Periodic homeomorphisms on T-like continua

by

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Abstract. Suppose that T is a tree with exactly k end points and there is no element in the group G of permutations on k elements with order n. We show that if the T-like continuum admits a period n homeomorphism then it contains an indecomposable continuum. We give a construction for a period $2i$ homeomorphism for each positive integer i so that there is an element in G of order i.

I. The results in this paper stem from our attempt to answer the following question: Does there exist an arc-like continuum which admits a period n homeomorphism? The answer is affirmative in case $n = 2$ or $n = 4$ and remains unknown otherwise [1] (1). We prove a general result about T-like continua which implies as a special case that any arc-like continuum which admits a period n homeomorphism $n>2$ has to contain an indecomposable continuum. For definitions see [3].

Throughout this paper the term continuum is used to mean a compact connected metric space. A monotone decomposition of a continuum M is an upper semi-continuous decomposition of M into continua. If M is a continuum and G is an upper semi-continuous decomposition of M then G^*/G denotes the decomposition space. For theorems concerning upper semi-continuous decompositions the reader should consult [4]. A tree is a locally connected continuum irreducible about a finite set of points.

DEFINITION. If M is a continuum, f is a homeomorphism of M onto itself, and n is an integer greater than 1; then f is a *period n homeomorphism* means that n is the smallest integer k such that f^k is the identity.

DEFINITION. Suppose that M is a continuum and G is a monotone decomposition of M. Then G is said to be a *minimal monotone decomposition of M into a tree* if it is true that G^*/G is a tree and if J is a monotone decomposition of M so that J^*/J is a tree then every element of G is a subset of some element of J.

The following theorems are used in this paper; Theorems A and B are well known, Theorem C is due to Mary Russell [5].

(1) Wayne Lewis has recently shown [Notices AMS, V. 26, No. 1 (Jan 1979), p. A 124] that for each prime p the pseudoarc admits a period p homeomorphism.

THEOREM A. *If I is a tree then I has the fixed point property.*

THEOREM B. *If M is a continuum, G is an upper semi-continuous decomposition of M and h is a homeomorphism of M onto itself then* $H = \{h(g)|\ g \in G\}$ *is an upper semi-continuous decomposition of M.*

THEOREM C. *If T is a tree irreducible about k end points and M is a hereditarily decomposable T-like continuum then there exists a minimal monotone decomposition G of M into a tree so that* G^*/G *is a tree irreducible about k or fewer end points.*

II.

LEMMA 1. *Suppose M is a continuum, G is a minimal monotone decomposition of M into a tree and h is a homeomorphism of M onto itself. Then if* $g \in G$ $h(g) = \{h(x)|\ x \in g\}$ *is an element of G and h induces a homeomorphism* h^* *of* G^*/G *onto itself where* $h^*(g) = \{h(x)|\ x \in g\}$ *for each* $g \in G$.

Proof. The collection $H = \{h(g)|\ g \in G\}$ is a monotone decomposition of M. Further the decomposition space H^*/H is a tree because (1) if g is not an end element of G then $h(g)$ separates H^*/H and (2) H^*/H is arcwise connected. Let $g \in G$ and let I be an element of G which intersects $h(g)$. Since G is minimal $I \subseteq h(g)$. Now h^{-1} is also a homeomorphism, so the collection $H^{-1} = \{h^{-1}(g)|\ g \in G\}$ is also a monotone decomposition of M into a tree. But $I \subseteq h(g)$ so $h^{-1}(I) \subseteq g$; and since G is minimal and $h^{-1}(I)$ is an element of H^{-1} intersecting g then $g \subseteq h^{-1}(I)$. Thus $g = h^{-1}(I)$ and so $h(g) = I$. So for each $g \in G$, $h(g)$ is an element of G. Similarly for each $g \in G$, $h^{-1}(g)$ is an element of G. Thus h^* is 1-1, the continuity of h^* follows from the continuity of h. Therefore h^* is a homeomorphism of G^*/G onto itself.

