Lindenbaum algebras and model companions

by

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Abstract. We study a property of first order theories suggested by Zeitler. It turns out to be stronger than companionability and weaker than companionability plus amalgamation. (A semantic version of this property was communicated to me by Felgner, with additional observations discussed in § 3).

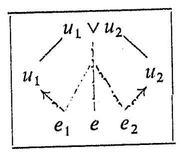
A basic algebraic invariant associated to any theory T is the sequence of Lindenbaum algebras $\{B_n T\colon n = 0, 1, \ldots\}$, where $B_n T$ is the Boolean algebra of formulas $\varphi(x_1, \ldots, x_n)$ in n free variables modulo the equivalence relation:

$$\varphi_1 \equiv \varphi_2 \quad \text{iff } T \vdash \varphi_1 \equiv \varphi_2 .$$

As is well known, many important model-theoretic properties of T are equivalent to structural properties of the corresponding Lindenbaum algebras; for example T is complete iff $B_0 T = 2$ (the Boolean algebra of order 2), and T is $\aleph_0$-categorical iff $B_n T$ is finite for all n (assuming T is complete).

In the theory of model companions and model completions one considers the algebras $B_n T$ with the sublattices E_{1n} of existential formulas as distinguished subsets (notice that we systematically confuse formulas with elements of $B_n T$). More generally we might wish to distinguish the sublattices E_{kn} of formulas φ of type E_k in $B_n T$ (a formula is of type E_k iff it is in prenex normal form with k alternations of quantifiers, and the first quantifier is existential). When the danger of confusion is reasonably slight we will write E_k rather than E_{kn}, and BT rather than $B_n T$. Notice that if we distinguish E_k then we have also distinguished the dual class $A_k = \{-e\colon e \in E_k\}$.

Numerous properties of theories T may be phrased in terms of the way the sublattices E_{1n} sit inside the algebras B_n. For example, T has the amalgamation property if for all $e \in E_1$, and $u_1, u_2 \in A_1$, if $e \leqslant u_1 \vee u_2$ then there are $e_1, e_2 \in E_1$ such that $e \leqslant e_1 \vee e_2$, $e_1 \leqslant u_1$, $e_2 \leqslant u_2$.



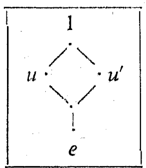
Amalgamation

Similarly, Robinson's test for model completeness says that the following are equivalent:

1. T is model complete.
2. $E_1 = A_1$.
3. $E_1 = BT$ (all n).
4. $A_1 = BT$.

On the final page of [1] the following condition for companionability (i.e. existence of a model companion) is given: a theory T has a model companion if every $u \in A_1$ has a weak complement $u' \in A_1$ satisfying:

1. $u \vee u' = 1$.
2. For $e \in E_1$, if $e \leqslant u \wedge u'$ then $e = 0$.



$e \in E_1 \Rightarrow e = 0$

Companionability

As is well known, the existence of a model completion is equivalent to the conjunction of amalgamation and companionability.

The subject matter of the present article is a structural property of $\langle BT, E_1\rangle$ which is stronger than companionability but does not imply amalgamation:

Definition. A theory T has *property* F iff for each $u \in A_1$ the set of $e \in E_1$ such that $e \leqslant u$ has a maximal element.

This property arose in connection with an unpublished result of Felgner concerning a possible variant of Łoś' theorem on ultraproducts proposed by Zeitler. We have included a discussion of this result in § 3, based on a handwritten note of Felgner, whom I thank for stimulating conversation.

Property F is analyzed in §1 from various points of view, and examples are given in § 2.

§ 1. Property F

Definition 1.1. The theory T has *property* FP (where $P \in B_nT$) iff the set of existential formulas $e \leqslant P$ has a greatest element.

We give a semantical version of this property as well.

Definition 1.2. The theory T has *property* GP ($P \in B_nT$) iff

For every set $\{M_i\colon i \in I\}$ of models of T, for every ultrafilter D over I, and for all $\bar{y} = y_1, \ldots, y_n$ in $\prod M_i$, *if* $T \cup \mathrm{Diag}(M_i) \vdash P(\bar{y}(i))$ for almost all i *then* $T \cup \mathrm{Diag}(\prod M_i/D) \vdash P(\bar{y})$.

(Diag denotes the Robinson diagram, $\bar{y}(i) = y_1(i), \ldots, y_n(i)$ and we systematically confuse elements of $\prod M_i$ with elements of $\prod M_i/D$.)

