A reflection phenomenon in descriptive set theory

by

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Abstract. We show that properties of an analytic set E expressible by suitable "universal" formulas are shared by (many of) the Borel "approximations" to E. E.g., any analytic equivalence relation is an intersection of $\aleph_1$ Borel equivalences. Further corollaries include a theorem of Solovay on invariant prewellorderings and a theorem of Cenzer and Mauldin on extending plane analytic sets with small cross-sections.

§ 1. Preliminaries. In this section we will state, after several preliminary definitions, a general principle to the effect that Borel "approximations" to an analytic set E share many "universal" properties of E. This Reflection Principle will be proved in § 2. In § 3 we will exhibit several recent results in descriptive set theory as special cases of the Reflection Principle. We are grateful to R. D. Mauldin, Douglas Miller, Jack Silver, and R. M. Solovay for helpful exchanges on the subject of this paper.

Throughtout let $Y = 2^{\omega\times\omega}$, the countable product, indexed by pairs of natural numbers, or copies of the two-point discrete space $2 = \{0, 1\}$. Elements of Y are (characteristic functions of) binary relations on the set ω of natural numbers. Y is a Polish space (separable topological space admitting a complete metric) and indeed a homeomorph of the Cantor middle-third set. Let $Z \subseteq Y$ be the set of (characteristic functions of) linear orders on ω. Being a G_δ subspace of Y, Z is also Polish. Let $W \subseteq Z$ be the set of wellorders, and let $\Omega\colon W \to \omega_1$ be the order-type function. (As is usual in modern set theory, we are identifying an ordinal with the set of all smaller ordinals, so the set ω_1 of all countable ordinals is the least uncountable ordinal.) For $\alpha < \omega_1$, let $W^\alpha = \Omega^{-1}[\alpha]$, the set of wellorders of type $<\alpha$. Define three relations on Z by:

$$z_0 \preccurlyeq z_1 \leftrightarrow z_0, z_1 \in W \,\&\, \Omega(z_0) \leqslant \Omega(z_1)\,,$$
$$z_0 \prec z_1 \leftrightarrow z_0, z_1 \in W \,\&\, \Omega(z_0) < \Omega(z_1)\,,$$
$$z_0 \preccurlyeq^* z_1 \leftrightarrow z_1 \notin W \vee \big(z_0, z_1 \in W \,\&\, \Omega(z_0) \leqslant \Omega(z_1)\big)\,.$$

The last of these is analytic (Σ_1^1) and the other two CA (Π_1^1).

Every analytic subset E of Z may be represented in the form $\{z \in Z\colon \sigma(t, z)\}$, where σ is a Σ_1^1 formula of 2nd order arithmetic in two free variables, and $t \in Y$

is a parameter. Let us fix a reasonable enumeration $\{\sigma_i: i \in \omega\}$ of such formulas. We can regard $y \in Y$ as coding a natural number and another element of Y by:

$$i(y) = \begin{cases} \text{the least } i \text{ such that } y(0, i) = 0 \text{ if such exists}, \\ 0 \text{ otherwise}, \end{cases}$$

$$t(y) \in Y \,\&\, t(y)(m, n) = y(m+1, n).$$

Let $U = \{(y, z) \in Y \times Z: \sigma_{i(y)}(t(y), z)\}$. U is an analytic subset of $Y \times Z$. For $y \in Y$ the *y-section* $U_y = \{z \in Z: (y, z) \in U\}$ of U is an analytic subset of Z. Moreover *every* analytic subset E of Z has form U_y for some y. Such a y is called a *U-index* for E. U is called a *universal* analytic set.

Moschovakis [11] has proved the following important Uniform Boundedness Theorem:

There is a continuous function Γ: $Y \to Z$ such that for all $y \in Y$, if $U_y \subseteq W$, then $\Gamma(y) \in W$ and $z \prec \Gamma(y)$ for all $z \in U_y$.

Now let X be an arbitrary Polish space, $A \subseteq X$ analytic. A *sifting function* for A in X is a Borel measurable function f: $X \to Z$ such that $A = X - f^{-1}[W]$. Such an f *induces* the sequence $(X - f^{-1}[W^\alpha]: \alpha < \omega_1)$, and sequences induced in this way are called *covering sequences* for A in X. If $(A^\alpha: \alpha < \omega_1)$ is a covering sequence for A in X, then (i) each A^α is Borel, (ii) $A = \bigcap_{\alpha < \omega_1} A^\alpha$, (iii) at limit ordinals $A^\lambda = \bigcap_{\alpha < \lambda} A^\alpha$, and (iv) for any analytic $A' \subseteq X$ disjoint from A, A' is disjoint from some A^α.

An *analytic structure* is a relational structure in the usual model-theoretic sense of the special form $\mathfrak{A} = (X, A_a, B_b, C_c, D_d)_{a,b,c,d \in \omega}$ where: (i) X is a Polish space, (ii) each A_a is an analytic subset of some finite Cartesian power $X^{q(a)}$ of X, (iii) each B_b is a Borel subset of some finite power of X, (iv) each C_c is a Borel measurable function from some finite power of X to X, and (iv) each $D_d \in X$. By a *sifting sequence* for such an $\mathfrak{A}$ we mean a sequence $(f_a: a \in \omega)$ such that f_a is a sifting function for A_a, inducing a covering sequence $(A_a^\alpha: \alpha < \omega_1)$. Such a sifting sequence *induces* a sequence $(\mathfrak{A}^\alpha: \alpha < \omega_1)$ where $\mathfrak{A}^\alpha = (X, A_a^\alpha, B_b, C_c, D_d)$, and such sequences we call *covering sequences* for $\mathfrak{A}$.

