On a coincidence of mappings of compact spaces
in topological groups

by

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Abstract. Let $X$ be a weak-oligosocial space of finite type and $p, q : X \to Y$ single-valued continuous mappings. If $p$ is a Vietoris mapping and the number of coincidences

$$
\Lambda(p, q) = \sum (-1)^i \alpha_{pq} i^i
$$

is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$.

1. Let $X$ and $Y$ be metric compact spaces and $p, q : X \to Y$ continuous mappings. The mappings $p$ and $q$ have a coincidence if there exists a point $x \in X$ such that $p(x) = q(x)$. It is necessary to impose some conditions on the space $X$, $Y$ and on the mappings $p$ and $q$, in order to be sure that the mappings $p$ and $q$ have a coincidence. For example, S. Lefschetz gave in [6] a sufficient condition for mappings $p$ and $q$ to have a coincidence in the case where $X$ and $Y$ are $n$-dimensional closed manifolds with triangulation. This condition is given in the frame work of the homology and the cohomology of the spaces $X$ and $Y$, [6], Ch. 8, 29.12.

S. Eilenberg and D. Montgomery proposed in [2] another way to obtain a sufficient condition for a coincidence of mappings $p$ and $q$. In this case $X$ and $Y$ are metric compact spaces, $Y$ is an absolute neighbourhood retract and the mapping $p : X \to Y$ is a Vietoris mapping, i.e. $p(X) = Y$, and for every point $y \in Y$ the space $p^{-1}(y)$ is connected and the homology groups of Alexandrov-Cech $H_i(p^{-1}(y))$ with rational coefficients are zero for $i > 1$. S. Eilenberg and D. Montgomery gave in [2] the following arguments. If the mapping $p$ is a Vietoris mapping, then the homomorphism

$$
\rho_p = \{ \rho_p \} : H_\alpha(X) \to H_\alpha(Y)
$$

is an isomorphism, [1]; here the homomorphism $\rho_p$ is induced by the mapping $p$.

Let us consider the linear mapping

$$
q_{h\alpha p^{-1}} : H_\alpha(X) \to H_\alpha(Y).
$$

The space $H_\alpha(Y)$ is a finite-dimensional vector space over the rationals and, for almost all $h$, has dimension zero (this follows because the space $Y$ is a compact ANR space).
In this case we can consider the trace $\text{tr} q_p q^{-1}$ of the linear mapping $q_p q^{-1}$.
The number of coincidence, $A(p, q)$, of mappings $p$ and $q$ is
\[
A(p, q) = \sum (-1)^i \text{tr} q_p q^{-1}.
\]
The following theorem is proved in [2]: If $Y$ is a compact metric ANR space, $p$ is a Vietoris mapping and $A(p, q) \neq 0$, then the mappings $p$ and $q$ have a coincidence, i.e., there exists a point $x \in X$ such that $p(x) = q(x)$.

L. Górničevic gave a modern proof and a generalization of this theorem of S. Eilenberg and D. Montgomery in [4]. A further generalization of this theorem is proposed in [5].

Let us recall that the theorem of S. Eilenberg and D. Montgomery is closely related to the problem of existence of fixed points for multi-valued mappings.

Let $\Phi$ be an upper semi-continuous multi-valued mapping of a compact metric space $X$ in itself. Suppose also that the mapping $\Phi$ is acyclic. This means that the set $\Phi(x)$ is connected and the homology groups of Alexandrov-Čech $H_i(\Phi(x))$ with rational coefficients are zero for $i \geq 1$ and for every point $x \in X$.

The mapping $\Phi$ has a fixed point if there exists a point $x \in X$ such that $x \in \Phi(x)$.

Let $\Gamma(\Phi)$ be the graph of the mapping $\Phi$, i.e.,
\[
\Gamma(\Phi) = \{(y, z) \in X \times X \mid z \in \Phi(y)\}.
\]

By $p$ and $q$ we shall denote mappings
\[
p, q: \Gamma(\Phi) \to Y,
\]
such that $p(y, z) = y$ and $q(y, z) = z$ for $(y, z) \in \Gamma(\Phi)$. Then it follows that
\[
\Phi(y) = q p^{-1}(y)
\]
for every point $y \in X$.

The mapping $\Phi: \Gamma(\Phi) \to Y$ is a Vietoris mapping because $\Phi$ is an acyclic and upper semi-continuous mapping.

The mapping $\Phi$ has a fixed point if and only if the mappings $p$ and $q$ have a coincidence.

We shall consider the following problem. Let $Y$ be a metric compact space and $G$ — a compact, connected and finite dimensional topological group. Let $p, q: Y \to G$ be continuous single valued mappings, and $p$ — a Vietoris mapping.

