Generalized paths and pointed 1-movability

by

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Abstract. In this note we present a characterization of pointed 1-movability and give some of its applications.

It is still unknown whether movable continua must be pointed movable. In case of compacta the answer is negative. It is known that for contina this problem reduces to the problem whether movability implies pointed 1-movability. Therefore investigations of pointed 1-movability seem to be interesting and important. In this note we generalize, in the spirit of shape theory, the notion of a path in a space. It turns out that the "pathwise connectedness" in this new sense characterizes pointed 1-movability (see Theorem 3.1). Using this characterization one can easily prove that, for instance, arcwise connected continua are pointed 1-movable (see Problem 11 from [1] and Problem 4 from [5]). This characterization clarifies also why pointed 1-movability is an invariant of continuous mappings of continua (see [7] and [11]).

1. Generalized paths. In this section we define a notion generalizing the notion of a path in a space. We shall show that such "nice" spaces as tree-like continua are "path-connected" in this new sense, but such "singular" ones as solenoids and the Case-Chamberlin curve are not. It will be clear from the definition that "path-connectedness" in this new sense is also invariant under continuous mappings.

Let $\underline{X} = \{X_n, f_{nm}\}$ be an ANR-sequence, where $f_{nm}\colon X_m \to X_n$, $1 \leqslant n \leqslant m$. Let $X = \underleftarrow{\lim}\, \underline{X}$ and for each $n \geqslant 1$ let $f_n\colon X \to X_n$ denote the projection. For $x, y \in X$ let $x_n = f_n(x)$ and $y_n = f_n(y)$ be their nth coordinates. Let $I = [0, 1]$ and $\dot{I} = \{0, 1\}$. Consider a sequence of paths (in the ordinary sense)

$$\underline{\omega} = \{\omega_n\}_{n=1}^{\infty},$$

where $\omega_n\colon (I, 0, 1) \to (X_n, x_n, y_n)$.

A. $\underline{\omega}$ is said to be an $(X_n, \underline{X})$-*approximative path from x to y* if for each $m \geqslant n$ we have

$$\omega_n \simeq f_{nm} \circ \omega_m$$

(all homotopies between paths are relatively $\dot{I}$).

B. $\underline{\omega}$ is said to be an $\underline{X}$*-approximative path from x to y* if $\underline{\omega}$ is an $(X_n, \underline{X})$-approximative path from x to y for each $n \geqslant 1$. $\underline{\omega}$ will be also called an *approximative path from x to y in* $\underline{X}$.

C. The points $x, y \in X$ are said to be $(X_n, \underline{X})$*-joinable* (or: X_n*-joinable in* $\underline{X}$) provided that there exists an $(X_n, \underline{X})$-approximative path from x to y.

The following proposition can be proved using some standard techniques employed in many papers on shape theory (comp. for instance [9] and [10]).

1.1. Proposition. *Let* $\underline{X}' = \{X'_n, f'_{nm}\}$ *be another sequence associated with* X. *Then we have*:

(i) *if for each* $n \geqslant 1$ *there exists an* $(X_n, \underline{X})$*-approximative path from* x *to* y, *then for each* $n \geqslant 1$ *there exists an* $(X'_n, \underline{X}')$*-approximative path from* x *to* y,

(ii) *if there exists an* $\underline{X}$*-approximative path from* x *to* y, *then there exists an* $\underline{X}'$*-approximative path from* x *to* y.

The above proposition enables us to define the following notions independent on the choice of a particular ANR-sequence associated with X.

D. The points $x, y \in X$ are said to be *weakly joinable* (*in* X) if for each $n \geqslant 1$ there exists an $(X_n, \underline{X})$-approximative path from x to y (equivalently: if they are $(X_n, \underline{X})$-joinable for each $n \geqslant 1$).

E. The points $x, y \in X$ are said to be *joinable* (*in* X) if there exists an $\underline{X}$-approximative path from x to y.

Note that if X is joinable between x and y then it is weakly joinable between these points.

Now we give three examples illustrating those notions.

1.2. Example. *Continua with trivial shape are joinable between any two points.*

This is an immediate consequence of the fact that such continua can be presented as inverse limits of absolute retracts (comp. [5]). Hence, in particular, tree-like continua have this property.

A continuum X is said to be *spreadable* if there exist a continuum Y and a surjection $f\colon Y \to X$ such that for each map $g\colon X \to Z$, where $Z \in$ ANR, we have $g \circ f \simeq 0$, i.e. $g \circ f$ is homotopic to a constant map.

Note that locally connected continua and continua with trivial shape are spreadable (comp. [5]).