Proposition 1.3. *The following are equivalent*:

1. *T has GP.*
2. *T has FP.*

Proof. We first show that 2⇒1. Assume therefore that T has FP, but that T does not have GP, so that there are models $\{M_i\colon i \in I\}$, an ultrafilter D on I, and functions $y_1, \ldots, y_n$ in $\prod M_i$ such that

$$T \cup \mathrm{Diag}(M_i) \vdash P(\bar{y}(i))$$

but $T \cup \mathrm{Diag}(\prod M_i/D) \cup \{\neg P(\bar{y})\}$ is consistent. Then there are existential sentences $e_i(\bar{y}(i))$ satisfied in M_i such that $T \vdash \forall \bar{x}(e_i(\bar{x}) \Rightarrow P(\bar{x}))$, in other words $e_i \leqslant P$ in B_nT. Applying property F we have an existential formula $e_0 \leqslant P$ in B_nT greater than all the formulas e_i. Thus $M_i \models e_0(\bar{y}(i))$ for almost all i, and therefore $\prod M_i/D \models e_0(\bar{y})$. Since $e_0 \leqslant P$ this contradicts the consistency of $T \cup \mathrm{Diag}(\prod M_i/D) \cup \{\neg P\}$.

Consider now the direction 1⇒2. We assume that T has property GP but not FP, and from the nonexistence of a largest existential formula below P we will construct a particular ultraproduct violating the conditions given by GP. Take as the index set I the set of all existential formulas e below P. For each $e \in I$ choose M_e satisfying $T \cup \neg e(\bar{y}(e))$ and some $e'(\bar{y}(e)) \in I$. It suffices to find an ultrafilter D on I such that $T \cup \mathrm{Diag}(\prod_I M_e/D)$ does not prove $P(\bar{y})$.

It suffices to let D be any ultrafilter extending the filter F generated by the sets $C_e = \{e'\colon M_e \vdash e'(\bar{y}(e))\}$. We need merely show that F is a proper filter. Indeed, if $e_1, \ldots, e_n \in I$ let $e = \bigvee_i e_i'$; then $e \in \bigcap C_{e_i}$.

We shall not need the semantical characterization of property F until the end of § 2. Our immediate concern is the relationship of property F to the theory of model completions and model companions.

Theorem 1.4. *Consider the following conditions on a theory T*:

1. *T has a model completion T^*.*
2. *T has property F.*
3. *T has a model companion.*

Then 1⇒2⇒3.

Proof. It is convenient (though not at all necessary) to work with the syntactic characterizations of 1, 3 given in the introduction.

2⇒3. Let $u \in A_1$ be given, and let $u' = -e$, where e is the largest existential formula below u. Then clearly $u \vee u' = 1$, and we must show that for any $e' \leqslant u \wedge u'$, if $e' \in E_1$ then $e' = 0$. This is clear: if $e' \leqslant u$ then $e' \leqslant e$. If also $e' \leqslant -e$ then $e' = 0$.

1⇒2. Let u be a universal formula. We assume that T has a model completion, i.e., T is companionable and has the amalgamation property. Let u' be the weak complement of u afforded by the companionability condition. Then $1 \leqslant u \vee u'$, so the amalgamation property yields existential formulas e, e' satisfying: $e \leqslant u$, $e' \leqslant u'$, $1 \leqslant e \vee e'$. Then $e \wedge e' \leqslant u \wedge u'$, so $e \wedge e' = 0$. Thus $e' = -e$. Now if $e_1 \in E_1$ and $e_1 \leqslant u$: then $e_1 \wedge e' \leqslant u'$, so $e_1 \wedge e' = 0$. Since $e' = -e$, $e_1 \leqslant e$, as desired.

COROLLARY 1.5 1. *T has a model completion iff T has the amalgamation property and the property F.*

2. *T is $\aleph_0$-categorical ⇒ the lattice E_{1n} is finite (all n) ⇒ T has property **FP** for all **P**⇒T has property F⇒T has a model companion.*

3. *If T is inductive and complete then T is model-complete iff T has property F.*

Remarks.

1. Statements 1,3 become obvious if "property F" is weakened to companionability.

2. The theorem: "If T is $\aleph_0$-categorical then T has a model companion" is Saracino's [3]. Statement 2 is intended to elucidate the syntactical content of this result.

Recently Weispfenning observed that $\aleph_0$-categoricity also implies that the class of amalgamation bases (pregeneric structures) is elementary (private communication). This fact is also a consequence of the weaker assumption that T has property F. To see this let e, e_1, e_2; u_1, u_2 vary over existential and universal formulas respectively and set

$$\varrho_n = \bigwedge_{T \vdash e \Rightarrow u_1 \vee u_2} \forall \bar{y}(e(\bar{y}) \Rightarrow \bigvee_{\substack{T \vdash e_1 \Rightarrow u_1 \\ T \vdash e_2 \Rightarrow u_2}} e_1 \vee e_2(\bar{y})) .$$

The amalgamation bases are always just the models of $\{\varrho_n\}$ (cf. our remarks on amalgamation in the introduction), and if T has property F then this produces a first order axiomatization of the class of amalgamation bases (however it is possible for the amalgamation bases to be first order axiomatizable even if some of the conjuncts of ϱ_n are not equivalent to a set of first order sentences).