Let I be a countable set of symbols with relation symbols $\bar{A}_a$, $\bar{B}_b$, and function symbols $\bar{C}_c$, and individual constants $\bar{D}_d$. A *generalized universal formula in vocabulary I* is an infinitary sentence Φ of the special form:

$$\bigwedge_{m \in \omega} \forall v_0 \forall v_1 \forall v_2 \ldots \bigvee_{n \in \omega} \varphi_{mn}$$

where each φ_{mn} is an atomic or negation-atomic formula in vocabulary I, and:

(*) For each m, the number of n such that φ_{mn} is atomic (rather than negation-atomic) and involves one of the $\bar{A}_a$ (rather than one of the $\bar{B}_b$ or the logical symbol $=$) is finite.

Note that any ordinary universal sentence φ of 1st order logic is a generalized universal formula, or is logically equivalent to one. To see this, first put φ in prenex normal form (a string of universal quantifiers followed by a quantifier-free matrix). Then put the matrix in conjunctive normal form (a conjunction of disjunctions of atomic and negation-atomic formulas). Finally distribute universal quantification over conjunction.

We write $\mathfrak{A} \models \Phi$ to indicate that the structure $\mathfrak{A}$ is a model of the sentence Φ.

Recall that $\Delta \subset \omega_1$ is *closed* if for all $\alpha < \omega_1$, $\sup(\alpha \cap \Delta) = \alpha$ implies $\alpha \in \Delta$, and *unbounded* if $\sup \Delta = \omega_1$, and CUB if both closed and unbounded. An intersection of countably many CUB sets is CUB.

At last we are ready to state our main result:

Reflection Principle. Let $\mathfrak{A}$ be an analytic structure, and Φ a generalized universal formula, such that $\mathfrak{A} \models \Phi$. Then:

(a) For any covering sequence $(\mathfrak{A}^\alpha: \alpha < \omega_1)$ for $\mathfrak{A}$, $\{\alpha < \omega_1: \mathfrak{A}^\alpha \models \Phi\}$ contains a CUB set.

(b) There exists a covering sequence $(\mathfrak{A}^\alpha: \alpha < \omega_1)$ for $\mathfrak{A}$ such that for *every* α, $\mathfrak{A}^\alpha \models \Phi$.

§ 2. Proof of the Reflection Principle.

Lemma 1. *Let X be a Polish space, $A_r \subset X$ analytic for $r \in \omega$. Suppose that for all r, $n \in \omega$, ${}^n f_r$ is a sifting function for A_r in X, inducing the covering sequence $({}^n A_r^\alpha: \alpha < \omega_1)$. Then there exist a CUB $\Delta \subseteq \omega_1$ and for each r a sifting function f_r for A_r in X inducing a covering sequence $(A_r^\alpha: \alpha < \omega_1)$ such that:*

(a) *For all $\alpha \in \Delta$ and all $r, m, n \in \omega$:*

$${}^m A_r^\alpha = {}^n A_r^\alpha .$$

(b) *For all $\alpha < \omega_1$ and all $r, n \in \omega$:*

$$A_r^\alpha = {}^n A_r^{\sup(\alpha \cap \Delta)} .$$

Proof. Let us fix continuous Θ: $Z^2 \to Z$ and Λ: $Z^\omega \to Z$ such that for $x, y \in Z$ of order-types ξ, η respectively, $\Theta(x, y)$ has order-type $\xi\omega + \eta$; and for $x_i \in Z$, $i \in \omega$, of order-types ξ_i, $\Lambda((x_i: i \in \omega))$ has order-type $\xi_0 + \xi_1 + \xi_2 + \ldots$

For Borel measurable g: $Z \to Z$ and for $y \in Y$, set:

$$\mathfrak{S}(g, y) = \{g({}^m f_r(x)): x \in X \,\&\, r, m \in \omega \,\&\, \exists n \in \omega\, {}^n f_r(x) \preccurlyeq^* g(y)\} \cup \{g(z): z \preccurlyeq^* g(y)\}.$$

Clearly $\mathfrak{S}(g, y)$ is always an analytic set. Indeed if we fix g, and fix a Borel isomorphism F: $Y \simeq X$, and fix Σ_1^1 formulas with parameters defining the analytic sets $\preccurlyeq^*$, graph g, and $\{(r, n, y, z): {}^n f_r(F(y)) = z\}$, then using these formulas we can effectively find for any $y \in Y$ a Σ_1^1 formula with parameters defining $\mathfrak{S}(g, y)$. This means that we may associate to g a Borel measurable function $h = \mathfrak{H}(g)$ such that for all $y \in Y$, $h(y)$ is a U-index for $\mathfrak{S}(g, y)$.

Notice that if $g[W] \subseteq W$, then for any $y \in W$, $\mathfrak{S}(g, y) \subseteq W$, and hence $\Gamma h(y) \in W$ where $h = \mathfrak{H}(g)$ is as above. (For on these assumptions, ${}^n f_r(x) \preccurlyeq^* g(y)$ implies

${}^{n}f_r(x) \in W$, hence $x \notin A_r$, hence for any $m \in \omega$, ${}^{m}f_r(x) \in W$ and $g({}^{m}f_r(x)) \in W$.) Let us now define functions g_i, $i \in \omega$, and g on Z:

$$g_0(z) = z\,,$$
$$g_{i+1}(z) = \Theta(g_i(z), \Gamma h_i(z)) \text{ where } h_i = \mathfrak{H}(g_i) \text{ and } \Gamma \text{ is Moschovakis' function,}$$
$$g(z) = \Lambda((g_i(z)\colon i \in \omega)).$$