Moreover, it is evident that

1.3. Proposition. *Continuous images of spreadable continua are spreadable.*

1.4. Example. *Spreadable continua are joinable between any two points.*

Proof. Let X be a spreadable continuum and let $x, y \in X$. We shall show that X is joinable between x and y. Let Y and $f\colon Y \to X$ be as in the definition of spreadability. Since f is a surjection, there exist $\bar{x}, \bar{y} \in Y$ such that $f(\bar{x}) = x$ and $f(\bar{y}) = y$. Let

$$X = \varprojlim \{X_n, f_{nm}\}$$

be a presentation of X as the limit of an ANR-sequence and let $f_n\colon X \to X_n$ denote the projection.

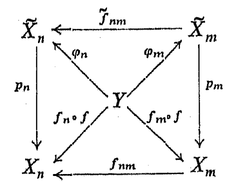
Let $p_n\colon \tilde{X}_n \to X_n$ be the universal covering projection. By our assumption we have

$$f_n \circ f \simeq 0 \quad \text{for each } n \geqslant 1 .$$

Now, by an analogous procedure as in ([5], § 4) one can construct a collection of mappings:

$$\varphi_n\colon Y \to \tilde{X}_n \quad \text{and} \quad \tilde{f}_{nm}\colon \tilde{X}_m \to \tilde{X}_n$$

such that for each $1 \leqslant n \leqslant m$ the following diagram commutes:



Let $\tilde{x}_n = \varphi_n(\bar{x})$ and $\tilde{y}_n = \varphi_n(\bar{y})$ for $n \geqslant 1$. Let $\tilde{\omega}_n\colon (I, 0, 1) \to (\tilde{X}_n, \tilde{x}_n, \tilde{y}_n)$ be an arbitrary path and let

$$\omega_n = p_n \circ \tilde{\omega}_n .$$

Note that $\omega_n\colon (I, 0, 1) \to (X_n, x_n, y_n)$, where x_n, y_n are the nth coordinates of x and y resp., i.e., $x_n = f_n(x)$ and $y_n = f_n(y)$. We shall show that

$$\underline{\omega} = \{\omega_n\}_{n=1}^{\infty}$$

is an approximative path from x to y (in the system $\{X_n, f_{nm}\}$).

Fix $n \geqslant 1$ and let $m \geqslant n$ be an arbitrary index. Since $\tilde{\omega}_n$ and $\tilde{f}_{nm} \circ \tilde{\omega}_m$ have the same endpoints and the space $\tilde{X}_n$ is simply connected we have

$$\tilde{\omega}_n \simeq \tilde{f}_{nm} \circ \tilde{\omega}_m \quad \text{in } \tilde{X}_n \text{ (rel. } \dot{I}) .$$

Since the diagram is commutative we conclude that

$$\omega_n = p_n \circ \tilde{\omega}_n \simeq p_n \circ \tilde{f}_{nm} \circ \tilde{\omega}_m = f_{nm} \circ p_m \circ \tilde{\omega}_m = f_{nm} \circ \omega_m .$$

This proves that $\underline{\omega}$ is an approximative path, which completes the proof.

1.5. Example. *No solenoid is weakly joinable between any two points belonging to different composants.*

Proof. We shall show this in the case of dyadic solenoid X. (The proof in the general case is almost the same). Let x and y be two points of X from distinct composants of X.

We prove that X is not weakly joinable between these points.

Denote by S the unit circle in the complex plane, i.e. $S = \{z \in \mathbf{C}\colon |z| = 1\}$; and let $\varphi\colon S \to S$ be the map given by

$$\varphi(z) = z^2 .$$

Represent X as the limit of the inverse sequence $\underline{X} = \{X_n, f_{nm}\}$, where $X_n = S$ and $f_{n,n+1} = \varphi$ for each $n \geqslant 1$. Suppose there is an $(X_1, \underline{X})$-approximative path $\underline{\omega} = \{\omega_n\}_{n=1}^{\infty}$ from x to y. Since $(f_{nm})_{\#}$ is a monomorphism and f_{nm} is a covering projection it follows that $\underline{\omega}$ is an $\underline{X}$-approximative path from x to y. Lifting successively ω_1 to $X_2, X_3, \ldots$ we obtain a path in X between x and y. It follows that x and y belong to the same composant of X, a contradiction.

Now we describe the above notions more geometrically treating X as a subset of a space $M \in \mathrm{ANR}(\mathfrak{M})$.

Let $\underline{\omega} = \{\omega_n\}_{n=1}^{\infty}$ be a sequence of paths, where

$$\omega_n \colon (I, 0, 1) \to (M, x, y)$$

such that for each neighborhood U of X in M there is an index n_U such that

$$\omega_n(I) \subset U \quad \text{for each } n \geqslant n_U .$$

Let W be a neighborhood of X in M.