We believe that the following concept merits further attention:

DEFINITION 1.6. The theory T is *finite at level k* iff for all n the lattice A_{kn} is finite. (A weaker notion, of possible interest because it implies $\forall \boldsymbol{P}\ \boldsymbol{FP}$ if $k = 1$, is the following: for all n A_{kn} is wellfounded.)

There are a number of open questions connected with the following problems:

1. Give semantic equivalents for all the syntactical properties we have considered.

2. What implications hold among these various properties?

It is likely that Problem 2 reduces at this point to the study of pathology. We will give enough examples in the next section to clarify most of the relationships among these concepts, but we still do not know the following:

1. Is there a theory finite at every level which is not ω-categorical?

2. If T has property Fu for all universal u, does it follow that T has property $\boldsymbol{FP}$ for all $\boldsymbol{P}$? In particular does the theory of fields have property $\boldsymbol{FP}$ for all $\boldsymbol{P}$?

§ 2. Some examples. For open problems see the last paragraphs of the preceding section. We will give examples clarifying the relationships among the following notions:

> ω-categoricity, finiteness at level k, $\forall \boldsymbol{P}\ \boldsymbol{FP}$, $\forall u\ Fu$, existence of a model companion or a model completion, and definability of the class of amalgamation bases.

In connection with property F we will also look briefly at a similar property F', for reasons explained in § 3.

EXAMPLE 2.1. We show by examples that the implications in Theorem 1.4 are irreversible.

1. The theory of formally real fields has a model companion but does not satisfy $\forall u Fu$ (u universal). ($u(x)$ = "x is not a square",)

2. A theory T satisfying $\forall u Fu$ and having no model completion (and either ω-categorical or inductive) is given by the theory, or the A_2 part of the theory, of an equivalence relation E consisting of infinitely many infinite classes, together with unary predicates P, Π, both infinite, such that with precisely one exception every equivalence class contains a single point of $P \cup \Pi$ — the exceptional class is to contain no point of $P \cup \Pi$. The failure of amalgamation is evident.

EXAMPLE 2.2. We will give a theory T finite at level 2 and having a model completion but not finite at level 3.

Let X_i be a dense linear ordering without endpoints and let Y_i be a linear ordering consisting of a dense set of gaps. Form the linear ordering $S = X_1 Y_1 X_2 Y_2 \ldots$ The natural formalization of the statement "y is in $\bigcup_{k \geqslant n} (X_k \cup Y_k)$" is an E_3 formula of $B_1 T$. Thus T is infinite at level 3. On the other hand inspection of E_2 formulas in $B_n T$ reveals that they assert only the existence of at least one gap which is either located in an open interval determined by the variables $y_1, \ldots, y_n$ or which has one of these points as an endpoint (one may form propositional combinations of such assertions). Thus T is finite at level 2.

EXAMPLE 2.3. Concerning definability of the pregenerics it follows from results of Hirschfeld that the class of pregenerics for complete arithmetic is not elementary, since ultrapowers of existentially complete models are not pregeneric (see [2]).

A simpler example with many nice properties is the following: Let T be the theory of (undirected) graphs without cycles. Then the pregenerics are just the connected models of T and the existentially complete models of T are the connected models with infinite branching at each vertex. In particular: the classes of pregenerics and generics are both nonelementary, the classes of existentially complete, finitely generic, and infinitely generic models cleancide (so that the forcing companion T^* is unambiguously defined), the models of T^* are disjoint unions of existentially complete models, T^* is ω-stable and inductive but has no model-companion, $B_1 T^* = 2$ but there is a universal formula in $B_2 T^*$ which is a limit from below of existential formulas, and the class of existentially complete models is ω-categorical (it can be made categorical in all powers by expanding the language to include a transitive family of automorphisms).

EXAMPLE 2.4. We say the theory T has property $G'\boldsymbol{P}$ ($\boldsymbol{P} \in B_n T$) iff:

for all models M of T, for all ultrafilters D on an index set I, and for all $\bar{y} \in M^I$ if $T \cup \mathrm{Diag}(M) \vdash \boldsymbol{P}(\bar{y}(i))$ for almost all i then $T \cup \mathrm{Diag}(M^I/D) \vdash \boldsymbol{P}(\bar{y})$.