It is evident that $W = g_i^{-1}[W] = g^{-1}[W]$ for all i. Moreover for $z \in W$ we have $g_0(z) \prec g_1(z) \prec g_2(z) \prec \ldots$ and:

$$\Omega(g(z)) = \sup_{i\in\omega} \Omega(g_i(z))\,.$$

(This last is true because, setting $\zeta_i = \Omega(g_i(z))$, ζ_{i+1} has form $\zeta_i\omega + \eta_i$, whence $\zeta_0 + \zeta_1 + \zeta_2 + \ldots = \sup_i \zeta_i$.) Note also that if $y, z \in W$ and $y \preccurlyeq g_i(z)$ for some i (in particular if $y \preccurlyeq z$) then for all $j > i$, $g_j(y) \prec g_{j+1}(z)$, and hence $g(y) \preccurlyeq g(z)$. This means that $\Omega(g(z))$ depends only on $\Omega(z)$, and so we may define a function $G\colon \omega_1 \to \omega_1$ by:

$$G(\alpha) = \Omega(g(z)) \quad \text{for some/any } z \text{ with } \Omega(z) = \alpha\,.$$

It is evident that $\alpha < G(\alpha)$ for all α. Moreover if $\alpha \leqslant \beta < G(\alpha)$, then $G(\beta) = G(\alpha)$. Let Δ be the closure of the range of G, i.e. $\{\alpha < \omega_1\colon \forall \beta < \alpha\; G(\beta) \leqslant \alpha\}$. For $r \in \omega$ define $f_r\colon X \to Z$ by $f_r(x) = g({}^{0}f_r(x))$. It is readily verified that Δ and the f_r satisfy the conclusions of the lemma, if we note that $g({}^{m}f_r(x)) \preccurlyeq g({}^{n}f_r(x))$ for all $r, m, n \in \omega$. ∎

LEMMA 2. *Let X be a Polish space, $A_r \subseteq X$ analytic for $r \in \omega$. Suppose $R < \omega$ is such that*:

$$\bigcap_{r<R} (X - A_r) \cap \bigcap_{r \geqslant R} A_r = \emptyset\,.$$

Then there exists for each r a sifting function f_r for A_r in X, inducing a covering sequence $(A_r^\alpha\colon \alpha < \omega_1)$ such that for all α:

$$\bigcap_{r<R} (X - A_r^\alpha) \cap \bigcap_{r \geqslant R} A_r^\alpha = \emptyset\,.$$

Proof. Fix arbitrary sifting functions e_r for A_r in X, inducing a covering sequence $(E_r^\alpha\colon \alpha < \omega_1)$. For Borel measurable $g\colon Z \to Z$ and $y \in Y$ set:

$$\mathfrak{T}(g, y) = \{g(z)\colon \exists x \in X(\forall r < R(e_r(x) \preccurlyeq^* g(y))\; \&\; \forall r \geqslant R(z \preccurlyeq^* e_r(x)))\} \cup$$
$$\cup \{g(z)\colon z \preccurlyeq^* g(y)\}\,.$$

Much as in the proof of Lemma 1 we can associate to every Borel measurable g a Borel measurable $k = \mathfrak{K}(g)$ such that for all $y \in Y$, $k(y)$ is a U-index for $\mathfrak{T}(g, y)$.

Notice that if $g[W] \subseteq W$, then for any $y \in W$, $\mathfrak{T}(g, y) \subseteq W$ and hence $\Gamma k(y) \in W$ where $k = \mathfrak{K}(g)$ is as above. (For on these assumptions, for any $x \in X$ if $e_r(x) \preccurlyeq^* g(y)$ for all $r < R$, then $x \in \bigcap_{r<R} (X - A_r)$, and hence for some $r \geqslant R$, $x \notin A_r$, so that if $z \preccurlyeq^* e_r(x)$ for this r, then $z \in W$ and $g(z) \in W$.)

Define functions $Z \to Z$ by:

$$g_0(z) = z\,,$$
$$g_{i+1}(z) = \Theta(g_i(z), \Gamma k_i(z)) \text{ where } k_i = \mathfrak{K}(g_i)\,,$$
$$g(z) = \Lambda((g_i(z)\colon i \in \omega))\,.$$

Finally for $r \in \omega$, define $f_r\colon X \to Z$ by $f_r = g e_r$. Since, as is evident, $g^{-1}[W] = W$, each f_r is a sifting function for A_r in X, inducing a covering sequence $(A_r^\alpha\colon \alpha < \omega_1)$.

Much as in the proof of Lemma 1, for $z \in W$, $\Omega(g(z))$ depends only on $\Omega(z)$, and we may define a function G on countable ordinals by:

$$G(\alpha) = \Omega(g(z)) \quad \text{for some/any } z \in W \text{ with } \Omega(z) = \alpha\,.$$

Let Δ be the closure of the range of G. It is readily verified that:

(a) For all $\alpha \in \Delta$:

$$\bigcap_{r<R} (X - E_r^\alpha) \cap \bigcap_{r \geqslant R} E_r^\alpha = \emptyset\,.$$

(b) For all $\alpha < \omega_1$ and all $r \in \omega$:

$$A_r^\alpha = E_r^{\sup(\alpha \cap \Delta)}\,.$$

From (a) and (b) it is immediate that the f_r satisfy the conclusion of the lemma. ∎

A special case. We will next prove the Reflection Principle in the following special case $\mathfrak{A}$ is an analytic structure of form $(X, A_a)_{a\in\omega}$ without B's, C's, or D's. Φ is a generalized universal sentence not containing the logical symbol $=$ and of the special form:

$$(**) \qquad \forall v_0 \forall v_1 \forall v_2 \ldots \bigvee_{n\in\omega} \varphi_n\,.$$

For convenience assume the φ_n are atomic for $n < N$ and negation-atomic for $n \geqslant N$, for some $N < \omega$. (It is part of the definition of generalized universality that only finitely many φ_n are atomic.)