A′. $\underline{\omega}$ is said to be an (W, M, X)-*approximative path from x to y* if there exists an index n_0 such that for each $m, n \geqslant n_0$ we have

$$\omega_n \simeq \omega_m \quad \text{in } W.$$

B′. $\underline{\omega}$ is said to be an (M, X)-*approximative path from x to y* if for each neighborhood V of X in M $\underline{\omega}$ is an (V, M, X)-approximative path from x to y. In this situation $\underline{\omega}$ will be also called an *approximative path (of X) from x to y (in M)*.

C′. The points $x, y \in X$ are said to be (W, M, X)-*joinable* (or shortly: *W-joinable*, if M and X are fixed under considerations) provided that there exists a (W, M, X)-approximative path from x to y. In this case we say also that X is (W, M, X)-*joinable between x and y*.

For these notions one can establish a proposition similar to 1.1, where the inverse sequences are replaced by ANR-spaces containing X and the coordinates of the inverse sequences are replaced by neighborhoods of X. Hence the notions formulated at D' and E' are independent on the choice of a particular ANR-space containing X. Furthermore, one can easily show that the notions formulated at points D' and E' below are equivalent to the corresponding notions from D and E.

D′. X is said to be *weakly joinable between* $x, y \in X$ if X is (V, M, X)-joinable between these points for each neighborhood V of X in M.

E′. X is said to be *joinable between* $x, y \in X$ if there exists an (M, X)-approximative path from x to y.

F′. If $\underline{\omega}^1 = \{\omega_n^1\}$ is a (W, M, X)-approximative path ((M, X)-approximative path) from x to y and $\underline{\omega}^2 = \{\omega_n^2\}$ is a (W, M, X)-approximative path ((M, X)-approximative path) from y to z, $z \in X$, then we define the product of $\underline{\omega}^1$ and $\underline{\omega}^2$ by the formula:

$$\underline{\omega}^1 * \underline{\omega}^2 = \{\omega_n^1 * \omega_n^2\} .$$

Clearly, $\underline{\omega}^1 * \underline{\omega}^2$ is a (W, M, X)-approximative path ((M, X)-approximative path) from x to z.

It is evident how to define an analogous product in the inverse sequences.

By this remark the relations of joinability and weak joinability are equivalence relations in X. The classes of equivalent elements of X under these relations will b, called *approximative path components* and *weak approximative path components of X*, respectively.

1.6. Proposition. *If x and y are (weakly) joinable points of X and f: $X \to Y$ is continuous, then $f(x)$ and $f(y)$ are (weakly) joinable in Y.*

It follows from 1.6 that the notions of weak approximative path component and approximative path component behave under continuous mappings similarily to the notion of path component (in the ordinary sense).

By a chain we mean a finite collection of sets $(A_1, \ldots, A_n)$ satisfying the condition:

$$A_i \cap A_j \neq \emptyset \Leftrightarrow |i-j| \leqslant 1 \quad \text{for each } i, j = 1, \ldots, n .$$

The chain is an ε-chain if $\max(\operatorname{diam} A_i \colon i = 1, \ldots, n) < \varepsilon$.

1.7. Example. *Let X be a continuum and let x, y be two points of X satisfying the following condition:*

for each $\varepsilon > 0$ there exists an ε-chain $(A_1, \ldots, A_n)$ of subcontinua of X scuh that $x \in A_1$ and $y \in A_n$.

Then X is weakly joinable between x and y.

Proof. Consider X as a subset of the Hibert space H and let U a neighborhood of X in H. We shall show that there is a (U, H, X)-approximative path from x to y.

Let $\mathscr{V} = \{V_1, V_2, \ldots, V_k\}$ be a collection of simply connected open subsets of U which covers X. Let $\varepsilon > 0$ be such that each subset of X of diameter less then ε is contained in a member of $\mathscr{V}$. Take an ε-chain $(A_1, \ldots, A_n)$ of subcontinua of X with $x \in A_1$ and $y \in A_n$. Let $x_0 = x$, $x_n = y$ and for each $1 \leqslant i < n$ let $x_i \in A_i \cap A_{i+1}$ be an arbitrary point. For each $j = 1, \ldots, n$ let $V_{r(j)}$ be a member of $\mathscr{V}$ containing A_j. For each such a j there is a $(V_{r(j)}, H, A_j)$-approximative path $\underline{\omega}^j$ from x_{j-1} to x_j. Clearly, $\underline{\omega}^j$ is a (U, H, X)-approximative path. Now taking the product $\underline{\omega} = \underline{\omega}^1 * \underline{\omega}^2 * \ldots * \underline{\omega}^n$ we obtain a (U, H, X)-approximative path from x to y.

The next proposition explains relationship between arc-connectedness, pointed 1-movability and the notions we have defined.