The space $G$ is a metric compact space, because $G$ is a compact, connected group, [8]. Therefore the homology groups $H_i(G)$ of Alexandrov-Čech with coefficients are finite dimension-vector spaces over the field of rational numbers and for almost all $k$ the space $H_k(G)$ is zero-dimensional, [9]. Then the number of coincidence,
\[
A(p, q) = \sum (-1)^i \text{tr} q_p q^{-1},
\]
of the mappings $p$ and $q$ exists.

In the case where $G$ is a locally connected topological space, $G$ is an ANR space, because $G$ is a Lie group, [7], Ch. 8. Then, if the number of coincidence is not zero, the mappings $p$ and $q$ have a coincidence, [2].

In this paper we consider the question of existence of a coincidence of the mappings $p$ and $q$ without the assumption that $G$ is a locally connected space. It is known that $G$ is locally homeomorphic to the product of a finite-dimensional ball and the Cantor discontinuum, [8].

We shall prove the following theorem:

**Theorem 1.** Let $G$ be a compact, connected, finite-dimensional topological group and $Y$ a metric compact space. Let $p, q: Y \to G$ be single-valued mappings and $p$ — a Vietoris mapping. If the number of coincidence,
\[
A(p, q) = \sum (-1)^i \text{tr} q_p q^{-1},
\]
is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$, i.e. the mappings $p$ and $q$ have a coincidence.

From this theorem follows Lefschetz's fixed point theorem of [9].

Actually we shall prove a more general theorem than Theorem 1. To formulate that theorem we need one definition.

The compact, connected space $X$ is called a weak-solenoidal space if $X$ is the limit of an inverse system
\[
\{X_k, \pi(k+1, k) \mid k = 1, 2, ...\}
\]
such that

1) $X_k$ is a connected finite polyhedron.
2) $\pi(k+1, k): X_{k+1} \to X_k$ is a finite sheet covering space.
An example of a weak-solenoidal space is the following. Let $A$ be a compact, connected, finite-dimensional topological group and $B$ a closed subgroup in $A$.

Then the quotient space $A/B$ is a weak-solenoidal space, [8].

Let us recall that the space $Y$ is called a space of finite type if dim$H_k(Y) < \infty$.

We shall prove the following theorem:

**Theorem 1’.** Let $X$ be a weak-solenoidal space of finite type and $p, q: Y \to X$ single-valued continuous mappings. If $p$ is a Vietoris mapping and the number of coincidence, $A(p, q)$, is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$, i.e., the mappings $p$ and $q$ have a coincidence.

The paper is divided into eight sections. In Section 1 we give a construction of P. S. Alexandrov: a realization of a given $\varepsilon$-mapping as an $\varepsilon$-translation. In Section 2 we give an embedding of a given inverse system in a Banach space with the construction given in Section 1. In Section 4 we consider the chain homomorphisms induced by the bonding mappings of this inverse system. In Section 5 we define the chain homomorphisms inducing in homology the inverse homomorphisms of the homomorphisms induced by projections of the space $X$ on $X_k$. In Section 6 we recall a lemma of E. Begle from [1] and Section 8 contains the proof of Theorem 1’.

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2. Let $H$ be a Banach space. We shall denote by $||x||$ the norm of the element $x$ in the space $H$. Let $X$ be a compact subspace in the space $H$. The compact space $X$ is said to be linearly independently imbedded (l.i.i.) in the space $H$ if the points $\{x_0, ..., x_s\}$ are vertices of a $s$-dimensional simplex in the space $H$ for every $x$ and $x_i \in X$, $i = 0, ..., s$.

Suppose that the space $X$ is l.i.i. in the space $H$.

Let $f: X \rightarrow K$ be a mapping of the space $X$ in the compact polyhedron $K$ and let $f$ be an $s$-mapping, i.e., the diameter of the set $f^{-1}(y)$ is less than $s$ for every $y \in K$.

Suppose that $\tau$ is a triangulation of the polyhedron $K$ such that

$$\text{diam } f^{-1}(\text{St}(a, \tau)) < \epsilon$$

of the triangulation $\tau$ of $K$. Here $\text{St}(a, \tau)$ is the open star of the vertex $a$ in the triangulation $\tau$. For $A \subset X$ we denote by $\text{diam } A$ the diameter of the set $A$.

Let us denote by $\{e_1, ..., e_s\}$ all vertices of the triangulation $\tau$ and by $V(Y)$ the set $f^{-1}(\text{St}(e_k, \tau))$. The family of open sets $\omega = \{V_1, ..., V_s\}$ is an open covering of the space $X$ and the diameter of every set $V_i$ is less than $\epsilon$, by (1).