So suppose $\mathfrak{A}$, Φ are as above and $\mathfrak{A} \models \Phi$. We begin with a combinatorial definition. To each $r \in \omega$ and each partition Q of r into disjoint pieces $\{0, 1, \ldots, r-1\} = Q_0 \cup Q_1 \cup \ldots \cup Q_{q(Q)}$ we associate a Borel subset X_Q of X^r, viz.

$$X_Q = \{(x_0, x_1, \ldots, x_{r-1})\colon \forall i, j < r(x_i = x_j) \leftrightarrow i, j$$
belong to the same piece Q_k of the partition $Q\}$. Clearly

$$X^r = \bigcup \{X_Q\colon Q \text{ a partition of } r\}\,.$$

Now each A_a is a subset of some finite power $X^{\varrho(a)}$ of X. Fix arbitrary sifting functions f_a for A_a in $X^{\varrho(a)}$. Let w^* be an arbitrary element of W. For each a and each partition Q of $\varrho(a)$ define a sifting function for $A_a \cap X_Q$ by:

$$f_{a,Q}(x) = \begin{cases} f_a(\boldsymbol{x}) & \text{if } x \in X_Q\,,\\ w^* & \text{otherwise}\,.\end{cases}$$

Here $\boldsymbol{x}$ represents a "vector" $(x_0, \ldots, x_{\varrho(a)-1})$.

Now let φ_n have form $(\neg)A_{a(n)}(v_{I_n(0)}, \ldots, v_{I_n(r_n-1)})$ where the negation sign $\neg$ is present if and only if $n \geqslant N$, and where of course r_n must equal $\varrho(a(n))$. Let $Q(n)$ be the partition of r_n with i, j in the same piece if and only if $I_n(i) = I_n(j)$. Set:

$$E_n = \{(x_i\colon i \in \omega) \in X^\omega\colon (\neg)\big((x_{I_n(0)}, \ldots, x_{I_n(r_n-1)}) \in A_{a(n)}\big)\}.$$

The fact that $\mathfrak{A} \models \Phi$ translates, working through the above definitions, into the assertion that:

(1) $$\bigcap_{n<N}(X^\omega - E_n) \cap \bigcap_{n\geqslant N} E_n = \emptyset.$$

By Lemma 2 there are sifting functions g_n for the E_n in X^ω inducing covering sequences $(E_n^\alpha\colon \alpha<\omega_1)$ such that for all α:

(2) $$\bigcap_{n<N}(X^\omega - E_n^\alpha) \cap \bigcap_{n\geqslant N} E_n^\alpha = \emptyset.$$

Each g_n induces a ranking function g_n^* for $A_{a(n)} \cap X_{Q(n)}$ defined by fixing $x^* \in X$ and setting:

$$g_n^*(x) = \begin{cases} g_n(y) \text{ if } x \in X_{Q(n)} \text{ where } x = (x_0, \ldots, x_{r_n-1}) \text{ and } y = (y_i\colon i \in \omega) \\ \qquad \text{and } y_{I_n(j)} = x_j \text{ and } y_i = x^* \text{ for } i \notin \operatorname{range} I_n, \\ w^* \quad \text{if } x \notin X_{Q(n)}. \end{cases}$$

(Note that $I_n(i) = I_n(j)$ implies $x_i = x_j$ for $x \in X_{Q(n)}$.)

Thus for each relevant a and Q we have several sifting functions for $A_a \cap X_Q$: The original $f_{a,Q}$ inducing $(A_{a,Q}^\alpha\colon \alpha<\omega_1)$ plus all the g_n^* inducing $(B_n^\alpha\colon \alpha<\omega_1)$ for n with $a(n) = a$, $Q(n) = Q$. Applying Lemma 1 we obtain a CUB $\Delta \subset \omega_1$ and sifting functions $h_{a,Q}$ for $A_a \cap X_Q$ inducing $(C_{a,Q}^\alpha\colon \alpha<\omega_1)$ such that:

(a) For all $\alpha \in \Delta$ and all $n \in \omega$:

$$A_{a(n),Q(n)}^\alpha = B_n^\alpha.$$

(b) For all α and all n:

$$C_{a(n),Q(n)}^\alpha = A_{a(n),Q(n)}^{\sup(\alpha\cap\Delta)} = B_n^{\sup(\alpha\cap\Delta)}.$$

Finally we can define a sifting function h_a for each A_a by setting:

$$h_a(x) = h_{a,Q}(x) \quad \text{for the (unique) partition } Q \text{ with } x \in X_Q.$$

The h_a induce covering sequences $(C_a^\alpha\colon \alpha<\omega_1)$ with $C_a^\alpha = \bigcup\{C_{a,Q}\colon Q \text{ a partition of } \varrho(a)\}$.