LEMMA 2 (Hamilton). *Suppose that T is a tree irreducible about k end points, M is a hereditarily decomposable T-like continuum, and h is a homeomorphism of M onto itself. Then M has a fixed point.*

Proof. Let G be the collection to which the subcontinuum I of M belongs if and only if $h(I) = I$. Let J be a maximal monotonic subcollection of G and let $L = \bigcap \{j|\ j \in J\}$. Then $h(L) = L$. Suppose that L is non-degenerate. Thus by Corollaries 2.5 and 3.1 of [5] there exists a minimal monotonic decomposition W of L into a tree. From Theorem A and Lemma 1 it follows that there is an element $g \in W$ so that $h^*(g) = g$. But then $g \in J$ and g is a proper subset of L which is a contradiction. Thus the lemma is proven.

DEFINITION. If k is a positive integer, then $p(k)$ denotes the set to which i belongs if and only if i is an integer greater than one and the symmetric group of permutations on k objects contains an element of order i.

The following two lemmas follow easily from properties of trees and homeomorphisms of trees.

LEMMA 3. *If I is an arc with end points p and q, and h is a period n homeomorphism on I, then* $n = 2$, $h(p) = q$ *and* $h(q) = p$.

LEMMA 4. *If T is a tree with exactly k end points and h is a period n homeomorphism on T, then* $n \in p(k)$.

THEOREM. *Suppose that T is a tree irreducible about k end points, M is a T-like continuum, n is an integer greater than 1 which is not in* $p(k)$, *and h is a period n homeomorphism on M. Then M contains an indecomposable continuum.*

Proof. Suppose that n is an integer greater than 1, $n \notin p(k)$, M is hereditarily decomposable, and h is a period n homeomorphism on M. Let G be a minimal monotone decomposition of M into a tree (Theorem C). Then by Lemma 1 h induces a homeomorphism h^* of G^*/G onto itself. Now h^n is the identity so $h^{*n}(g) = g$ for all $g \in G$. But by Lemma 4 h^* cannot be a period n homeomorphism. So there exists an integer t in $p(k)$ which divides n such that h^* is a period t homeomorphism. Therefore if $g \in G$, $h^t(g) = g$. If $g \in G$ then by Lemma 2 there is a fixed point p_g in g of the homeomorphism h^t. Let $W = \{p|\ h^t(p) = p\}$, then W is closed. The set W cannot be connected for if it were it would be a subcontinuum of M which intersects each of the end continua of G and hence must be M. But if $W = M$ then h^t is the identity which is a contradiction.

Thus W is the union of two mutually exclusive closed point sets H and K. There exists a subcontinuum I of M which is irreducible from H to K. Let $P \in H \cap I$ and $Q \in K \cap I$. Now $h^t(I)$ is a subcontinuum of M irreducible from $h^t(P)$ to $h^t(Q)$, $h^t(P) = P$ and $h^t(Q) = Q$; so by unicoherence $h^t(I) = I$. Let J be the minimal monotone decomposition of I into an arc. Since h^t is a period n/t homeomorphism then $(h^t)^*$ is either the identity or it is a period r homeomorphism for some integer $r \leqslant n/t$. If E_p and E_Q are the end continua of J containing P and Q respectively then $(h^t)^*(E_P) = E_P$ and $(h^t)^*(E_Q) = E_Q$. So by Lemma 3 $(h^t)^*$ is the identity. Thus $(h^t)^*(g) = g$ for all $g \in J$. But then if $g \in J$ there is a fixed point P_g of h^t which lies in g. So W intersects each element of J which contradicts the definition of I. Thus the theorem is proven.

III. Note that for the case of the arc we have $k = 2$, $p(2) = \{2\}$ and thus if $n > 2$ and M is an arc-like continuum which admits a period n homeomorphism, then M contains an indecomposable continuum. It should also be observed that for each $k \geqslant 2$, $\{2, 3, \ldots, k\} \subset p(k) \subset p(k+1)$ but also $6 \in p(5)$ and $30 \in p(10)$.

Using and extending the construction method used in [1], we obtain a class of examples described in the following:

THEOREM. *If* $k \geqslant 2$ *and* $i \in p(k)$, *then there exists a k-od like continuum which admits a period* $2i$ *homeomorphism.*

Proof. Let M be the continuum constructed by joining a collection G of K pseudoarcs at a common point 0. That is, the intersection of any two elements of G is the intersection of all of the elements of G and is $\{0\}$. Since the pseudoarc is homogeneous, each element $g \in G$ is chainable between 0 and some other point of g. Any finite open cover of M can be refined by another open cover the nerve of which is a k-od. Therefore M is k-od like.