1.8. Proposition. *Let X be a continuum and let x and y be two points of X. Consider the following conditions:*

(i) *there is an arc in X between x and y,*

(ii) *there is a pointed 1-movable subcontinuum of X containing x and y,*

(iii) *x and y are joinable in X,*

(iv) *x and y are weakly joinable in X.*

Then the following implications are satisfied:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)},$$

and none of them can be reversed.

Proof. Consider X as a subset of $M \in \mathrm{ANR}(\mathfrak{M})$ and let Y be a subcontinuum of X. The following condition is equivalent to pointed 1-movability of Y:

For each $p, q \in Y$ and for each neighborhood U of Y in M there is a neighborhood U_0 of Y in M such that each path in U_0 with endpoints at p and q can be shrunk inside U (rel. $\dot{I}$) into any neighborhood of Y (see [7]).

This proves the second implication; the remaining ones are trivial. Obviously, the first implication is not reversible. Now we give two examples showing an analogous fact for the remaining ones.

1.9. Example. *There exists a continuum X joinable between two points $x, y \in X$ such that no pointed 1-movable subcontinuum of X contains these points.*

Proof. Let Y be the dyadic solenoid and let $f: (0, 1] \to Y$ be a map such that $f((0, 1])$ is dense in Y. Let X be a subset of $I \times Y$ given by

$$X = \{0\} \times Y \cup \{(t, f(t)):\ 0 < t \leqslant 1\}.$$

Set $x = (0, f(1))$ and $y = (1, f(1))$. Clearly, X is joinable between these points. The remaining property of X follows from the following observations:

1) X is irreducible between x and y, 2) Y is a continuous image of X, 3) Y is not pointed 1-movable (comp. Ex. 1.4 and Prop. 1.5) and 4) pointed 1-movability is an invariant of continuous maps.

1.10. Example. *There exists a continuum X weakly joinable between two points $x, y \in X$ but not joinable between them.*

Proof. Let X be defined as the union

$$X = \bigcap_{n=0}^{\infty} Y_n,$$

where each Y_n is a copy of the dyadic solenoid and the following conditions are satisfied:

(1) $Y_i \cap Y_j = \{x\} = \bigcap_{n=0}^{\infty} Y_n$ for $i, j \geqslant 1$ and $i \neq j$,

(2) $Y_0 \cap Y_i = \{x, y_i\}$ for $i \geqslant 1$,

(3) $\operatorname{diam} Y_n \xrightarrow[n\to\infty]{} 0$,

(4) all y_i's belong to a single composant of Y_0 which does not contain x,

(5) x and y_i belong to distinct composants of Y_i for each $i \geqslant 1$.

Set $y = y_1$. The fact that X is weakly joinable between x and y follows from arc-connectedness of composants of a solenoid and from Example 1.7. The idea of the argument that X is not joinable between x and y is as follows. If there existed and approximative path of X from x to y, then pushing it stepwise away from $Y_1, Y_2, \ldots$ we could obtain an approximative path of Y_0 from x to y. Such a deformation is possible according to (1), (2) and (5). The complete argument is not very difficult but the details are rather complicated and therefore we omit them.

2. The fundamental lemma. For the original and geometric definitions of movability and (pointed) n-movability the reader is referred to [2]. Let us briefly recall an equivalent definition of pointed 1-movability expressed in the language of inverse systems.

Suppose

$$(X, x_0) = \varprojlim\{(X_n, x_n), f_{nm}\}, \quad \text{where} \quad X_n \in \mathrm{ANR}.$$

Continuum X is pointed 1-movable provided the inverse sequence of fundamental groups $\{\pi(X_n, x_n), (f_{nm})_{\#}\}$ is a Mittag-Leffler sequence (ML-sequence), i.e., for each $n \geqslant 1$ there is an index $n_0 \geqslant n$ such that

$$\operatorname{im}(f_{nn_0})_{\#} = \operatorname{im}(f_{nm})_{\#} \quad \text{for each } m \geqslant n_0 \text{ (see [7]).}$$

It is known that this notion does not depend on a particular choice of an ANR-sequence associated with X.

The following lemma establishes the fundamental result of this paper.

2.1. Lemma. *Let X be a continuum. If X is not pointed 1-movable and $\underline{X} = \{X_n, f_{nm}\}$ is an ANR-sequence associated with X, then there exist an index $n_0 \geqslant 1$ and a subset A of X such that $\operatorname{Card} A = \mathfrak{c}$ and no two distinct points of A are X_{n_0}-joinable in $\underline{X}$.*

Proof. Let x_0 be a point of X and let $x_n \in X_n$ be the nth coordinate of x_0, $n \geqslant 1$. Since (X, x_0) is not 1-movable, $\{\pi(X_n, x_n), (f_{nm})_{\#}\}$ is not an ML-sequence (f_{nm} is regarded as a map between pointed spaces). Hence there exists an index $n_0 \geqslant 1$ such that the sequence $\{\operatorname{im}(f_{n_0 n})_{\#}\}_{n=n_0}^{\infty}$ does not stabilizes.