Setting $\mathfrak{A}^\alpha = (X, A_a^\alpha)_{a\in\omega}$, $\mathfrak{C}^\alpha = (X, C_a^\alpha)_{a\in\omega}$, it is readily verified using (2), (a), (b), and working backwards through the definitions, that $\mathfrak{A}^\alpha = \mathfrak{C}^\alpha$ for all $\alpha \in \Delta$, that $\mathfrak{C}^\alpha \models \Phi$ for *all* α, and hence that $\mathfrak{A}^\alpha \models \Phi$ for all $\alpha \in \Delta$. This is precisely what is required by the Reflection Principle. ■

The general case. Still restricting our attention to $\mathfrak{A}$ of form (X, A_a), we may extend the above argument to cover arbitrary generalized universal Φ not involving the logical symbol $=$. (Any such Φ is a conjunction of sentences of the special form (**) considered above, and the extension of the arguments above to this case is only notationally more awkward.) Next we may allow $\mathfrak{A}$ to have Borel relations B_b, simply by treating each such B_b and its complement as a new analytic relation A with the trivial sifting function:

$$f(x) = \begin{cases} z^* & \text{if } x \in A, \\ w^* & \text{if } x \notin A \end{cases}$$

where w^*, z^* are arbitrary fixed elements of W and $Z = W$ respectively. (This f is Borel measurable when A is Borel, and induces a covering sequence with all A^α equal to A itself.) Since identity is a Borel relation, we can now allow $=$ to appear in Φ. Borel measurable functions C_c can be handled by the usual model-theoretic trick of treating an r-ary function as an $(r+1)$-ary relation. Of course the presence of distinguished elements D_d can cause no difficulties. We leave the details of the extension of our arguments to prove the Reflection Principle in its full generality to the interested reader.

§ 3. Examples of the Reflection Phenomenon. The Reflection Principle has consequences in several areas of descriptive set theory. We are confident that the illustrations presented in this section do not exhaust the interesting instances of the phenomenon. (In particular we have not investigated applications of the analogous of the Reflection Principle for higher levels of the projective hierarchy.)

Equivalence relations.

COROLLARY 1. *Any analytic equivalence relation on a Polish space is an intersection of $\aleph_1$ Borel equivalences.*

Proof. Let X be Polish, E an analytic equivalence relation on X, $\mathfrak{A} = (X, E)$, and Φ the conjunction of the reflexivity, symmetry, and transitivity axioms. Φ is a universal 1st order sentence and $\mathfrak{A} \models \Phi$. Hence by Reflection:

(a) For any covering sequence $(E^\alpha\colon \alpha<\omega_1)$ for E, $\Delta = \{\alpha<\omega_1\colon E^\alpha \text{ is an equivalence relation}\}$ is CUB in ω_1.

(b) There is a covering sequence $(E^\alpha\colon \alpha<\omega_1)$ for E such that *every* E^α is an equivalence relation.

Thus E is an intersection of Borel equivalences E^α. ■

Our original proof of Corollary 1 essentially involved proving (b) above. Later Solovay gave a proof via (a) above, and we incorporated this proof into our thesis [1], Chapter II. Silver suggested to us the project of extracting from the proof(s) of Corollary 1 a general Reflection Principle.

The proof of Corollary 1 given above can be used to show that a CPCA (Π_2^1) equivalence of the special form $\forall z(x, y, z) \in D$, where D is analytic and for each fixed z $\{(x, y)\colon (x, y, z) \in D\}$ is an equivalence relation, can be represented as an intersection of $\aleph_1$ CA (Π_1^1) equivalences. We leave this result as an exercise to the interested reader.

Elsewhere [2] we have shown using a theorem of Silver, that:

Any equivalence relation on a Polish space which is an intersection of $\aleph_\alpha$ CA *equivalences has either $\leqslant\aleph_\alpha$ or else exactly $2^{\aleph_0}$ equivalence classes.*

(Silver's theorem is the case $\alpha = 0$.) As remarked in [2] this, together with Corollary 1 yields bounds on the number of classes in analytic and special CPCA equivalences, including the main result of our thesis [1]. In fact we get the following, which rightly viewed (see [13]) is a generalization of a theorem of Morley on the number of isomorphism types of countable models of a first-order theory:

COROLLARY 2. *Let X be a Polish space, E an analytic (or special* CPCA) *equivalence on X, A a* PCA (Σ_2^1) *subset of X. Then the number of E-equivalence classes in A is either $\leqslant\aleph_1$ or else exactly $2^{\aleph_0}$.*

Proof. If A is all of X, Corollary 2 is immediate from Corollary 1 and the result from [2] cited above. For $A\subseteq X$ analytic, A is a continuous image under a map f of a Polish space Y. Pulling back E by f yields an analytic (or special CPCA) equivalence on (all of) Y, so again we have $\leqslant\aleph_1$ or $= 2^{\aleph_0}$ classes. Since any PCA set is a union of $\aleph_1$ Borel (hence analytic) sets, we get Corollary 2 in its full generality. ∎

Invariant descriptive set theory. Let X be a Polish space, $P\subseteq X$ a CA set. A binary relation $R\subseteq P^2$ is a CA *prewellordering* (PWO) of P if (i) R is reflexive, transitive, connected, and wellfounded, and (ii) R itself is CA (as a subset of X^2) while $R^* = \{(x, y)\colon y\in X-P \vee (x, y\in P \,\&\, xRy)\}$ is analytic. Every CA subset P of X possesses a CA PWO, for if f is any sifting function for $X-P$, then we get a CA PWO for P by defining xRy to hold if and only if $x, y\in P$ and:

$$\text{least } \alpha\big(f(x)\in W^\alpha\big)\leqslant \text{least } \alpha\big(f(y)\in W^\alpha\big).$$

In the works [0], [5], and [11], it is shown how the existence of CA PWO's can be used to prove the classical Reduction Principle for the class $\mathfrak{P}$ of CA subsets of X: If $A, B\in\mathfrak{P}$ then there exist $A_0, B_0\in\mathfrak{P}$ with $A_0\subseteq A$, $B_0\subseteq B$, $A_0\cup B_0 = A\cup B$, and $A_0\cap B_0 = \emptyset$. We also get the classical First Separation Theorem: Any two disjoint analytic subsets of a Polish space X can be separated by a Borel set.

