On a coincidence of mappings of compact spaces in topological groups

by

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Abstract. Let X be a weak-solenoidal space of finite type and $p, q\colon Y \to X$ single-valued continuous mappings. If p is a Vietoris mapping and the number of coincidences

$$\Lambda(p, q) = \sum (-1)^k \operatorname{tr} q_k p_k^{-1}$$

is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$.

1. Let X and Y be metric compact spaces and $p, q\colon X \to Y$ continuous mappings. The mappings p and q have a coincidence if there exists a point $x \in X$ such that $p(x) = q(x)$. It is necessary to impose some conditions on the space X, Y and on the mappings p and q, in order to be sure that the mappings p and q have a coincidence. For example, S. Lefschetz gave in [6] a sufficient condition for mappings p and q to have a coincidence in the case where X and Y are n-dimensional closed manifolds with triangulation. This condition is given in the frame work of the homology and the cohomology of the spaces X and Y, [6], Ch. 8, 29.12.

S. Eilenberg and D. Montgomery proposed in [2] another way to obtain a sufficient condition for a coincidence of mappings p and q. In this case X and Y are metric compact spaces, Y is an absolute neighbourhood retract and the mapping $p\colon X \to Y$ is a Vietoris mapping, i.e. $p(X) = Y$, and for every point $y \in Y$ the space $p^{-1}(y)$ is connected and the homology groups of Alexandroff–Čech $H_i(p^{-1}(y))$ with rational coefficients are zero for $i \geqslant 1$. S. Eilenberg and D. Montgomery gave in [2] the following arguments. If the mapping p is a Vietoris mapping, then the homomorphism

$$p_* = \{p_n\}\colon H_*(X) \to H_*(Y)$$

is an isomorphism, [1]; here the homomorphism p_* is induced by the mapping p.

Let us consider the linear mapping

$$q_k p_k^{-1}\colon H_k(Y) \to H_k(Y).$$

The space $H_k(Y)$ is a finite-dimensional vector space over the rationals and, for almost all k, has dimension zero (this follows because the space Y is a compact ANR space).

In this case we can consider the trace $\operatorname{tr} q_k p_k^{-1}$ of the linear mapping $q_k p_k^{-1}$.

The number of coincidence, $\Lambda(p, q)$, of mappings p and q is

$$\Lambda(p, q) = \sum (-1)^k \operatorname{tr} q_k p_k^{-1} .$$

The following theorem is proved in [2]: If Y is a compact metric ANR space, p is a Vietoris mapping and $\Lambda(p, q) \neq 0$, then the mappings p and q have a coincidence, i.e., there exists a point $x \in X$ such that $p(x) = q(x)$.

L. Górniewicz gave a modern proof and a generalization of this theorem of S. Eilenberg and D. Montgomery in [4]. A further generalization of this theorem is proposed in [5].

Let us recall that the theorem of S. Eilenberg and D. Montgomery is closely related to the problem of existence of fixed points for multi-valued mappings.

Let Φ be an upper semi-continuous multi-valued mapping of a compact metric space X in itself. Suppose also that the mapping Φ is acyclic. This means that the set $\Phi(x)$ is connected and the homology groups of Alexandroff–Čech $H_i(\Phi(x))$ with rational coefficients are zero for $i \geqslant 1$ and for every point $x \in X$.

The mapping Φ has a fixed point if there exists a point $x \in X$ such that $x \in \Phi(x)$. Let $\Gamma(\Phi)$ be the graph of the mapping Φ, i.e.

$$\Gamma(\Phi) = \{(y, z) \in X \times X \mid z \in \Phi(y)\} .$$

By p and q we shall denote mappings

$$p, q \colon \Gamma(\Phi) \to X ,$$

such that $p(y, z) = y$ and $q(y, z) = z$ for $(y, z) \in \Gamma(\Phi)$. Then it follows that

$$\Phi(y) = qp^{-1}(y) \quad \text{for every point } y \in X .$$

The mapping $p\colon \Gamma(\Phi) \to Y$ is a Vietoris mapping because Φ is an acyclic and upper semi-continuous mapping.