Restricting, if necessary, the system $\underline{X}$ to a subsystem we may assume that $n_0 = 1$. Thus the above condition changes to the following

(1) $\{\operatorname{im}(f_{1n})_{\#}\}_{n\geqslant 1}$ is a decreasing sequence such that $\bigcap_{n=1}^{\infty} \operatorname{im}(f_{1n})_{\#} \neq \operatorname{im}(f_{1i})_{\#}$ for each $i \geqslant 1$.

The relation of X_1-joinability in the system $\underline{X}$ is an equivalence relation in X. Thus there is a subset A of X with the following property

(2) for each $x \in X$ there exists exactly one point $a(x) \in A$ such that x and $a(x)$ are X_1-joinable in $\underline{X}$.

To complete the proof it suffices to show that

$$(*) \qquad \operatorname{Card} A = \mathfrak{c}.$$

Now we perform several auxiliary constructions leading to an argument for (*).

Since X_1 is an ANR-space, by the West theorem [15] there is a (compact) polyhedron $\tilde{X}_1$ and two mappings $\alpha\colon X_1\to\tilde{X}_1$, $\beta\colon\tilde{X}_1\to X_1$ such that $\beta\circ\alpha\simeq\mathrm{id}_{X_1}$ and $\alpha\circ\beta\simeq\mathrm{id}_{\tilde{X}_1}$. Having this result it is easy to verify that without loss of generality we may assume that X_1 is a polyhedron (precisely: replaceing f_{12} by $\alpha\circ f_{12}$ and X_1 by $\tilde{X}_1$ we obtain an ANR-system associated with X in which the first space is a polyhedron and which satisfies conditions analogous to (1) and (2), where A and $a(x)$ are unaltered).

Let T be a triangulation of X_1 such that x_1 is a vertex of T. Let D be a maximal tree in $T^{(1)}$, where $T^{(1)}$ denotes as usual the 1-skeleton of T. Denote by (B, b) the space obtained from (X_1, x_1) pinching D to a point. Let

$$q\colon (X_1, x_1)\to(B, b)$$

be the quotient map.

Set

(3) $$G_n = \operatorname{im}(q\circ f_{1n})_{\#} \quad\text{for}\quad n\geqslant 1\,.$$

Since $(q)_{\#}$ is an isomorphism by (1) we have

(4) $G_1, G_2, \ldots$ is a decreasing sequence of subgroups of $\pi(B, b)$ such that $\bigcap_{n=1}^{\infty} G_n\neq G_i$ for each $i\geqslant 1$.

There is a covering projection

$$p_n\colon (\tilde{Y}_n, y_n)\to(B, b)$$

such that

(5) $$G_n = \operatorname{im}(p_n)_{\#} \quad\text{for}\quad n\geqslant 1$$

(see [13, p. 82, Th. 13]). From (3) it follows that $G_1 = \pi(B, b)$, hence (5) implies that

(6) p_1 is a homeomorphism.

By (3), (5) and the lifting theorem [13, p. 76, Th. 5], there is a mapping

$$\tilde{g}_n\colon (X_n, x_n)\to(\tilde{Y}_n, y_n)$$

such that

(7) $$q\circ f_{1n} = p_n\circ\tilde{g}_n \quad\text{for}\quad n\geqslant 1\,.$$

By (4) and the lifting theorem there is a mapping

$$\tilde{h}_{n,n+1}\colon (\tilde{Y}_{n+1}, y_{n+1})\to(\tilde{Y}_n, y_n)$$

such that

(8) $$p_{n+1} = p_n\circ\tilde{h}_{n,n+1} \quad\text{for}\quad n\geqslant 1\,.$$

It follows also that

(9) $h_{n,n+1}$ is a covering projection for $n\geqslant 1$ (see [13, p. 79, Lemma 1]).