If $i \in p(k)$ then there exists an order i permutation φ on k objects. The permutation φ can be expressed as the product of disjoint cycles and i is the least common multiple of the orders of the disjoint cycles. Let A denote a pseudoarc and choose a point $a \in A$. For each $g \in G$, let h_g denote a homeomorphism of A onto g which takes a to 0. Assume that φ permutes the elements of G and defines the homeomorphism $\hat{\varphi}$ induced by φ by $\hat{\varphi}(x) = h_{\varphi(g)}h_g^{-1}(x)$ for all $x \in g$ and $g \in G$.

The homeomorphism $\hat{\varphi}$ is period i. We will now define a period 2 homeomorphism r on M and the product $\hat{\varphi}r$ will be the desired period $2i$ homeomorphism. From each cycle c of φ choose an element g_c, of G which is operated on by c. There may be only one cycle. On each of the elements g_c, r is a period two homeomorphism which fixes only the point 0. On each other element of G, r is the identity.

Now consider $\hat{\varphi}r$. If c is a cycle of order j and $x \in g_c$, then $(\hat{\varphi}r)^j(x) = \hat{\varphi}^j r(x) = r(x)$ and $(\varphi r)^{2j}(x) = r^2(x) = x$. The period of φr must therefore be an even multiple of the order of each cycle of φ. The smallest such integer is $2i$. This completes the proof.

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Accepté par la Rédaction le 15. 11. 1976

Eigentlich operierende Gruppen von Isometrien

by

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Abstract. It is known that the group $I(X)$ of isometries of a locally compact, connected, metric space X is a locally compact (topological) group (Theorem of van Dantzig–van der Waerden) and that the action $(I(X), X)$ is proper. Both properties hold true also if X has a finite number of (connected) components, but neither of them remains still true, in general, if X has infinitely many components.

In this paper a necessary and sufficient condition is given in order to answer the question, when a subgroup of $I(X)$ is a locally compact group, which acts properly on X, allthough X may have infinitely many components. In the proof of the corresponding result is the Theorem of van Dantzig–van der Waerden not used, so that the main result is a strict generalization of this theorem.

As a corollary is shown, that the aforementioned two properties of $I(X)$ are valid, if the space $\bar{Z}(X)$ of the components of X is compact; this assertion does not hold (in general) in the absence of the compactness of $\bar{Z}(X)$. Further, it is indicated how the theories of non-compact proper actions on connected spaces and on spaces with infinitely many components are related.

Einleitung. Fragen über Gruppen von Isometrien lokal-kompakter, metrischer Räume stehen in unmittelbarem Zusammenhang mit der Untersuchung der eigentlichen Transformationsgruppen [16; Kor. 5.2]. Es ist bekannt, daß die Gruppe der Isometrien eines derartigen Raumes mit endlich vielen Zusammenhangskomponenten eine lokal-kompakte (topologische) Gruppe ist, die eigentlich auf diesem Raum operiert [15; Lemma 2]. Durch Beispiele wird gezeigt, daß (im allgemeinen) weder die Gruppe der Isometrien lokal-kompakt ist (vgl. 2.1) noch die lokal-kompakten Untergruppen davon eigentlich auf dem Raum operieren, wenn dieser Raum unendlich viele Zusammenhangskomponenten hat (vgl. 2.2). Die daraus resultierende Frage nach der Untergruppen der Gruppe der Isometrien eines derartigen Raumes, die eigentlich auf diesem Raum operieren (und damit notwendigerweise lokal-kompakt nach 1.3 sind), wird durch eine notwendige und hinreichende Bedingung beantwortet (vgl. 2.4). Folgerung des entsprechenden Satzes, die den Satz von van Dantzig–van der Waerden (echt) verallgemeinert, ist die nachfolgende Aussage:

I. *Sei X ein lokal-kompakter, metrischer Raum mit kompaktem Raum von Zusammenhangskomponenten* $\bar{Z}(X)$; *dann ist die Gruppe der Isometrien* $I(X)$ *von* X *eine lokal-kompakte topologische Gruppe, die eigentlich auf* X *operiert* (*vgl.* 2.5); *das gilt im allgemeinen nicht mehr, wenn* $\bar{Z}(X)$ *nicht kompakt ist.*