Let $a_k$ be a point in the set $f^{-1}(e_k)$ for $k = 1, ..., s$. Then we have

$$||a_k - a_i|| < 2\epsilon \quad \text{if} \quad V_k \cap V_i \neq \emptyset.$$ (2)

Let us consider the points $\{a_1, ..., a_s\}$. If the space $X$ is l.i.i. in the space $H$, the points $\{a_1, ..., a_s\}$ are vertices of an $(s-1)$-dimensional simplex in $H$.

By definition, $f(x) \in \text{St}(a_k, \mu)$ for $x \in V_k$ (here $\text{St}(a_k, \mu)$ is the open star of the vertex $a_k$ in the triangulation $\mu$).

From (2) we have

$$||f(x) - a_k|| < 2\epsilon.$$ (3)

It follows from (1) and (3) that

$$||x - f(x)|| < 3\epsilon,$$ (4)

i.e., the mapping $f$ is a $3\epsilon$-mapping of the space $X$ in the space $H$.

3. Let $X$ be a weak-solenoidal space, i.e., $X$ is a compact space and $X$ is the limit of the inverse system

$$\{X_k, \pi(k+1, k), k = 1, 2, ...\}$$

where

1) $X_k$ is a finite connected polyhedron.

2) $\pi(k+1, k): X_{k+1} \rightarrow X_k$ is a finite sheet covering map.

We can also assume that

3) the projections $\pi_i: X \rightarrow X_i$ are $2^{-i}$-mappings, i.e.,

$$\text{diam } \pi_i^{-1}(x) < 2^{-i}$$

for every point $x \in X_i$.

4) in the space $X_i$, we choose a triangulation $\tau_i$ such that

$$\text{diam } \pi_i^{-1}(\text{St}(a, \tau_i)) < 2^{-i}$$

for every vertex $a$ of the triangulation $\tau_i$.

5) $\pi(i+1, i): X_{i+1} \rightarrow X_i$ is a simplicial mapping of the triangulation $\tau_{i+1}$ on the triangulation $\tau_i$ for $i > 2, i = 1, ...$

Here $\tau_i^{\text{bary}}$ is an $s_i$-barycentric subdivision of the triangulation $\tau_i$.

The mappings $\pi_i: X \rightarrow X_i$ (the projections) and $\pi(i+1, i): X_{i+1} \rightarrow X_{i}$ induce the homomorphisms of the homology groups

$$\pi_i^*: H_i(X) \rightarrow H_i(X_i),$$

$$\pi(i+1, i)^*: H_i(X_{i+1}) \rightarrow H_i(X_i).$$

We shall prove later (Section 4, Lemma 1) that the homomorphisms $(\pi(i+1, i))_*$ are epimorphisms. If

$$H_i(X) = \lim \{H_i(X_n), (\pi(i+1, i))_*\}$$

and

$$\text{dim } H_i(X) < \infty,$$

there exists an $i_0$ such that the homomorphisms $(\pi_i)_*$ and $(\pi(i+1, i))_*$ are isomorphisms for $i > i_0$.

We suppose that

6) the homomorphisms $(\pi_i)_*$ and $(\pi(i+1, i))_*$ are isomorphisms for $i = 1, ...$.

Let $l_i$ be the number of points in the set $\pi(i+1, i)^{-1}(x)$ for $x \in X_i$.

Let us recall that the compact space $X$ is l.i.i. in the space $H$ (by a theorem of K. Kuratowski every compact space can be l.i.i. in some Banach space, [10]).

Let us consider the mappings

$$\pi(i+1, i): X_{i+1} \rightarrow X_i.$$ (5)

The mapping $\pi(i+1, i)$ is a locally trivial bundle. Therefore for every vertex $a$ of the triangulation $\tau_i$, the mapping

$$\pi(i+1, i)|\pi(i+1, i)^{-1}(\text{St}(a, \tau_i)): \pi(i+1, i)^{-1}(\text{St}(a, \tau_i)) \rightarrow \text{St}(a, \tau_i)$$

is a trivial bundle, and the mapping $\pi(i+1, i)$ is a simplicial isomorphism on the components of the set $\pi(i+1, i)^{-1}(\text{St}(a, \tau_i))$. So the mapping $\pi(i+1, i)$ and the triangulation $\tau_i$ induce the triangulation $\tau_{i+1}$ on the space $X_{i+1}$.
Let us consider the open coverings of the space $X_{i+1}$

$s(\tau_{i+1}) = \{St(a, \tau_{i+1})|a\}$ — a vertex of the triangulation $\tau_{i+1}$,

$s(\tau_{i+1}) = \{St(b, \tau_{i+1})|b\}$ — a vertex of the triangulation $\tau_{i+1}^r$.