Let us prove that

(10) $$\tilde{g}_n\circ f_{n,n+1} = \tilde{h}_{n,n+1}\circ\tilde{g}_{n+1} \quad\text{for}\quad n\geqslant 1\,.$$

By (7) and (8) we have

$$\begin{aligned}\gamma &= p_n\circ\tilde{g}_n\circ f_{n,n+1} = q\circ f_{1n}\circ f_{n,n+1} = q\circ f_{1,n+1} = p_{n+1}\circ\tilde{g}_{n+1}\\ &= p_n\circ\tilde{h}_{n,n+1}\circ\tilde{g}_{n+1}\,.\end{aligned}$$

Thus the maps appearing in (8) are liftings to $(\tilde{Y}_n, y_n)$ of the map $\gamma\colon (X_{n+1}, x_{n+1})\to(B, b)$. Since they agree at x_{n+1} and X_{n+1} is connected, condition (10) follows from [13, p. 67, Th. 2]. Now we shall prove the following proposition

(11) if $\Delta^k\in T$ and for each $n\geqslant 1$ we have a map $\varphi_n\colon \Delta^k\to\tilde{Y}_n$ such that $q|\Delta^k = p_n\circ\varphi_n$ and $\varphi_n = \tilde{h}_{n,n+1}\circ\varphi_{n+1}$ then there is an index n_0 such that for $m\geqslant n_0$ the map $\tilde{h}_{m,m+1}$ is a homeomorphism between $\varphi_{m+1}(\Delta^k)$ and $\varphi_m(\Delta^k)$.

Let us note that $\varphi_n|\Delta^k\setminus D\colon \Delta^k\setminus D\to\varphi_n(\Delta^k\setminus D)$ is a homeomorphism because q is a homeomorphism between $\Delta^k\setminus D$ and $q(\Delta^k\setminus D)$. It follows that

$$\tilde{h}_{n,n+1}|\varphi_{n+1}(\Delta^k\setminus D)\colon \varphi_{n+1}(\Delta^k\setminus D)\to\varphi_n(\Delta^k\setminus D)$$

is a homeomorphism. Let r be the number of components of the set $\Delta^k\cap D$. Observe that each such a component is mapped by φ_n in the fibre $p_n^{-1}(b)$ (because $q(D) = \{b\}$). Since the fibres are discrete, the set $\varphi_n(\Delta^k\cap D)$ is finite and contains at most r points. Since $\tilde{h}_{n,n+1}(\varphi_{n+1}(\Delta^k\cap D)) = \varphi_n(\Delta^k\cap D)$ there is an index $n_0\geqslant 1$ such that for each $n\geqslant n_0$ the set $\varphi_n(\Delta^k\cap D)$ has the same number of points as the set $\varphi_{n_0}(\Delta^k\cap D)$. Now it is easily seen that the conclusion of (11) holds true.

For each $n\geqslant 1$ let Y_n be a subspace of $\tilde{Y}_n$ defined as follows:

$$Y_n = \bigcup\{\varphi(\Delta^k)\colon\ \Delta^k\in T,\ \varphi\colon\ \Delta^k\to\tilde{Y}_n,\ q|\Delta^k = p_n\circ\varphi,\ \varphi(\Delta^k)\cap\tilde{g}_n(X_n)\neq\emptyset\}\,.$$

It is easy to see that Y_n is a compact connected polyhedron with a CW complex structure determined by the cells $\varphi(\Delta^k)$. Let $Y_n^{(1)}$ denote the 1-skeleton of Y_n with respect to this cell structure.

By (8) and (10) we infer that

(12) $$\tilde{h}_{n,n+1}(Y_{n+1})\subset Y_n\supset\tilde{g}_n(X_n) \quad\text{and}\quad \tilde{h}_{n,n+1}(Y_{n+1}^{(1)})\subset Y_n^{(1)}\,.$$

Let $g_n\colon (X_n, x_n)\to(Y_n, y_n)$ be defined by $\tilde{g}_n$ and let $h_{n,n+1}\colon (Y_{n+1}, y_{n+1})\to(Y_n, y_n)$ be the restriction of $\tilde{h}_{n,n+1}$. Furthermore, let $h_{nn} = 1_{Y_n}$ and $h_{nm} = h_{n,n+1}\circ\ldots\circ h_{m-1,m}$ for $1\leqslant n<m$. By (10) we obtain an inverse sequence of pointed connected polyhedra $\{(Y_n, y_n), h_{nm}\}$. Denote its limit by (Y, y_0). Since, by (8), the diagram

$$\begin{array}{ccc}(X_n, x_n) & \xleftarrow{\ f_{nm}\ } & (X_m, x_m)\\ \big\downarrow{\scriptstyle g_n} & & \big\downarrow{\scriptstyle g_m}\\ (Y_n, y_n) & \xleftarrow{\ h_{nm}\ } & (Y_m, y_m)\end{array}$$

commutes for each $n\leqslant m$, the maps g_n induce the mapping

$$g\colon (X, x_0)\to(Y, y_0)$$

such that

(13) $$h_n \circ g = g_n \circ f_n ,$$

where f_n and h_n are the projections. The map g need not be surjective. But we have the following property:

(14) each point of Y can by joined by an arc to some point of $g(X)$.