The mapping Φ has a fixed point if and only if the mappings p and q have a coincidence.

We shall consider the following problem. Let Y be a metric compact space and G — a compact, connected and finite dimensional topological group. Let $p, q\colon Y \to G$ be continuous single valued mappings, and p — a Vietoris mapping.

The space G is a metric compact space, because G is a compact, connected group, [8]. Therefore the homology groups $H_k(G)$ of Alexandroff–Čech with rational coefficients are finite dimensional-vector spaces over the field of rational numbers and for almost all k the space $H_k(G)$ is zero-dimensional, [9]. Then the number of coincidence,

$$\Lambda(p, q) = \sum (-1)^k \operatorname{tr} q_k p_k^{-1} ,$$

of the mappings p and q exists.

In the case where G is a locally connected topological space, G is an ANR space, because G is a Lie group, [7], Ch. 8. Then, if the number of coincidence is not zero, the mappings p and q have a coincidence, [2].

In this paper we consider the question of existence of a coincidence of the mappings p and q without the assumption that G is a locally connected space. It is known that G is locally homeomorphic to the product of a finite-dimensional ball and the Cantor discontinuum, [8].

We shall prove the following theorem:

THEOREM 1. *Let G be a compact, connected, finite-dimensional topological group and Y a metric compact space. Let $p, q\colon Y \to G$ be single-valued mappings and p — a Vietoris mapping. If the number of coincidence,*

$$\Lambda(p, q) = \sum (-1)^k \operatorname{tr} q_k p_k^{-1} ,$$

is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$, i.e. the mappings p and q have a coincidence.

From this theorem follows Lefschetz's fixed point theorem of [9].

Actually we shall prove a more general theorem than Theorem 1. To formulate that theorem we need one definition.

The compact, connected space X is called a *weak-solenoidal space* if X is the limit of an inverse system

$$\{X_k, \pi(k+1, k) \mid k = 1, 2, \ldots\}$$

such that

1) X_k is a connected finite polyhedron.

2) $\pi(k+1, k)\colon X_{k+1} \to X_k$ is a finite sheet covering space.

An example of a weak-solenoidal space is the following. Let A be a compact, connected, finite-dimensional topological group and B a closed subgroup in A. Then the quotient space A/B is a weak-solenoidal space, [8].

Let us recall that the space Y is called a *space of finite type* if $\dim H_*(Y) < \infty$.

We shall prove the following theorem:

THEOREM 1′. *Let X be a weak-solenoidal space of finite type and $p, q\colon Y \to X$ single-valued continuous mappings. If p is a Vietoris mapping and the number of coincidence, $\Lambda(p, q)$, is not zero, then there exists a point $y \in Y$ such that $p(y) = q(y)$, i.e., the mappings p and q have a coincidence.*

The paper is divided into eight sections. In Section 1 we give a construction of P. S. Alexandroff: a realization of a given ε-mapping as an ε-translation. In Section 2 we give an embedding of a given inverse system in a Banach space with the construction given in Section 1. In Section 4 we consider the chain homomorphisms induced by the bonding mappings of this inverse system. In Section 5 we define the chain homomorphisms inducing in homology the inverse homomorphisms of the homomorphisms induced by projections of the space X on X_k. In Section 6 we recall a lemma of E. Begle from [1] and Section 8 contains the proof of Theorem 1′.

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2. Let H be a Banach space. We shall denote by $||x||$ the norm of the element x in the space H. Let X be a compact subspace in the space H. The compact space X is said to be linearly independently imbedded (l.i.i.) in the space H if the points $\{x_0, \dots, x_s\}$ are vertices of a s-dimensional simplex in the space H for every s and $x_i \in X$, $i = 0, \dots, s$.

Suppose that the space X is l.i.i. in the space H.

Let $f\colon X \to K$ be a mapping of the space X in the compact polyhedron K and let f be an ε-mapping, i.e., the diameter of the set $f^{-1}(y)$ is less than ε for every $y \in K$.