Let now E be an equivalence relation on X. $A\subseteq X$ is *E-invariant* if whenever $x\in A$ and xEy we have $y\in A$. $R\subseteq X^2$ is *E-invariant* if whenever xRy, $x'Ex$, and $y'Ey$ we have $x'Ry'$. Our proof of the Reflection Principle was in part inspired by Solovay's unpublished proof of Corollary 3(a) below. A version of this proof appears in Miller's thesis [9], and we are grateful to him for explaining it to us. A different proof of Corollary 3(a) is given in our thesis [1]. For more on invariant descriptive set theory, including connections with early work of Vaught and others, see also [3].

COROLLARY 3. *Let X be a Polish space, E an analytic equivalence relation on X.*

(a) (Solovay) *Every E-invariant* CA *subset $P\subseteq X$ possesses an E-invariant* CA PWO.

(b) (Burgess, [3] § 1) *The class of E-invariant* CA *subsets of X satisfies the Reduction Principle.*

(c) (Ryll-Nardzewski, see [7]) *Any two disjoint E-invariant analytic subsets of X can be separated by an E-invariant Borel set.*

Proof. (b) and (c) follow from (a) much as in the classical case (cf. Moschovakis' work cited above). To prove (a), let $P\subseteq X$ be an E-invariant CA set, $A = X-P$, $\mathfrak{A} = (X, A, E)$. $\mathfrak{A}$ is a model of the following sentence Φ:

$$\forall v_0\forall v_1\big(v_0\bar{E}v_1 \,\&\, \bar{A}(v_0)\to\bar{A}(v_1)\big).$$

By Reflection there is a sifting function f for A inducing a covering sequence $(A^\alpha\colon \alpha<\omega_1)$ and a covering sequence $(E^\alpha\colon \alpha<\omega_1)$ for E such that for all α $(X, A^\alpha, E^\alpha)\models\Phi$, i.e. A^α is E^α- and hence E-invariant. Define xRy to hold if $x, y\in P$ and:

$$\text{least } \alpha\ \big(f(x)\in W^\alpha\big)\leqslant \text{least } \alpha\ \big(f(y)\in W^\alpha\big).$$

Then R is an E-invariant CA PWO for P as required. ∎

Separating analytic sets by Borel sets with special properties. Let X be a Polish space. A property $\mathfrak{P}$ of subsets of X is called a *faithful separation* (FS) *property* if for any disjoint analytic $A, E\subseteq X^2$, if there exists any $B\subseteq X^2$ separating A, E (i.e. such that $A\subseteq B$, $B\cap E = \emptyset$) such that every I-section $B_y = \{x\colon (y, x)\in B\}$ of B has $\mathfrak{P}$, then there exists a Borel set with the same properties. Clearly if $\mathfrak{P}$ is an FS property, so is the property of being the complement of a set with property $\mathfrak{P}$. $\mathfrak{P}$ is a CA-*monotone* property if there exists an analytic $T\subseteq X^\omega$ (product of countably many copies of X) such that for any $A\subseteq X$, A has $\mathfrak{P}$ if and only if $A^\omega\cap T = \emptyset$. Such a T is called a *test set* for $\mathfrak{P}$. A CA-monotone property $\mathfrak{P}$ is "monotone" in the sense that whenever A has $\mathfrak{P}$ and $A'\subseteq A$, then A' has $\mathfrak{P}$.

COROLLARY 4 (Cenzer & Mauldin [4]). *Every* CA-*monotone property is* FS.

Proof. Let X be a Polish space, and to avoid trivialities assume X uncountable. Let $\mathfrak{P}$ be a CA-monotone property of subsets of X, $T\subseteq X$ an analytic test set for $\mathfrak{P}$. Let A, E be disjoint analytic subsets of X^2 which can be separated by a set B all of whose I-sections have $\mathfrak{P}$. Since $A\subseteq B$ and $\mathfrak{P}$ is monotone, it follows all I-sections of A have $\mathfrak{P}$. Fix a Borel isomorphism $F\colon X\to X^\omega$ and for $n\in\omega$ let $F_n = \pi_n F$, where $\pi_n\colon X^\omega\to X$ is projection to the nth coordinate. Let $S = F^{-1}(T)$. Let

$$\mathfrak{A} = (X, A, E, S, F_n)_{n\in\omega}.$$

Let Φ be:

$$\forall v_0\forall v_1\big(\neg\bar{S}(v_1)\vee\bigvee_{n\in\omega}\neg\bar{A}(v_0, \bar{F}_n(v_1))\big)\wedge\forall v_0\forall v_1\big(\neg\bar{A}(v_0, v_1)\vee\neg\bar{E}(v_0, v_1)\big).$$

Then $\mathfrak{A}\models\Phi$, and by Reflection any covering sequences for A, E, and S will yield Borel $A^\alpha\supseteq A$, $E^\alpha\supseteq E$, $S^\alpha\supseteq S$ such that $(X, A^\alpha, E^\alpha, S^\alpha, F_n)\models\Phi$. It follows A^α is a Borel set separating A, E all of whose I-sections have $\mathfrak{P}$. ∎

Cenzer and Mauldin show that, *inter alia*, the properties of being:

(FS 1) finite,

(FS 2) scattered,

(FS 3) totally bounded (closure compact), (FS4) nowhere dense (NWD),

(FS5) of Jordan content 0 (for X = unit interval)

are CA-monotone, hence FS. (So, of course, are the complementary properties; see [4].)