Let $z \in Y$. Thus $z = (z_1, z_2, \ldots)$, where $z_i = h_i(z)$. Since T is finite, by the description of Y_n, there is a simplex $\Delta^k \in T$ and a sequence of mappings $\varphi_n\colon \Delta^k \to Y_n$ such that $q|\Delta^k = p_n \circ \varphi_n$, $\varphi_n(\Delta^k) \cap g_n(X_n) \neq \emptyset$ and $z_n \in \varphi_n(\Delta^k)$ for $n = 1, 2, \ldots$ By (11) the set

$$M = \varprojlim \{\varphi_n(\Delta^k), h_{nm}\}$$

is a continuous image of Δ^k containing z. Note also that M is a subset of Y meeting $g(X)$. This completes the proof of (14).

Note that the sets $(Y_n^{(1)}, y_n)$ form an inverse sequence of connected 1-dimensional polyhedra with the bonding maps being restrictions of the maps h_{nm}. Denote by $(Y^{(1)}, y_0)$ the inverse limit of that sequence.

By (3) and (7) we have

$$\operatorname{im}(p_n \circ \tilde{g}_n)_{\#} = G_n .$$

By (12) it follows that

$$\operatorname{im}(p_n|(Y_n, y_n))_{\#} = G_n .$$

Since $Y_n^{(1)}$ is the 1-skeleton of the CW complex Y_n we infer that

$$\operatorname{im}(p_n|(Y_n^{(1)}, y_n))_{\#} = G_n .$$

Hence by (4), (8) and (12) we infer that

(15) $Y^{(1)}$ is not pointed 1-movable.

Consider the sets $C = h_1^{-1}(y_1)$ and $C_n = h_{1n}^{-1}(y_1) = p_n^{-1}(b) \cap Y_n$. It is easy to see that the maps h_{nm} restricted to C_m form an inverse sequence of finite sets with C as its limit. Note that $C_n \subset Y_n^{(1)}$. By the description of q and by (6) we infer that $Y_1^{(1)}$ is the one-point union of a finite number of circles with y_1 being the centre. According to (9) the map h_{1n} restricted to $Y_n^{(1)}$ is an immersion, i.e. a local embedding. It follows that h_{1n} restricted to each component of $Y_n^{(1)} \setminus C_n$ is an embedding. Obviously, the same holds for the maps h_{nm}. Therefore, the closure of each component of $Y^{(1)} \setminus C$ is an arc or a simple closed curve. Using this fact we now prove that

(16) $$\operatorname{Card} C = \mathfrak{c} .$$

Suppose $\operatorname{Card} C < \mathfrak{c}$. Since C is compact it follows that C is countable.

Consider a subcontinuum Z of $Y^{(1)}$. We shall show that Z is decomposable. Otherwise, there is a composant E of Z missing C. Thus E is a subset of a component of $Y^{(1)} \setminus C$, hence $\bar{E}$ is an arc or a simple closed curve. But $\bar{E} = Z$, a contradiction. Therefore $Y^{(1)}$ is hereditarily decomposable, and by [7] we infer that $Y^{(1)}$ is pointed 1-movable, contrary to (15). This proves (16).

Let $\underline{Y}$ be the system $\{Y_n, h_{nm}\}$. Consider the following relation in Y: $x \sim y$ if and only if x and y are Y_1-joinable in $\underline{Y}$. This is an equivalence relation. Let $\mathscr{F}$ denote the set of the equivalence classes of that relation. By (13) it follows that if $x, y \in X$ are X_1-joinable in $\underline{X}$, then $g(x)$ and $g(y)$ are Y_1-joinable in $\underline{Y}$. Hence conditions (14) and (2) imply that each point of Y is Y_1-joinable in $\underline{Y}$ to some point of $g(A)$. Thus

(17) $$\operatorname{Card} \mathscr{F} \leqslant \operatorname{Card} A .$$

Now we are ready to prove (*); i.e. that $\operatorname{Card} A = \mathfrak{c}$, which will complete the proof.

Suppose, to the contrary, that $\operatorname{Card} A < \mathfrak{c}$. Then, by (17), we have $\operatorname{Card} \mathscr{F} < \mathfrak{c}$. Since $\bigcup \mathscr{F} = Y$, by (16) we infer that there is an element $F \in \mathscr{F}$ such that

(18) $F \cap C = C_0$ is uncountable.

Now we define a function

$$\psi\colon C_0 \to \pi(B, b)$$

as follows. Pick a point $c_0 \in C_0$. For $y \in C_0$ let $\{\omega_n^y\}$ be a $(Y_1, \underline{Y})$-approximative path from c_0 to y. Observe that $p_1 \circ \omega_1^y$ is a loop in (B, b). Define $\psi(y)$ to be the element of $\pi(B, b)$ with representative $p_1 \circ \omega_1^y$; i.e. $\psi(y) = [p_1 \circ \omega_1^y]$.