Suppose that τ is a triangulation of the polyhedron K such that

$$\operatorname{diam} f^{-1}(\operatorname{St}(a, \tau)) < \varepsilon \quad \text{for every vertex } a \tag{1}$$

of the triangulation τ of K. Here $\operatorname{St}(a, \tau)$ is the open star of the vertex a in the triangulation τ. For $A \subset X$ we denote by $\operatorname{diam} A$ the diameter of the set A.

Let us denote by $\{e_1, \dots, e_s\}$ all vertices of the triangulation τ and by V_k the set $f^{-1}(\operatorname{St}(e_k, \tau))$. The family of open sets $\omega = \{V_1, \dots, V_k\}$ is an open covering of the space X and the diameter of every set V_i is less than ε, by (1).

Let a_k be a point in the set $f^{-1}(e_k)$ for $k = 1, \dots, s$. Then we have

$$||a_k - a_l|| < 2\varepsilon \quad \text{if} \quad V_k \cap V_l \neq \emptyset . \tag{2}$$

Let us consider the points $\{a_1, \dots, a_s\}$. If the space X is l.i.i. in the space H, the points $\{a_1, \dots, a_s\}$ are vertices of an $(s-1)$-dimensional simplex in H.

We denote by r a simplicial homeomorphism of the triangulation τ in the simplex with vertices $\{a_1, \dots, a_s\}$ such that $r(e_k) = a_k$ for $k = 1, \dots, s$. By Z we shall denote the set $r(K)$, and by $\tilde{r}$ the mapping $rf\colon X \to Z$. By μ we denote the triangulation of the polyhedron Z, the image of the triangulation τ under the homeomorphism r.

By definition, $\tilde{r}(x) \in \operatorname{St}(a_k, \mu)$ for $x \in V_k$ (here $\operatorname{St}(a_k, \mu)$ is the open star of the vertex a_k in the triangulation μ).

From (2) we have

$$||\tilde{r}(x) - a_k|| < 2\varepsilon . \tag{3}$$

It follows from (1) and (3) that

$$||x - \tilde{r}(x)|| < 3\varepsilon , \tag{4}$$

i.e., the mapping $\tilde{r}$ is a 3ε-mapping of the space X in the space H.

3. Let X be a weak-solenoidal space, i.e., X is a compact space and X is the limit of the inverse system

$$\{X_k, \pi(k+1, k), k = 1, 2, \dots\}$$

where

1) X_k is a finite connected polyhedron.

2) $\pi(k+1, k)\colon X_{k+1} \to X_k$ is a finite sheet covering map.

We can also assume that

3) the projections $\pi_i\colon X \to X_i$ are 2^{-i}-mappings, i.e.,

$$\operatorname{diam} \pi_i^{-1}(x) < 2^{-i} \quad \text{for every point } x \in X_i . \tag{5}$$

4) in the space X_i we choose a triangulation τ_i such that

$$\operatorname{diam} \pi_i^{-1}(\operatorname{St}(a, \tau_i)) < 2^{-i} \quad \text{for every vertex } a \tag{6}$$

of the triangulation τ_i.

5) $\pi(i+1, i)\colon X_{i+1} \to X_i$ is a simplicial mapping of the triangulation τ_{i+1} on the triangulation $\tau_i^{(s_i)}$ for $s_i \geqslant 2$, $i = 1, \dots$

Here $\tau_i^{(s_i)}$ is an s_i-barycentric subdivision of the triangulation τ_i.

The mappings $\pi_i\colon X \to X_i$ (the projections) and $\pi(i+1, i)\colon X_{i+1} \to X_i$ induce the homomorphisms of the homology groups

$$(\pi_i)_{*s}\colon H_s(X) \to H_s(X_i) ,$$

$$(\pi(i+1, i))_{*s}\colon H_s(X_{i+1}) \to H_s(X_i) .$$

We shall prove later (Section 4, Lemma 1) that the homomorphisms $(\pi(i+1, i))_{*s}$ are epimorphisms. If

$$H_s(X) = \varprojlim \{H_s(X_i), (\pi(i+1, i))_{*s}\}$$

and

$$\dim H_s(X) < \infty ,$$

there exists an i_0 such that the homomorphisms $(\pi_i)_{*s}$ and $(\pi(i+1, i))_{*s}$ are isomorphisms for $i \geqslant i_0$.