The analog of $\boldsymbol{FP}$ is as follows. Any extension T' of T induces a family of homomorphisms h_n: $B_n T \to B_n T'$. Call such a homomorphism a *completion homomorphism* iff it is induced by a complete extension of T (e.g. a completion homomorphism of $B_0 T$ is just any homomorphism h: $B_0 T \to 2$). We say that a theory T has *property* $F'\boldsymbol{P}$ ($\boldsymbol{P} \in B_n T$) iff for any completion homomorphism h: $B_n T \to B_n T'$ the lattice $h[E_1 \cap \{x\colon x \leqslant \boldsymbol{P}\}]$ has a greatest element. Then as the reader may verify, $G'\boldsymbol{P}$ is equivalent to $F'\boldsymbol{P}$.

We will now define a class of theories satisfying $\forall \boldsymbol{P} F'\boldsymbol{P}$ and not $\mathrm{A}u\, Fu$ (u universal). Some of these theories have model companions, some do not.

Let $\varkappa$, λ be ordinals and let $\{\Pi_{\alpha\beta}\colon \alpha < \varkappa, \beta < \lambda\}$ be unary predicates. Let P be another unary predicate.

Take as axioms:

1. $\Pi_{\alpha\beta} x \Rightarrow \Pi_{\alpha\beta'} y$ if $\beta \leqslant \beta'$.
2. $\Pi_{\alpha\beta} x \Rightarrow \neg \Pi_{\alpha'\beta'} y$ if $\alpha' \neq \alpha$.
3. $\Pi_{\alpha\beta} x \Rightarrow Py$.

Then $T = \{1, 2, 3\}$ has property $F'\boldsymbol{P}$ for all $\boldsymbol{P}$ but fails to have $F(\forall x Px)$ if $\varkappa$ is a limit. There is a model companion if λ is finite, and otherwise there is not.

EXAMPLE 2.5. In our last example we will have $\forall \boldsymbol{P} \boldsymbol{FP}$, but T is not finite at level 1.

We take $T = \mathrm{Th}(Z, \leqslant)$. The existential assertions are limited to statements concerning the cardinalities of the intervals determined by the parameters $y_1, \ldots, y_n$, and hence the lattices A_{1n} are all wellfounded, which implies $\forall \boldsymbol{P} \boldsymbol{FP}$. But E_{11} is already infinite.

§ 3. Felgner's problem. We begin by stating an inelegant version of Łoś' Theorem:

THEOREM. *Let $\underline{M} = M^I/D$ be an ultrapower of a model of the theory T, $\boldsymbol{P}$ a formula in the language of T, $\bar{y} \in \underline{M}$. Then:*

if $\mathrm{Th}(M) \vdash \boldsymbol{P}(\bar{y}(i))$ *for almost all i, then* $\mathrm{Th}(\underline{M}) \vdash \boldsymbol{P}(\bar{y})$.

If T is a model complete theory then $\mathrm{Th}(\underline{M})$ is equivalent to $T \cup \mathrm{Diag}(\underline{M})$. Hence we consider the following, valid for model complete theories by Łoś' Theorem:

POTENTIAL THEOREM. *Let $\underline{M} = M^I/D$ be an ultrapower of a model of the theory T, $\boldsymbol{P}$ a formula in the language of T, $\bar{y} \in \underline{M}$. Then:*

if $T \cup \mathrm{Diag}(M) \vdash \boldsymbol{P}(\bar{y}(i))$ *for almost all i, then* $T \cup \mathrm{Diag}((\underline{M}) \vdash \boldsymbol{P}(\bar{y}))$.

In 2 we said that T has *property $G'\boldsymbol{P}$* iff our potential theorem is true for T, $\boldsymbol{P}$, and all appropriate M, I, D, $\bar{y}$. We also indicated an equivalent syntactical condition $F'\boldsymbol{P}$. It is clear that for complete theories T $F'\boldsymbol{P}$ is equivalent to $\boldsymbol{FP}$.

The starting point for this work was the following result of Felgner (unpublished):

THEOREM. *Let T be the theory of infinitely generic division rings of characteristic zero. Then T provides an example of a complete theory such that for some universal formula u, T does not have property $G'u$, and the formula u is a minimal example in the following sense: any theory clearly has property $G'e$ for all existential e whereas u can be taken to have one free variable, one bound variable, and an atomic matrix — namely we use the formula $\forall x(xy = yx)$.*

(Proof. The center of an infinitely generic division ring of characteristic zero is just the prime field Π, from which the claim follows. The completeness of T is well known.)

One of the consequences of our general considerations above is that any complete theory without a model companion will fail to have property $G'u$ for some (possibly quite complex) universal u.

References

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[2] J. Hirschfeld and W. Wheeler, *Forcing, arithmetic, and division rings*, Lecture Notes in Mathematics 454, Springer Verlag 1975.

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