Now we prove that

(19) $$\psi(z) \neq \psi(y) \quad \text{for} \quad z \neq y \ (y, z \in C_0) .$$

There is an index m such that

(20) $$h_m(z) \neq h_m(y) .$$

Suppose $\psi(z) = \psi(y)$. Hence $p_1 \circ \omega_1^z \simeq p_1 \circ \omega_1^y$. By (8) we have $p_1 \circ h_{1m} = p_m$. Since $\omega_1^z \simeq h_{1m} \circ \omega_m^z$ and $\omega_1^y \simeq h_{1m} \circ \omega_m^y$, then

$$p_m \circ \omega_m^z \simeq p_m \circ \omega_m^y .$$

Since $\omega_m^z(0) = h_m(c_0) = \omega_m^y(0)$ and p_m is a covering projection we infer that

$$\omega_m^z \simeq \omega_m^y .$$

In particular, $\omega_m^z(1) = \omega_m^y(1)$ contrary to (20). This completes the proof of (19).

By (18) and (19) we conclude that $\pi(B, b)$ is uncountable, a contradiction. This completes the proof of the lemma.

3. Main results. From Lemma 2.1 it follows that continua with countably many weak approximative path components are pointed 1-movable. Thus by Proposition 1.8 we have the following theorem.

3.1. THEOREM. *Let X be a continuum. Then the following are equivalent*:

(i) *X is pointed* 1-*movable*,

(ii) *each two points of X lie in a pointed* 1-*movable, subcontinuum of X*,

(iii) *X is joinable,*

(iv) *X has countably many approximative path components,*

(v) *X is weakly joinable,*

(vi) *X has countably many weak approximative path components.*

As an application of the above result we have

3.2. COROLLARY. *Continua with countably many arc-components are pointed 1-movable. In particular, arcwise connected continua are pointed 1-movable.*

The second assertion settles Problem 4 from [5] (comp. Problem 11 from [1]).

A continuum X is said to be *λ-connected* if every two points of X lie in a hereditarily decomposable subcontinuum of X. Since hereditarily decomposable continua a pointed 1-movable [7], we have also

3.3. COROLLARY. *λ-connected continua are pointed 1-movable.*

Combining the above theorem with the fact established in Example 1.4 we obtain the following corollary which has been recently obtained by J. Dydak using different methods.

3.4. COROLLARY. *Spreadable continua are pointed 1-movable.*

3.5. Remark. Note that this result for curves follows from [5] because, us can be easily shown (comp. [5, § 4]), spreadable curves coincide with continuous images of tree-like continua.

The characterization of pointed 1-movability contained in Theorem 3.1 together with Proposition 1.6 offer an alternative proof of the invariance of pointed 1-movability under continuous mappings of continua. For the original proofs of this result see [7] and [11].

We still have much to learn on pointed 1-movability. In connection with Corollaries 3.2 and 3.3 we have the following

PROBLEM 1. Given a not pointed 1-movable continuum X, can we map X onto an (nondegenerate) indecomposable continuum?

A strengthened version of this question is the following

PROBLEM 2. Let X be a continuum which is not pointed 1-movable. Does there exist an indecomposable not pointed 1-movable continuum Y being a continuous image of X?

Let us remark that in Problem 2 we can not insist Y to be a curve, i.e. 1-dimensional continuum. In fact, in [4] it is given an example of a (2-dimensional) 1-movable and not pointed 1-movable continuum X_0. According to [14], for curves the notions of movability, 1-movability and pointed 1-movability coincide. By the theorem presented below it now follows that the continuum X_0 can not be mapped onto any non-movable curve.

3.6. THEOREM. *Let $f\colon X \to Y$ be a continuous surjection from a 1-movable continuum X onto a curve Y. Then Y is movable.*

Proof. By the Whyburn factorization theorem there exist a continuum Z and two surjections $g\colon X \to Z$ and $h\colon Z \to Y$ such that g is monotone, h is 0-dimensional and $f = h \circ g$. By [3] it follows that Z is 1-movable. By the Hurewicz theorem [8, p. 114, Th. 1] we infer that Z is a curve. By the quoted theorem from [14], Z is pointed 1-movable. Since this notion invariantly behaves under continuous mappings, Y is movable. This completes the proof.

In particular, Theorem 3.6 implies the following

3.7. COROLLARY. *One-dimensional image of a movable continuum is movable.*

This result settles Problem 1 from [5] (comp. also Problem 10 from [1]). Also, it extends some results established in [6] and [12].

In connection with Remark 3.5 we have the following

PROBLEM 3. *Is every spreadable continuum a continuous image of a continuum with trivial shape?*

We are indebted to dr J. Dydak for a simplification of Example 1.5.

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