If $s(\tau_{i+1})$ is a star refinement of the covering $s(\tau_{i+1})$.

Let the vertices of the triangulations $\tau_i$ and $\tau_{i+1}$ be

$\{e_{1}, ..., e_{l}\}, \{e'_{1}, ..., e'_{l}\}$.

Then the triangulation $\tau_{i+1}$ has the vertices $\{e_{1}^{k}, ..., e_{l}^{k+1}\}$, where $k = m_{i}-r$, and

$\pi(i+1, l)(e_{v}^{k+1}) = e'_{v}$ for $v = m_{i}-r, 0 < r < l$.

By $Y_{r}$ we shall denote the set $\pi_{i+1}(St(a, \tau_{i}))$ for a given vertex $a$ of the triangulation $\tau_i$. So we have a finite, open covering

$\omega_i = \{Y_{r} | a \}$ — a vertex of the triangulation $\tau_i$)

of the space $X$. It follows from (6) that

$diam Y_{r} < 2^{-i}$ for every $Y_{r} \in \omega_i$.

If the covering $s(\tau_{i+1})$ is a star refinement of the covering $s(\tau_{i+1})$, then $\omega_{i+1}$ is a star refinement of the covering $\omega_{i}$ for $i = 1, ..., t$.

Let us consider the covering $\omega_i$ of the space $X$ and let $a_i$ be a point in the set $\pi_{i+1}(St(a, \tau_{i}))$, for $a_i$ a vertex of the triangulation $\tau_i$. So we obtain a subset

$E_i = \{a_{1}, ..., a_{l}\}$

in the space $X$. It follows from (2) that

$diam E_{i} < 2^{-i}$.

$E_{i}$ and $e_{i}$ being vertices of a one dimensional simplex of the triangulation $\tau_i$. Also

$\pi_{i}(a_{i}) = e_{i}$ for $k = 1, ..., l$.

The points of the set $E_i$ are vertices of a $(t_i - l)$-dimensional simplex in the space $X$, because the space $X$ is l.i.i. in the space $H$.

Let us define a simplicial homomorphism $k_i: X \rightarrow H$ by $k_i(e_i') = a_{l}$ for

$s = 1, ..., t_l$.

Let $p_i = k_i; \pi_i: X \rightarrow A_i$, here $A_i = k_i(X)$.

It follows from (4) that

$||x - p_i(x)|| < 3 \cdot 2^{-i}$ for $x \in X$ and $i = 1, 2, ...$

Let us consider the mapping

$p(i+1, l): A_{i+1} \rightarrow A_i$

given by

$p(i+1, l) = k_i\pi(i+1, l)k_{i+1}^{-1}$.

It follows that

(9) $||p(i+1, l)(y) - y|| < 3 \cdot 2^{-i}$ for $y \in A_{i+1}$.

We denote by $\mu_i$ the image of the triangulation $\tau_i$ under the simplicial homomorphism $k_i$.

The mapping $p(i+1, l)$ is a simplicial $l_i$-sheet covering map of the triangulation $\mu_{i+1}$ on the triangulation $\mu_i$. Here $\mu_i$ is an $l_i$-barycentric subdivision of the triangulation $\mu_i$.

4. Let $K$ be a finite simplicial complex with a given triangulation $\tau$. We denote by

$C_{*}(K, \tau) = (C_{*}(K, \sigma) \mid \sigma = 0, 1, ...)$

the chain complex of the triangulation $\tau$. With rational coefficients.

The simplicial mapping $p(i+1, l)$ induces the chain homomorphism

$p(i+1, l)_*: C_{*}(A_{i+1}, \mu_{i+1}) \rightarrow C_{*}(A_i, \mu_i)$

for $i = 1, 2, ...$

We denote by

$\sigma(i+1, l)_* = \{\sigma(i+1, l)_{j} \mid \sigma = 0, 1, ...\}$

the chain homomorphism

$\sigma(i+1, l)_*: C_{*}(A_{i+1}, \mu_{i+1}) \rightarrow C_{*}(A_i, \mu_i)$

given by

$\sigma(i+1, l)_*(\sigma) = \sum \alpha_i e_i$

where

$\alpha_i = \{a_i, ..., a_i\}, \sigma = \{a_i, ..., a_i\}$

and

$p(i+1, l)(a_i) = a_i$.

Let us consider the homomorphism

$\sigma(i+1, l)_*: H_{*}(A_{i+1}) \rightarrow H_{*}(A_i)$

induced by the chain mapping $p(i+1, l)_*$.

