha \in A\} \leqslant f_\gamma(x) \leqslant \sup\{f_\alpha(x)\colon \alpha \in A\}$ for each $\gamma \in A$. We now show that $\mathscr{F}$ is equicontinuous.

Let $x \in X$ and $\varepsilon>0$ be given. Choose $1/2^n<\varepsilon$. For each integer m define

$$A_{mn} = \left\{\alpha\colon \alpha \in A \text{ and } \frac{m+1}{2^{n+1}} \leqslant f_\alpha(x) \leqslant \frac{m+2}{2^{n+1}}\right\}.$$

Note that $\bigcup_m A_{mn} = A$, and since $\mathscr{F}$ is bounded at x, $A_{mn} \neq \emptyset$ for only finitely many m. For each $A_{mn} \neq \emptyset$, we have that

$$s_{mn} = \sup\{f_\alpha\colon \alpha \in A_{mn}\}$$

and

$$l_{mn} = \inf\{f_\alpha\colon \alpha \in A_{mn}\}$$

are continuous since $\mathscr{F}$ is relatively complete.

Set

$$W_n(x) = \bigcap_m \left[s_{mn}^{-1}\left(-\infty, \frac{m+3}{2^{n+1}}\right) \cap l_{mn}^{-1}\left(\frac{m}{2^{n+1}}, \infty\right)\right].$$

Clearly $W_n(x)$ is open, and $x \in W_n(x)$. For each $y \in W_n(x)$, $|f_\alpha(x)-f_\alpha(y)|<1/2^n<\varepsilon$ for each $\alpha \in A$. Thus $\mathscr{F}$ is equicontinuous.

Now assume that $\mathscr{F}$ is equicontinuous and pointwise bounded. Let $x \in X$, $B \subset A$, and let $\varepsilon>0$. Since $\mathscr{F}$ is pointwise bounded, $m = \sup\{f_\beta(x)\colon \beta \in B\}$ exists. By equicontinuity, there is a nbd U of x such that $f_\gamma(x)-\frac{1}{2}\varepsilon<f_\gamma(y)<f_\gamma(x)+\frac{1}{2}\varepsilon$ for every $y \in U$ and for every $\gamma \in B$. Hence, $f_\gamma(x)-\frac{1}{2}\varepsilon<f_\gamma(y)<m+\frac{1}{2}\varepsilon$, and this implies that $f_\gamma(x)-\frac{1}{2}\varepsilon<\sup\{f_\beta(y)\colon \beta \in B\}<m+\varepsilon$ for every $y \in U$ and for every $\gamma \in B$. Thus, $m-\varepsilon<\sup\{f_\beta(y)\colon \beta \in B\}<m+\varepsilon$ and $\sup\{f_\beta\colon \beta \in B\}$ is continuous at x. Similarly, $\inf\{f_\beta\colon \beta \in B\}$ can be shown to be continuous, so $\mathscr{F}$ is relatively complete.

THEOREM 2. *Let $\mathscr{F} = \{f_\alpha\colon \alpha \in A\}$ be a collection of continuous real-valued functions, and consider the following statements:*

(1) *$\mathscr{F}$ is a partition of unity.*

(2) *$\mathscr{F}$ is relatively complete.*

(3) *$\mathscr{F}$ is equicontinuous.*

(4) *$\mathscr{F}$ is evenly continuous.*

Then (1)→(2)→(3)→(4) *and without additional hypotheses, none of the arrows is reversible.*

Proof. (1)→(2): By Theorem 1, it is sufficient to show that a partition of unity $\mathscr{F}$ is equicontinuous since, *a priori*, $\mathscr{F}$ is pointwise bounded. So let $x \in X$ and let $\varepsilon>0$. Now pick an integer n such that $1/n<\varepsilon$. The collection $\{f_\alpha^{-1}([1/n, 1])\colon \alpha \in A\}$ is locally finite and hence there is a nbd V of x such that $f_\alpha(V) \subset [0, 1/n)$ whenever $\alpha \neq \alpha_1, \ldots, \alpha_m$. Find nbds V_i such that $f_{\alpha_i}(V_i) \subset (f_{\alpha_i}(x)-\varepsilon, f_{\alpha_i}(x)+\varepsilon)$ and note that the nbd $U = (\bigcap_{i=1}^{m} V_i) \cap V$ of x satisfies the requirement that $|f_\alpha(y)-f_\alpha(x)|<\varepsilon$ for every $y \in U$ and for every $\alpha \in A$.