J. Saint-Raymond has shown that the property of being:

(FS6) σ-compact

is also FS. His proof establishes the same result for:

(FS7) F_σ,

(FS8) compact,

(FS9) closed.

(Compactness is also treated in [4].) Let us sketch a proof, using Reflection, of the easiest of these, (FS9): Let X be a Polish space, and to avoid trivialities let us assume X uncountable. Let $A, E \subseteq X^2$ be disjoint analytic sets which can be separated by a set all of whose I-sections are closed. Let $T = \{(x_i: i \in \omega) \in X^\omega: (x_i: i>0)$ converges to x_0 in $X\}$, a Borel set. Let F, F_n be as in the proof of Corollary 4, and set $S = F^{-1}(T)$. Then $B = \{(y, x): x \in \text{closure } A_y\} = \{(y, x): \exists z(z \in S \,\&\, F_0(z) = x \,\&\, \forall i \in \omega (y, F_i(z)) \in A\}$ is an analytic set separating A, E, all of whose I-sections are closed. Let $\mathfrak{A} = (X, B, E, S, F_n)_{n\in\omega}$, and let Φ be:

$$\forall v_0 \forall v_1 (\neg \bar{S}(v_1) \vee \bar{B}(v_0, \bar{F}_0(v_1)) \vee \bigvee_{i>0} \neg \bar{B}(v_0, \bar{F}_i(v_1))) \wedge \\ \wedge \forall v_0 \forall v_1 (\neg \bar{B}(v_0, v_1) \vee \neg \bar{E}(v_0, v_1)) .$$

Then $\mathfrak{A} \models \Phi$ and Reflection provides Borel $B^\alpha \supseteq B \supseteq A$, $E^\alpha \supseteq E$ such that $(X, B^\alpha, E^\alpha, S, F_n) \models \Phi$. Thus B^α is a Borel set separating A, E, all of whose I-sections are closed.

That the property of being:

(FS10) Δ_2^0 (simultaneously F_σ and G_δ)

is FS follows from the next theorem, which was inspired by an effective version of S. Raymond's theorem due to A. Louveau, and by a classical construction of Hausdorff. Recall that an ordinal is odd (even) if it is of form $\lambda+n$, λ a limit ordinal, n an odd (resp. even) integer.

Uniform Hausdorff Development Theorem. *Let X be a Polish space, and $A, E \subseteq X^2$ disjoint analytic sets which can be separated by a set all of whose I-sections are Δ_2^0. Then there exist a countable ordinal ν and a sequence B_η, $\eta \leqslant \nu$, of Borel subsets of X^2, such that:*

(0) *Each I-section of each B_η, $\eta \leqslant \nu$, is closed,*

(1) $B_0 = X^2$,

(2) $B_{\eta+1} \subseteq B_\eta$, *for all* $\eta < \nu$,

(3) $A \cap B_\eta \subseteq B_{\eta+1}$, *for all even* $\eta < \nu$, *and* $E \cap B_\eta \subseteq B_{\eta+1}$, *for all odd* $\eta < \nu$,

(4) $B_\lambda = \bigcap_{\eta<\lambda} B_\eta$, *for all limit ordinals* $\eta \leqslant \nu$,

(5) $B_\nu = \emptyset$.

Thus $B = \bigcup_{\eta<\nu, \eta \text{ odd}} (B_{\eta+1} - B_\eta)$ *is a Borel set separating A, E, all of whose I-sections are Δ_2^0.*

Proof. Fix a complete metric δ on X and define a sequence of analytic sets $C_\eta \subseteq X^2$, η a countable ordinal, as follows:

(1′) $C_0 = X^2$.

For even η, $C_{\eta+1} = \{(y, x): \forall n \in \omega\, \exists z \in X(\delta(x, z) < 2^{-n} \,\&\, (y, z) \in A \cap C_\eta)\}$; for odd η, $C_{\eta+1}$ is similarly defined, with E in place of A.

(4′) $C_\lambda = \bigcap_{\eta<\lambda} C_\eta$ for countable limit ordinals.

Clearly this construction guarantees:

(0′) Each I-section of each C_η is closed,

(2′) $C_{\eta+1} \subseteq C_\eta$, for all η,

(3′) $A \cap C_\eta \subseteq C_{\eta+1}$, for all even η, and $E \cap C_\eta \subseteq C_{\eta+1}$, for all odd η.

We can readily "formalize" the above construction: let T be the subset of $2^{\omega\times\omega} \times 2^\omega \times X \times X^{\omega\times\omega}$ consisting of all $(t, u, y, (x_{ij}: i, j \in \omega))$ such that:

(i) t is the characteristic function of a linear order $\prec$ on ω in which 0 occupies the first place, and 1 the last, and in which every element i but the last has an immediate successor Si.

(ii) u is the characteristic function of a subset e of ω containing all i which are not immediate successors in $\prec$, and containing the immediate successor Si of $i \in \omega$ if and only if it does not contain i. (So in case $\prec$ is a wellorder, e contains precisely those i occupying even positions in that order.)

(iii) Writing X_i for $\{x_{ij}: j \in \omega\}$, we have: Whenever $i \prec k$, then $X_k \subseteq X_i$. Whenever $k = \text{S}i$ and $i \in e$ (resp. $i \notin e$), then for every $x \in X_k$ and $n \in \omega$ there exists a $z \in X_i$ with $\delta(x, z) < 2^{-n}$ and $(y, z) \in A$ (resp. $(y, z) \in E$).