It follows that

(10) $p(i+1, l)_*\sigma(i+1, l)_* = \lambda_i: id$

here

$p(i+1, l)_*: H_{*}(A_{i+1}) \rightarrow H_{*}(A_i)$

is the homomorphism induced by the chain mapping $p(i+1, l)_*$.

It follows from (10) that the homomorphism $p(i+1, l)_*$ is an epimorphism for every $i = 1, 2, ...$. The map $p(i+1, l)_*$ is an epimorphism for every $i = 1, 2, ...$. 

Fundamenta Mathematicae T. Civ
Let us recall that
\[ p(i+1,i) = k_1 p(i+1,i) k_1^{-1} \]
where \( k_i \) and \( k_1^{-1} \) are homeomorphisms. So we have

**Lemma 1.** The homomorphisms \( \sigma(i+1,i) \) are epimorphisms for \( i = 1, 2, \ldots \)
and \( s = 0, 1, \ldots \)

It follows from (10) that the homomorphism \( \sigma(i+1,i) \) is an isomorphism because \( p(i+1,i) \) is an isomorphism.

Let
\[ \phi(i+1,i) = \phi(i+1,i) \mid s = 0, 1, \ldots \]
be a chain mapping such that
\[ \phi(i+1,i) = \phi^{-1}(i+1,i) \]

If \( \phi(i+1,i) \) is the homomorphism of the homology groups induced by the chain mapping \( \phi(i+1,i) \), then
\[ p(i+1,i) \phi(i+1,i) = \text{id} \]

5. Let \( M \) be a compact metric space and \( \nu_i \), \( i = 1, 2, \ldots \), a fundamental system of finite open covering of the space \( M \) such that

a) every element of the covering \( \nu_i \) has a diameter less than 2\(^{-i} \)

b) \( \nu_i \) is a star refinement of the covering \( \nu_i \) for \( i = 1, 2, \ldots \)

The ordered \((k+1)-tuple \( \sigma^k = (x_0, \ldots, x_k) \) of points of the space \( M \) is called a \( \text{\textit{k}} \)-dimensional Vietoris simplex of the space \( M \). We denote by \( M(\nu_i) \) the simplicial complex of all \( \nu_i \)-Vietoris simplexes of the space \( M \). If \( A \) is a subset of the space \( M \), then by \( M(\nu_i) \cap A \) we denote the closed subcomplex \( M(\nu_i) \) consisting of all simplexes of \( M(\nu_i) \) with vertices belonging to the set \( A \).

We denote by \( V_a(M, \nu_i) \) the chain complex with rational coefficients of the simplicial complex \( M(\nu_i) \). By \( [c] \) we denote the closed simplicial complex of all \( \nu_i \)-Vietoris simplexes of the chain \( \sigma \) for a given \( \sigma \in V_a(M, \nu_i) \).

Let
\[ V_a(M) = \prod \{ V_a(M, \nu_i) \mid i = 1, 2, \ldots \} \]

The elements of the group \( V_a(M) \) are the Vietoris chains of the space \( M \).

Let \( \tilde{a} = \{ a_i \mid i = 1, \ldots \} \) be the Vietoris chain of the space \( M \). The chain \( \tilde{a} \) is called a convergent Vietoris chain if the chain \( a_i \) is homologous to the chain \( a_{i-1} \) in \( V_a(M, \nu_i-1) \) for \( i = 2, 3, \ldots \)

By \( V_a(M) \) we shall denote the linear space over the rationals with the basis consisting of all convergent Vietoris chain of the space \( M \).

Let \( M \) and \( \tilde{M} \) be compact metric spaces and
\[ v = \{ v_i \mid i = 1, 2, \ldots \} \quad \text{and} \quad \tilde{v} = \{ \tilde{v}_i \mid i = 1, 2, \ldots \} \]

fundamental systems of finite open covering satisfying a) and b) in \( M \) and \( \tilde{M} \).

Let \( f : M \to \tilde{M} \) be a continuous mapping. We shall define the chain mapping \( f_* : V_a(M) \to V_a(\tilde{M}) \).

Let us consider the covering \( \tilde{v}, f^{-1} \tilde{v} \) is an open, finite covering of the space \( M \).

Let \( \nu_i \) be a covering which is a refinement of the covering \( f^{-1} \tilde{v} \) and \( \nu_i \in v \). Then the image of the complex \( M(\nu_i) \) under the mapping \( f_* \) belongs to the complex \( \tilde{M}(\tilde{v}) \).

So for \( \nu_i \in \tilde{v} \), we have
\[ f_*(\nu_i) = f(\nu_i) = \{ f(\nu_i) \mid i = 1, \ldots \} \in V_a(\tilde{M}) \]

By \( \tilde{N}(\nu_i) \) we shall denote the nerve of the covering \( \nu_i \) for a given finite, open covering of the space \( M \).