(2)→(3): This is Theorem 1.

(3)→(4): This was proved by Kelley [5, p. 237].

The following examples will show that none of the arrows are reversible.

EXAMPLE A. There is a relatively complete collection $\mathscr{F}$ of functions from R into $[0, 1]$ that is not a partition of unity.

For each $y \in R$ define

$$f_y(x) = \begin{cases} 1 & \text{if } x \leqslant y-1, \\ y-x & \text{if } y-1<x<y, \\ 0 & \text{if } y \leqslant x. \end{cases}$$

The method of proof used in Example 4 of [3] can be applied here to show that $\mathscr{F} = \{f_y\colon y \in R\}$ is relatively complete. The set $\{f_y^{-1}(0, 1]\colon y \in R\}$ is not point finite so $\mathscr{F}$ is not a partition of unity.

EXAMPLE B. There is an equicontinuous family $\mathscr{F}$ that is not relatively complete.

Let $\mathscr{F} = \{f_n\colon n = 1, 2, \ldots\}$ where $f_n(x) = n$.

EXAMPLE C. There is an evenly continuous family $\mathscr{F}$ that is not equicontinuous.

Let $\mathscr{F} = [f_n\colon n = 1, 2, \ldots\}$ where $f_n(x) = nx$. Then $\mathscr{F}$ is not equicontinuous at any $x \in R$. For if $x \in R$ and $(x-\delta, x+\delta)$ is any nbd of x, let $y = x+\delta/2$, let $n>\varepsilon/2\delta$ and observe that

$$|f_n(y)-f_n(x)| = |ny-nx| > \varepsilon/2\delta \cdot 2\delta = \varepsilon.$$

However, $\mathscr{F}$ is evenly continuous at each $x \in R$. To see this, let $x \in R$, let $y \in R$, and let $\varepsilon>0$. Since there are at most finitely many n with $|f_n(x)-y|<\varepsilon$, choose $\delta = \varepsilon$ and use continuity to find the required nbd V.

THEOREM 3. *Let $\mathscr{F}$ be a collection of continuous real-valued functions. Then the following are equivalent*:

(1) *$\mathscr{F}$ is relatively complete.*

(2) *$\mathscr{F}$ is equicontinuous and pointwise bounded.*

(3) *$\mathscr{F}$ is evenly continuous and pointwise bounded.*

Proof. Theorem 1, and Theorem 23 of [5, p. 237].

We now turn our attention towards investigating the question of when the aforementioned collections of real functions exist, given a space X, and what impact they have on X when their existence is hypothesized. Theorem 4 is the pseudometric analogue of Theorem 4 in [3], and will be utilized to determine when X is paracompact.

THEOREM 4. *A topological space X is pseudometrizable iff X has the weak topology induced by a σ-relatively complete collection.*

Proof. Assume X is pseudometrizable with d a bounded pseudometric. For each $x \in X$, define $d_x\colon X \to R$ by $d_x(y) = d(x, y)$, and observe that $\mathscr{F} = \{d_x\colon x \in X\}$ has the desired properties.

For the converse, assume X has the weak topology induced by the σ-relatively complete collection $\mathscr{F} = \bigcup \mathscr{F}_n$. For each n note that

$$d'_n(x, y) = \sup\{|f(x) - f(y)|\colon f \in \mathscr{F}_n\}$$

is a pseudometric for X. Define $d_n(x, y) = \min\{1, d'_n(x, y)\}$ and let

$$d(x, y) = \sum_{n=1}^{\infty} d_n(x, y) \cdot 2^{-n}.$$

We assert that d is a pseudometric for X that generates $\mathscr{T}$. That is, we will show that $\mathscr{T} = \mathscr{T}_d$ where $\mathscr{T}_d$ is the topology generated by d.