We suppose that

6) the homomorphisms $(\pi_i)_{*s}$ and $(\pi(i+1, i))_{*s}$ are isomorphisms for $i = 1, \dots$

Let l_i be the number of points in the set $\pi(i+1, i)^{-1}(x)$ for $x \in X_i$.

Let us recall that the compact space X is l.i.i. in the space H (by a theorem of K. Kuratowski every compact space can be l.i.i. in some Banach space, [10]).

Let us consider the mappings

$$\pi(i+1, i)\colon X_{i+1} \to X_i .$$

The mapping $\pi(i+1, i)$ is a locally trivial bundle. Therefore for every vertex a of the triangulation τ_i the mapping

$$\pi(i+1, i) | \pi(i+1, i)^{-1}(\operatorname{St}(a, \tau_i))\colon \pi(i+1, i)^{-1}(\operatorname{St}(a, \tau_i)) \to \operatorname{St}(a, \tau_i)$$

is a trivial bundle, and the mapping $\pi(i+1, i)$ is a simplicial isomorphism on the components of the set $\pi(i+1, i)^{-1}(\operatorname{St}(a, \tau_i))$. So the mapping $\pi(i+1, i)$ and the triangulation τ_i induce the triangulation τ'_{i+1} on the space X_{i+1}.

Let us consider the open coverings of the space X_{i+1}

$$s(\tau_{i+1}) = \{\mathrm{St}(a, \tau_{i+1}) | a \text{ — a vertex of the triangulation } \tau_{i+1}\},$$
$$s(\tau'_{i+1}) = \{\mathrm{St}(b, \tau'_{i+1}) | b \text{ — a vertex of the triangulation } \tau'_{i+1}\}.$$

If $s_i \geqslant 2$, the covering $s(\tau_{i+1})$ is a star refinement of the covering $s(\tau'_{i+1})$.

Let the vertices of the triangulations τ_i and $\tau_i^{(s_i)}$ be

$$\{e_1^i, \ldots, e_{t_i}^i\}, \ \{e_1^i, \ldots, e_{t_i}^i, \ldots, e_k^i\}.$$

Then the triangulation τ'_{i+1} has the vertices $\{e_1^{i+1}, \ldots, e_k^{i+1}\}$, where $k = t_i l_i$ and $\pi(i+1, i)(e_v^{i+1}) = e_m^i$ for $v = ml_i - r$, $0 \leqslant r < l$.

By V_a we shall denote the set $\pi_i^{-1}(\mathrm{St}(a, \tau_i))$ for a given vertex a of the triangulation τ_i. So we have a finite, open covering

$$\omega_i = \{V_a | \ a \text{ — a vertex of the triangulation } \tau_i\}$$

of the space X. It follows from (6) that

$$\operatorname{diam} V_a < 2^{-i} \quad \text{for every } V_a \in \omega_i.$$

If the covering $s(\tau_{i+1})$ is a star refinement of the covering $s(\tau'_{i+1})$, then ω_{i+1} is a star refinement of the covering ω_i for $i = 1, \ldots$

Let us consider the covering ω_i of the space X and let a_k^i be a point in the set $\pi_i^{-1}(e_k^i)$, for e_k^i a vertex of the triangulation τ_i. So we obtain a subset

$$E_i = \{a_1^i, \ldots, a_{t_i}^i\}$$

in the space X.

It follows from (2) that

(7) $$\|a_m^i - a_n^i\| < 2^{-i+1},$$

e_m^i and e_n^i being vertices of a one dimensional simplex of the triangulation τ_i. Also

$$\pi_i(a_k^i) = e_k^i \quad \text{for } k = 1, \ldots, t_i.$$

The points of the set E_i are vertices of a $(t_i - 1)$-dimensional simplex in the space H, because the space X is 1.i.i. in the space H.