Let \( \omega', \omega'' \) be a finite, open covering of the space \( M \)
\[ \omega' = \{ U'_1, \ldots, U'_s \} \quad \text{and} \quad \omega'' = \{ U''_1, \ldots, U''_t \} \]

and the covering \( \omega' \) is a refinement of the covering \( \omega'' \). Then we have the simplicial mapping \( i : N(\omega') \to N(\omega'') \) given by \( i(U'_1) = U''_1 \) for \( U'_1 \in \omega' \) and \( U''_1 \in \omega'' \) such that \( U'_1 \subseteq U''_1 \).

The covering \( \nu_i+1 \) is a star refinement of the covering \( \nu_i \). Let us define the chain mapping
\[ \zeta : V_a(M, \nu_i+1) \to C_\mathbb{Q}(N(\nu_i)) \]

Suppose \( \sigma^k = (x_0, \ldots, x_k) \) is a \( \nu_i+1 \)-Vietoris simplex of the space \( M \). The vertex \( x_k \) belongs to some element \( V_{\nu_i} \) of the covering \( \nu_i+1 \). Let \( W_{\nu_i} \) be an element of the covering \( \nu_i \) such that
\[ W_{\nu_i} = \text{St}(V_{\nu_i}, \nu_i+1) \]

Then we put
\[ \zeta(x_0, \ldots, x_k) = 0 \]

if the simplex \( (W_{\nu_i}, ..., W_{\nu_i}) \) degenerates and
\[ \zeta(x_0, \ldots, x_k) = (W_{\nu_i}, ..., W_{\nu_i}) \]

otherwise.

Here \( \text{St}(V_{\nu_i}, \nu_i+1) \) is the star of the set \( V_{\nu_i} \) in the covering \( \nu_i+1 \).

Also we have the chain mapping
\[ \phi : C_\mathbb{Q}(N(\nu_i+1)) \to V_a(M, \nu_i) \]

This mapping is defined as follows. For a given vertex \( U \) of the simplicial complex \( N(\nu_i+1), (U) \) is a point belonging to \( U \).

We shall use the mapping \( \phi \) in the following case:
\[ \phi : C_\mathbb{Q}(A_{\nu_i+1}, \nu_i+1) \to V_a(X, \omega_0) \]

In this case we put
\[ \phi(\sigma(\nu_i+1, \mu_{\nu_i+1})) = \sigma(\nu_i+1) \]

We also know that if \( a \in V_a(M) \) and \( \zeta(\sigma) \) is a Čech chain, then \( a \in V_a(X) \), [1].
Let us consider the chains
\[ \gamma = \{ \gamma_{l+1} \mid l = 0, 1, \ldots \} \]
and
\[ A = \{ A_{l+1} \mid l = 0, 1, \ldots \}. \]

We shall consider the chains \( A_{l+1} \) as chains of the nerves of the coverings \( \omega_{l+1} \).

The chain \( \gamma_{l+1} \) is a \( \omega_{l+1} \)-Vietoris chain of the space \( X \). Also the image of the chain \( A_{l+1} \) under the mapping

\[ \psi : C(A_{l+1}, \mu_{l+1}) \to V(A, \omega_{l+1}) \]

is the chain \( \gamma_{l+1} \).

The chain \( \gamma_{l+1} \) is a convergent Vietoris chain of the space \( X \). Indeed, we know that the mapping \( p(l+1, l+1-l-1) \) is a canonical simplicial mapping of the nerve of the covering \( \omega_{l+1} \) in the nerve of the covering \( \omega_{l+1}^{-1} \).

Let \( j(t_{l+1}) \) be the canonical simplicial mapping of the nerve of the covering \( \omega_{l+1}^{-1} \) in the nerve of the covering \( \omega_{l+1}^{-1} \).

It follows from (10) that

\[ p(l+1, l+1-l-1)(A_{l+1}) = \chi(t_{l+1}, t_{l+1-1}, A_{l+1}) \cdot \]

We have from (13)

\[ j(t_{l+1})p(l+1, l+1-l-1)(A_{l+1}) = j(t_{l+1})\chi(t_{l+1}, t_{l+1-1})(A_{l+1}) \cdot \]

It is well known that the chain \( A_{l+1} \) is homologous to the chain

\[ j(t_{l+1})\chi(t_{l+1}, t_{l+1-1}, A_{l+1}) \]

in the simplicial complex \( N(\omega_{l+1}) \) (the nerve of the covering \( \omega_{l+1}^{-1} \)). Therefore \( \gamma \) is a convergent Vietoris chain of the space \( X \), [1].