To show $\mathscr{T}_d \subset \mathscr{T}$ it is sufficient to establish the continuity of each d'_n, for then the continuity of d, and the continuity of the identity map $i\colon (X, \mathscr{T}) \to (X, \mathscr{F}_d)$ will follow. So, let (x_0, y_0) and $\varepsilon > 0$ be given. Since $\mathscr{F}_n$ is equicontinuous by Theorem 3, there exist neighborhoods $N(x_0)$ and $N(y_0)$ of x_0 and y_0, respectively, satisfying

$$f(N(x_0)) \subset (f(x_0) - \tfrac{1}{4}\varepsilon, f(x_0) + \tfrac{1}{4}\varepsilon) \quad \text{and} \quad f(N(y_0)) \subset (f(y_0) - \tfrac{1}{4}\varepsilon, f(y_0) + \tfrac{1}{4}\varepsilon)$$

for each $f \in \mathscr{F}_n$. Hence, for every $(x, y) \in N(x_0) \times N(y_0)$ we have

$$\begin{aligned} |d'_n(x, y) - d'_n(x_0, y_0)| &= \big|\sup\{|f(x) - f(y)|\colon f \in \mathscr{F}_n\} - \sup\{|f(x_0) - f(y_0)|\colon f \in \mathscr{F}_n\}\big| \\ &\leqslant \sup\{||f(x) - f(y)| - |f(x_0) - f(y_0)||\colon f \in \mathscr{F}_n\} \\ &\leqslant \sup\{||f(x) - f(x_0)| + |f(y) - f(y_0)||\colon f \in \mathscr{F}_n\} \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus, d'_n is continuous and $\mathscr{T}_d \subset \mathscr{T}$.

To prove that $\mathscr{T} \subset \mathscr{T}_d$, it is sufficient to show that for every n, each $f \in \mathscr{F}_n$ is continuous relative to $\mathscr{T}_d$. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Choose $\delta = \min\{\frac{1}{2}, \frac{1}{2}\varepsilon\}$ and note that if $d(x, x_0) < \delta$, then $d_n(x, x_0) < d(x, x_0) < \delta < 1$. Hence,

$$\begin{aligned} |f(x) - f(x_0)| &\leqslant \sup\{|f(x) - f(x_0)|\colon f \in \mathscr{F}_n\} \\ &= d_n(x, x_0) < \varepsilon. \end{aligned}$$

Thus, f is continuous, and $\mathscr{T} \subset \mathscr{T}_d$. The proof is complete.

Evidently, the pointwise boundedness of $\mathscr{F}$ in Theorem 4 is not important, because Theorem 5 shows that equicontinuity alone is sufficient.

THEOREM 5. *A topological space X is pseudometrizable iff X has the weak topology induced by a σ-equicontinuous collection.*

Proof. Suppose X has the weak topology induced by a σ-equicontinuous collection $\bigcup\{f_{n\alpha}\colon \alpha \in A_n\}$. Let $h\colon R \to (0, 1)$ be any nonexpansive homeomorphism and note that each $\{h \circ f_{n\alpha}\colon \alpha \in A_n\}$ is equicontinuous, pointwise bounded, and hence relatively complete by Theorem 1. Since the weak topology induced by $\bigcup\{h \circ f_{n\alpha}\colon \alpha \in A_n\}$ coincides with the weak topology induced by $\bigcup\{f_{n\alpha}\colon \alpha \in A_n\}$, the result follows from Theorem 4.

COROLLARY. *A T_0-space X is metrizable iff X has the weak topology induced by a σ-equicontinuous collection.*

We recall two definitions before stating the next theorem. A subset A of a space X is a *cozero set* if there exists a continuous real-valued function $f\colon X \to R$ such that $A = f^{-1}(R - \{0\})$. A collection $\mathscr{B} = \{\mathscr{B}_\alpha | \alpha \in A\}$ is *hereditarily closure-preserving* if $\{C_\alpha | \alpha \in A\}$ is closure-preserving whenever $C_\alpha \subset B_\alpha$ for every α.