A tedious but routine computation shows that T is analytic. A routine induction shows that conditions (i)-(iii) imply that if $\prec$ is a wellorder, $i \in \omega$ occupies the ηth place in that order, and $x \in X_i$, then $x \in C_\eta$. Conversely, if t, u satisfy (i), (ii) above, $\prec$ is a wellorder, and 1 occupies the ηth place in that order, and $x \in C_\eta$, then a Löwenheim–Skolem–style argument produces an $\mathbf{x} = (x_{jk}: j, k \in \omega) \in X^\omega$ such that $(t, u, y, \mathbf{x}) \in T$ and $\mathbf{x} \in X_1$.

Now we claim that the analytic set $T' = \{t: \exists u, y, x((t, u, y, x) \in T)\}$ contains only characteristic functions of wellorders. For say $(t, u, y, x) \in T$ and suppose, with other notation as above, that we have a descending sequence $\ldots i_2 \prec i_1 \prec i_0 \prec 1$ in $\prec$. Without loss of generality we may assume $i_r \in e$ if and only if r is even. Set $k_r = \text{S}i_r$. Now the I-sections A_y, E_y of A, E can be separated by a Δ_2^0 set. So let us fix open G_r and closed F_r, $r \in \omega$, such that $\bigcap_r G_r = \bigcup_r F_r$, and this set separates A_y, E_y.

Since $\emptyset \neq X_{k_0}$ and $i_0 \in e$, there exists $x_0 \in X_{i_0}$ with $(y, x_0) \in A$, by (iii) above.

Hence $x_0 \in \bigcap_r G_r$. Pick $n_0 \in \omega$ so that the open δ-disc D_0 of radius 2^{-n_0} about x_0 is contained in G_0. Since $x_0 \in X_{i_0}$ and $i_1 \in e$, there exists $x_1 \in X_{i_1}$ with $\delta(x_0, x_1) < 2^{-n_0}$ and $(y, x_1) \in F$. Hence $x_1 \in D_0 \cap \bigcap_r (X-F_r)$. Pick $n_1 > n_0$ so that the open δ-disc D_1 of radius 2^{-n_1} about x_1 is contained in $D_0 \cap (X-F_0)$. Iterating, we obtain a sequence x_r, $r \in \omega$, converging to a point $z \in \bigcap_r G_r \cap \bigcap_r (X-F_r) = \emptyset$, a contradiction which establishes our claim that T' contains only characteristic functions of wellorders.

Now by the Boundedness Theorem, there is a countable ordinal ν strictly greater than the order type of any wellorder whose characteristic function is in T'. In view of our remarks immediately following the definition (i)-(iii) of T above, it follows that:

(5′) $C_\nu = \emptyset$.

Now (0′)-(5′) may all be expressed by generalized universal sentences with symbols for A, E, and the C_η, $\eta \leqslant \nu$. For (0′) we also need symbols for an auxiliary Borel set S and some auxiliary Borel measurable functions F_n, as in our remarks on (FS9) above. Hence Reflection supplies Borel $B_\eta \supseteq C_\eta$ and $A' \supseteq A$, $E' \supseteq E$ satisfying these same conditions. In connection with (3), notice that $A' \cap B_\eta \subseteq B_{\eta+1}$ implies $A \cap B \subseteq B_{\eta+1}$, and similarly for E. Thus the B_η satisfy all of (0)-(5), completing the proof. ■

Kechris [6] introduces a general method which can be used to show the properties of being:

(FS11) countable,

(FS12) meager (1st category),

(FS13) of Lebesgue measure 0 (for X = unit interval),

(FS14) σ-bounded (a subset of a σ-compact set)

are FS (as well as: nonmeager, positive measure, etc.). For (FS11) this is a classical theorem of Lusin [8]; for (FS12) it is implicit in Vaught [13]. (FS13), (FS14), and other examples are new in [6]. We are grateful to Prof. Kechris for making this work of his available to us in advance of publication, and for bringing to our attention the work of S. Raymond and Louveau mentioned above.

We may mention one further group of FS properties. Let E be a CA equivalence relation on a Polish space X. The following properties of subsets of X are FS:

(FS15) (E a CA equivalence) Being a subset of an E-equivalence class,

(FS16) (E a CA equivalence) Containing representatives of only countably many E-equivalence classes.

The former is CA-monotone; FS for the latter is implicit in Silver's (unpublished) work on CA equivalences. If instead we take E analytic, the following are FS:

(FS17) (E an analytic equivalence) Containing no pair of E-equivalent elements,

(FS18) (E an analytic equivalence) Having countable intersection (possibly empty) with every E-equivalence class.

Again (FS17) is CA-monotone. We will not enter into the proof for (FS18) here.

If $\mathfrak{P}$ is a property of subsets of a Polish space X, by $\mathfrak{P}_\sigma$ we mean the property of being a union of countably many sets each having property $\mathfrak{P}$. If $\mathfrak{P}$ is the property of being finite (NWD, closed, compact, totally bounded), then $\mathfrak{P}_\sigma$ is the property of being countable (resp. meager, F_σ, σ-compact, σ-bounded). If $\mathfrak{P}$ is (FS15) or (FS17), $\mathfrak{P}_\sigma$ is (FS16) or (FS18) respectively. These examples suggest the question: Is $\mathfrak{P}_\sigma$ always FS when $\mathfrak{P}$ is? Affirmative examples, in addition to those just mentioned, are provided by a remarkable result recently announced by A. Louveau, which implies that the properties of being $G_{\delta\sigma}$, of being $F_{\sigma\delta\sigma}$, of being $G_{\delta\sigma\delta\sigma}$, &c. are all FS. One fairly simple case is open: For $X = 2^\omega$ consider the property $\mathfrak{P}$ of containing no two elements of comparable Turing degree. This $\mathfrak{P}$ is CA-monotone, hence FS. Is $\mathfrak{P}_\sigma$ FS?

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