So we have the chain mapping

\[ w_{l}^{*} = \{ w_{l}^{*} \mid l = 0, 1, \ldots \} \cdot \]

where

\[ w_{l}^{*} : C(A, \mu) \to V(A) \cdot \]

and

\[ w_{l}^{*} = \{ w_{l}^{*} \mid l = 0, 1, \ldots \} \cdot \]

This chain mapping induces the homomorphism

\[ H_{l}^{*} : H_{l}(A) \to H_{l}(X) \cdot \]

It follows from (14) that the homomorphism \( w_{l}^{*} \) is the inverse homomorphism of the homomorphism \( p_{l}^{*} \), which is induced by the mapping \( p_{l} : X \to A_{l} \).

7. Let us consider the sequence of coverings \( \{ \omega_{t} \mid t = 1, 2, \ldots \} \) in the space \( X \). We know that the covering \( \omega_{t+1} \) is a star refinement of the covering \( \omega_{t} \).
We have the Vietoris mapping \( p: Y \rightarrow X \) of the metric compact space \( Y \) onto the space \( X \).

Let us recall one definition which belongs to E. B. T. [1].

Let \( f: A \rightarrow B \) be a map of the compact metric space \( A \) onto the space \( B \). The mapping \( f \) is called an \( n \)-Vietoris mapping if for every open covering \( \omega \) of the space \( A \) and for every point \( y \in B \) there exists an open and finite covering \( \pi = \pi(\omega, y) \) of the space \( A \), such that

a) \( \pi \) is a refinement of the covering \( \omega \),

b) every \( k \) dimensional cycle in the complex \( A(\pi) \cap f^{-1}(y) \) is homologous to zero in the complex \( A(\omega) \cap f^{-1}(y) \) for \( q \leq k \leq n \).

It is proved in [1], § 6 that if we consider a Vietoris homology with rational coefficients, then every Vietoris mapping is an \( n \)-Vietoris mapping for \( m = 0, 1, \ldots \).

Therefore the mapping \( p: Y \rightarrow X \) is an \( n \)-Vietoris mapping for \( m = 0, 1, \ldots \).

8. Now we shall prove Theorem 1'. It is sufficient to prove that there exists a point \( z_i \) in the space \( Y \) such that

\[
\| p(z_i) - q(z_i) \| < 2^{-i+3} \quad \text{for every} \quad i = 1, 2, \ldots
\]

Indeed, let us have a sequence \( \{z_i\}_{i = 1, 2, \ldots} \) in the space \( Y \) and let every point \( z_i \) of the sequence satisfy (15).

If the space \( Y \) is a metric compact space, we can suppose without loss of generality that this sequence is convergent. Let \( z_\infty \) be the limit of the sequence \( \{z_i\}_{i = 1, 2, \ldots} \). It follows from (15) that \( \| p(z_\infty) - q(z_\infty) \| < 2^{-i+3} \).

Now we shall construct such a sequence.

Let us consider the covering \( \omega_0 \) of the space \( X \), and the covering \( q^{-1}(\omega_{i+1}) \) of the space \( Y \). The mapping \( p \) is an \( n \)-Vietoris mapping. Here \( n = \dim X = \dim X_i \) for \( i = 1, 2, \ldots, \) Lemma 2, § 4, [1] implies the following

**Lemma.** There exists an integer \( l \) and a chain mapping \( T \) of \((n + 1)\)-skeleton of \( V_\delta(X, \omega_{i+1}) \) in \( V_\delta(Y, q^{-1}(\omega_{i+1})) \) such that

a) the chain \( pT(\sigma^i) \) is a barycentric subdivision of the chain \( \sigma^i \) for every simplex \( \sigma^i \),

b) \( |pT(\sigma^i)| \leq n + 1 \) in \( V_\delta(X, \omega_{i+1}) \),

c) there exists a point \( q(\sigma^i) \) in \( X \) such that

\[
\text{St}(g(\sigma^i), \omega_0) = \| \sigma^i \|,
\]

\[
\text{St}(p^{-1}(g(\sigma^i)), q^{-1}(\omega_{i+1})) = |T(\sigma^i)|.
\]

Actually this lemma follows from the proof of Lemma 2, § 4, [1]. By \( T \) we denote the same chain mapping constructed in the proof of Lemma 2, § 4, [1].

We shall define a chain mapping

\[
a_\ast = \{a_s \mid s = 0, 1, \ldots\}: C_\ast(A_1, \mu_0) \rightarrow C_\ast(A_1, \mu_0)
\]

such that the Lefschetz number

\[
\lambda(a_\ast) = \sum (-1)^{i} \text{tr} a_i
\]

of the chain mapping \( a_\ast \) will be equal to the number of coincidence

\[
A(p, q) = \sum (-1)^{i} \text{tr} q_{i} p_{i}^{-1}
\]

of the mappings \( p \) and \( q \).