THEOREM 6. *Let $\mathscr{F}$ be a family of continuous functions from a space X into* $[0, 1]$. *If the cozero sets of the functions in $\mathscr{F}$ form a hereditarily closure-preserving collection, then $\mathscr{F}$ is relatively complete.*

Proof. Let $\mathscr{G} = \{f_\alpha\colon \alpha \in B\}$ be a subset of $\mathscr{F}$. If B is finite, we are done. So assume $|B| \geqslant \aleph_0$. Let $F = \sup\{f_\alpha\colon \alpha \in B\}$ and let G_α be the cozero set of f_α. Let $x_0 \in X$. Clearly, $F(x_0)$ is defined since $f_\alpha(x_0) \in [0, 1]$ for each α. For every $\varepsilon > 0$, there exists $f_\alpha \in \mathscr{G}$ such that $F(x_0) - f_\alpha(x_0) < \frac{1}{2}\varepsilon$. But f_α is continuous at x_0 so there exists a nbd U of x_0 such that $|f_\alpha(y) - f_\alpha(x_0)| < \frac{1}{2}\varepsilon$ for each $y \in U$. Hence, $F(y) \geqslant F(x_0) - \varepsilon$. Define $D_\alpha = f_\alpha^{-1}([F(x_0) + \varepsilon, 1])$. Then D_α is a closed subset of G_α and $\bigcup_{\alpha \in B} D_\alpha$ is closed. Let $V = X - \bigcup_{\alpha \in B} D_\alpha$. Then $x_0 \in V$ and if $y \in V$, $f_\alpha(y) < F(x_0) + \varepsilon$ for each α, so $F(y) \leqslant F(x_0) + \varepsilon$. Thus, $U \cap V$ is a nbd of x_0 such that

$$F(U \cap V) \subset (F(x_0) - \varepsilon, F(x_0) + \varepsilon).$$

Now we will show that $f = \inf\{f_\alpha\colon \alpha \in B\}$ is also continuous. Clearly, $f(x_0)$ is defined for each $x_0 \in X$. Let $\varepsilon > 0$ be given. If $f(x_0) = 0$, then there exists $f_\beta \in G$ such that $f_\beta(x_0) < \frac{1}{2}\varepsilon$, and since f_β is continuous at x_0 there exists a nbd U of x_0 such that for each $y \in U$, $|f_\beta(y)| < \frac{1}{2}\varepsilon$. Thus, $f(y) < \varepsilon$ for each $y \in U$, so f is continuous at x_0.

Suppose then that $f(x_0) \neq 0$. Then $x_0 \in \bigcap_\alpha G_\alpha$, for otherwise $f(x_0) = 0$. Let $L_n = X - \overline{\bigcup f_\alpha^{-1}((0, f(x_0) - 1/n))}$ for $n = 1, 2, \ldots$ Now $x_0 \in L_n$ for each n since $x_0 \notin \overline{f_\alpha^{-1}((0, f(x) - 1/n))} \subset G_\alpha$ for each α. So for each n, choose $f_{\alpha(n)}$ such that $f_{\alpha(n)}(x_0) < f(x_0) + 1/n$, and define $U_n = f_{\alpha(n)}^{-1}((f(x_0) - 1/n, f(x_0) + 1/n))$. Then $x_0 \in U_n$ for every n. Well-order B and define

$$H_\alpha = \begin{cases} G_\alpha \cap U_\alpha \cap L_\alpha & \text{for} \quad \alpha < w, \\ G_\alpha & \text{for} \quad \alpha \geqslant w. \end{cases}$$

Now let $K_\alpha = H_\alpha - H_{\alpha+1}$ and note that $\bigcup_{\alpha \in B} K_\alpha = H_1 - \bigcap_{\alpha \in B} H_\alpha$, and also that $x_0 \in \bigcap_{\alpha \in B} H_\alpha$. Suppose $y \neq x_0$ and $y \in \bigcap_{\alpha \in B} H_\alpha$. We have that $y \in \bigcap_n L_n$ so $f_\alpha(y) \geqslant f(x_0)$ for each α and $f(y) \geqslant f(x_0)$. Also, $y \in \bigcap_n U_n$, so $f(y) \leqslant f(x_0)$. Thus, $f(y) = f(x_0)$. We assert that $\bigcap_\alpha H_\alpha$ is a nbd of x_0. Suppose, to the contrary, that $\bigcap_{\alpha \in B} H_\alpha$ is not a nbd of x_0. Then $x_0 \in \overline{\bigcup_\alpha K_\alpha}$ but $x_0 \notin \overline{K}_\alpha$ for each α since $H_{\alpha+1}$ is a nbd of x_0 which misses K_α. This contradicts the hereditary closure-preserving property since $K_\alpha \subset G_\alpha$. Hence, $\bigcap_{\alpha \in B} H_\alpha$ is a nbd of x_0 and $f(\bigcap_{\alpha \in B} H_\alpha) = f(x_0)$. This completes the proof.

COROLLARY (Burke, Engelking, Lutzer). *A regular space is metrizable iff it has a σ-hereditarily closure preserving base.*

Proof. The necessity of the condition is obvious. For the sufficiency, the space is clearly an M_1-space and thus perfectly normal. For each basic open set in σ-hereditarily closure preserving base there exists a continuous function into $[0, 1]$ which is nonzero precisely on the basic open set. By Theorem 6, the functions form a σ-relatively complete family and hence the space is metrizable by Theorem 4.

Example A shows that the converse of Theorem 6 fails.

Finally, we will investigate conditions under which a space is paracompact. In [7], Morita lists several conditions for a cover of a space X to be a normal cover. By definition, a *normal sequence* in X is a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of X such that $\mathscr{U}_{n+1}$ star-refines $\mathscr{U}_n$ for $n = 1, 2, \ldots$, and any open cover of X which is $\mathscr{U}_1$ in some normal sequence in X is called a *normal cover*. If $\mathscr{G}$ is an open cover of X, a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers is called *locally starring* for $\mathscr{G}$ if for each $x \in X$, there exists a nbd $V(x)$ and an integer n such that $\mathrm{St}(V, \mathscr{U}_n) \subset \mathscr{G}$ for some $G \in \mathscr{G}$. The next theorem extends Morita's Theorem 1.2.

THEOREM 7. *Let $\mathscr{G}$ be an open cover of a topological space (X, τ). Then the following statements are equivalent*:

(1) *$\mathscr{G}$ is a normal cover.*

(2) *There is a continuous mapping f from X into a metric space Y such that $\mathscr{G}$ is refined by the inverse image of some open cover of Y.*

(3) *$\mathscr{G}$ has a partition of unity $\mathscr{F}$ subordinated to it.*

(4) *There is a σ-relatively complete collection $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$ and a refinement of $\mathscr{G}$ consisting of sets of the form $f_{n\alpha}^{-1}(U_{n\alpha})$ where $U_{n\alpha}$ is open in R.*

(5) *There is a σ-equicontinuous collection $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$ and a refinement of $\mathscr{G}$ consisting of sets of the form $f_{n\alpha}^{-1}(U_{n\alpha})$ where $U_{n\alpha}$ is open in R.*

(6) *There is a σ-evenly continuous collection $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$ and a refinement of $\mathscr{G}$ consisting of sets of the form $f_{n\alpha}^{-1}(U_{n\alpha})$ where $U_{n\alpha}$ is open in R.*

Proof. (1)→(2)→(3) is part of Theorem 2.1 of [7]. (3)→(4) follows immediately from Theorem 2 and the definition of $\mathscr{F}$ being subordinated to $\mathscr{G}$. (4)→(5)→(6) also follows from Theorem 2. (6)→(1): Let $\mathscr{G}$ be an open cover of (X, τ) and let $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$ be an evenly continuous collection with the given refinement. Let τ_d denote the weak topology induced by $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$. Since Theorem 5 is obviously true when equicontinuous families are replaced by evenly continuous ones, (X, τ_d) is pseudometrizable with pseudometric d, and hence the collections of balls $\mathscr{B}_n = \{B(x, 1/n): x \in X\}$ of radii $1/n$ are locally starring for each τ_d-open cover. Note that each $\mathscr{B}_n$ is also a τ-open cover since (X, τ_d) has the weak topology. Using the technique of Arhangel'skiĭ [2, Theorem 3.7], we can now produce a normal cover that star-refines $\mathscr{G}$, and hence $\mathscr{G}$ is a normal cover.