Let \( \Omega = (\omega_0, \ldots, \omega_1) \) be a simplex of the triangulation \( \mu_0 \) and

\[
\omega_i(\Omega) = \{ \eta_{i+1} \mid k = 0, 1, \ldots \}.
\]

We have

\[
\gamma_i = \gamma_i(\Omega) \in \mathcal{P}(\gamma_i(p_{i+1}^{-1})),
\]

and

\[
|\gamma_i| = O(\gamma_i(p_{i+1}^{-1})), \quad 2^{-i+1}.
\]

Then by definition

\[
a_\ast(\Omega) = \lambda(a_\ast) = \lambda(a_\ast).
\]

It follows from Lemma 1, [1] that

\[
\delta_{i+1} = P_{i+1} P_{i+2} \cdots P_{i+1} P_{i+2} \cdots
\]

Here \( \delta_{i} \) is the homomorphism of the homology groups induced by the chain mapping \( a_\ast \).

Therefore

\[
\lambda(\delta_{i+1}) = \lambda(p, q).
\]

We have \( \lambda(p, q) \neq 0 \), and so \( a_\ast \neq 0 \). Therefore there exists a simplex \( \gamma_i \) of the triangulation \( \mu_0 \) such that

\[
\gamma_i \in [a_\ast(\Omega)].
\]

Let \( \gamma_i = (c_0, \ldots, c_1) \). If

\[
\omega_i(\gamma_i) = \{ \eta_{i+1} \mid k = 0, 1, \ldots \},
\]

then

\[
|\gamma_i| = O(\gamma_i(p_{i+1}^{-1})), \quad 2^{-i+1}.
\]

Let \( \gamma = (x_0, \ldots, x_0) \) be a simplex of the chain \( \eta_{i+1} \) such that the simplex \( \gamma \) belongs to the chain \( \lambda(\gamma(p_{i+1}^{-1})) \). It follows from (19) that there exists a point \( y_0 \) in the set \( \gamma(p_{i+1}^{-1}(c_0)) \) such that

\[
|y_0 - x_0| < 2^{-i+1}.
\]

Therefore

\[
|y_0 - c_0| < 3 \cdot 2^{-i}.
\]
Also, there exists a point $g(\beta) \in X$ such that
\begin{align}
|\beta'| &\subseteq \text{St}(g(\beta'), \omega_i), \\
|T(\beta') &\subseteq \text{St}(p^{-1}(g(\beta')), q^{-1}(\omega_{i+1})).
\end{align}

It follows from (21) that
$$|x_0 - g(\beta')| < 2^{-i}.$$ 

From (20) and the last inequality we obtain
$$|p(\beta') - c_i| < 2^{-i+1}.$$ 

If $u = (u_0, ..., u_i)$ is a Victoris simplex in $V_k(X, \omega_{i+1})$, then
$$\xi(u) \in C_k(N(\omega_i)) = C_k(A_i, \mu_i).$$

Here the chain mapping $\xi$ is defined as follows. The vertex $u_k$ is in the set $V_k$ of the covering $\omega_{i+1}$. Let $W_k$ be the element of the covering $\omega_i$ containing $\text{St}(V_k, \omega_{i+1})$. Then
$$\xi(u) = (W_0, ..., W_i)$$
when the simplex $(W_0, ..., W_i)$ is not degenerate and $\xi(u) = 0$ otherwise. 

Let $\gamma = (\gamma_0, ..., \gamma_i)$ be such a simplex of the chain $qT(\beta')$, that the image of the chain $\gamma$ under the chain mapping $\xi$ is the simplex $(c_0, ..., c_i)$.

Then
$$||x_0 - c_i|| < 2^{-i}.$$ 

It follows from (22) that
$$\gamma_0 \subseteq \text{St}(q^{-1}(p(g(\beta'))), \omega_{i+1}).$$

From (22) and (23) we infer that the distance between the point $g(\beta')$ and the set $q^{-1}(g(\beta'))$ is less than $11 \cdot 2^{-i-1}$. Therefore there exists a point $w_i \in q^{-1}(g(\beta'))$ such that
$$||g(\beta') - w_i|| < 11 \cdot 2^{-i-1} < 2^{-i+3}.$$ 

Let $z_i$ be such a point in the set $p^{-1}(g(\beta'))$ that $q(z_i) = w_i$. Then
$$||p(z_i) - q(z_i)|| < 2^{-i+3},$$

Theorem 1' is proved.

References


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