The final theorem, which extends Theorem 9 of [3], is now a trivial consequence of A. H. Stone's characterization of paracompact spaces as fully normal spaces applied to Theorem 7.

THEOREM 8. *For a T_1-space X, the following are equivalent*:

(1) *X is paracompact.*

(2) *Given an open cover $\mathscr{G}$ of X, there is a partition of unity subordinated to it.*

(3) *Given an open cover $\mathscr{G}$ of X, there is a σ-equicontinuous (σ-relatively complete, σ-evenly continuous) collection $\bigcup_n \{f_{n\alpha}: \alpha \in A_n\}$ and a refinement of $\mathscr{G}$ consisting of sets of the form $f_{n\alpha}^{-1}(U_{n\alpha})$ where $U_{n\alpha}$ is open in R.*

References

[1] D. Burke, R. Engelking and D. Lutzer, *Hereditarily closure-preserving collections and metrization*, Proc. Amer. Math. Soc. 51 (1975), pp. 483–488.

[2] J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass. 1966.

[3] J. A. Guthrie and M. Henry, *Metrization, paracompactness, and real-valued functions*, Fund. Math. 95 (1977), pp. 49–53.

[4] — *Metrization and paracompactness in terms of real functions*, Bull. Amer. Math. Soc. 80 (1974), p. 720.

[5] J. L. Kelley, *General Topology*, Van Nostrand, Princeton 1955.
[6] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. 4 (1953), pp. 831–838.
[7] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.
[8] J. Nagata, *A contribution to the theory of metrization*, J. Inst. Polytech., Osaka City University, 8 (1957), pp. 185–192.

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Intersections of ANR's

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Abstract. Let $\{F_i\}$ be a sequence of compact ANR's such that each F_i is a retract of F_{i+1}. K. Borsuk conjectured that the intersection of this collection is a fundamental ANR. In this note, algebraic conditions which imply this conjecture are obtained. For example, the conjecture is verified if the fundamental group of some F_i is Abelian or finite. A partial converse is obtained if some F_i has the homotopy type of a 2-complex.

Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of compact ANR's such that $F_i \supseteq F_{i+1}$ and there exist retractions $r_i\colon F_i \to F_{i+1}$ for all i. If F_1 is contractible, then Borsuk [2] has shown that $\bigcap F_i$ is a fundamental AR and conjectured that, in general, $\bigcap F_i$ is a fundamental ANR. In this note we attempt to reduce this conjecture to an algebraic problem and we solve the algebraic problem in many cases. If F is a compact ANR, then F *satisfies* FIR if whenever $\{F_i\}_{i=1}^{\infty}$ is a sequence of subspaces of F with $F_1 = F$ and there exist retractions $r_i\colon F_i \to F_{i+1}$ for all i, then $\bigcap F_i$ is a fundamental ANR. If G is a group (R-module), then G *satisfies* FIR if whenever $\{G_i\}_{i=1}^{\infty}$ is a sequence of subgroups (submodules) of G with $G_1 = G$ and there exist retractions $r_i\colon G_i \to G_{i+1}$ for all i, then there exists n such that for all $i \geqslant n$, $G_i = G_n$.

THEOREM 1. *Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of compact connected* ANR'*s such that $F_i \supseteq F_{i+1}$ and there exist retractions $r_i\colon F_i \to F_{i+1}$ for all i. Let $x \in F_i$. The following are equivalent.*

(A) $(\bigcap F_i, x)$ *is a pointed fundamental* ANR [3].

(B) *For each $j \geqslant 1$, the induced system of groups $\{\pi_j(F_i, x)\}_{i=1}^{\infty}$ is equivalent in the category of pro-groups to a group.*

(C) *For each $j \geqslant 1$, there exists n_j such that if $i > n_j$, then the inclusion induced homomorphism $\pi_j(F_{i+1}, x) \to \pi_j(F_i, x)$ is an isomorphism.*

Proof. By West [12], F_1 has the homotopy type of a finite n-dimensional complex. By Theorem F of Wall [11], each F_i has the homotopy type of a finite complex of dimension $\leqslant \max\{3, n\}$. It follows that F_i has finite fundamental dimen-

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