Let us define a simplicial homeomorphism $k_i \colon X_i \to H$ by $k_i(e_s^i) = a_s^i$ for $s = 1, \ldots, t_i$.

Let $p_i = k_i \pi_i \colon X \to A_i$, here $A_i = k_i(X_i)$.

It follows from (4) that

(8) $$\|x - p_i(x)\| < 3 \cdot 2^{-i} \quad \text{for } x \in X \text{ and } i = 1, 2, \ldots$$

Let us consider the mapping

$$p(i+1, i) \colon A_{i+1} \to A_i$$

given by

$$p(i+1, i) = k_i \pi(i+1, i) k_{i+1}^{-1}.$$

It follows that

(9) $$\|p(i+1, i)(y) - y\| < 9 \cdot 2^{-i-1} \quad \text{for } y \in A_{i+1}.$$

We denote by μ_i the image of the triangulation τ_i under the simplicial homeomorphism k_i.

The mapping $p(i+1, i)$ is a simplicial l_i-sheet covering map of the triangulation μ_{i+1} on the triangulation $\mu_i^{(s_i)}$. Here $\mu_i^{(s_i)}$ is an s_i-barycentric subdivision of the triangulation μ_i.

4. Let K be a finite simplicial complex with a given triangulation τ. We denote by

$$C_*(K, \tau) = \{C_s(K, \tau) | \ s = 0, 1, \ldots\}$$

the chain complex of the triangulation τ with rational coefficients.

The simplicial mapping $p(i+1, i)$ induces the chain homomorphism

$$p(i+1, i)_* = \{p(i+1, i) | \ s = 0, 1, \ldots\} \colon C_*(A_{i+1}, \mu_{i+1}) \to C_*(A_i, \mu_i^{(s_i)})$$

for $i = 1, 2, \ldots$

We denote by

$$\sigma(i+1, i)_* = \{\sigma(i+1, i)_s | \ s = 0, 1, \ldots\}$$

the chain homomorphism

$$\sigma(i+1, i)_* \colon C_*(A_i, \mu_i^{(s_i)}) \to C_*(A_{i+1}, \mu_{i+1})$$

given by

$$\sigma(i+1, i)_s(\sigma^s) = \sum_{i=1}^{l_i} \sigma_i^s$$

where

$$\sigma^s = \{a_{i_0}^i, \ldots, a_{i_s}^i\}, \quad \sigma_i^s = \{a_{i_0}^{i+1}, \ldots, a_{i_s}^{i+1}\}$$

and

$$p(i+1, i)(a_{i_k}^{i+1}) = a_{i_k}^i.$$

Let us consider the homomorphism

$$\sigma(i+1, i)_{*s} \colon H_s(A_i) \to H_s(A_{i+1})$$

induced by the chain mapping $\sigma(i+1, i)_*$.

It follows that

(10) $$p(i+1, i)_{*s} \sigma(i+1, i)_{*s} = l_i \cdot \mathrm{id};$$

here

$$p(i+1, i)_{*s} \colon H_s(A_{i+1}) \to H_s(A_i)$$

is the homomorphism induced by the chain mapping $p(i+1, i)_*$.

It follows from (10) that the homomorphism $p(i+1, i)_{*s}$ is an epimorphism for every $i = 1, 2, \ldots$ and $s = 0, 1, \ldots$

Let us recall that

$$p(i+1,i) = k_i \pi(i+1,i) k_{i+1}^{-1}$$

where k_i, and k_{i+1}^{-1} are homeomorphisms. So we have

LEMMA 1. *The homomorphisms* $(\pi(i+1,i))_{*s}$ *are epimorphisms for* $i = 1, 2, \ldots$ *and* $s = 0, 1, \ldots$

It follows from (10) that the homomorphism $\sigma(i+1,i)_{*s}$ is an isomorphism because $p(i+1,i)_{*s}$ is an isomorphism.

Let

$$\varrho(i+1,i)_* = \{\varrho(i+1,i)_s| \ s = 0, 1, \ldots\}$$

be a chain mapping such that

$$\varrho(i+1,i)_s = l_i^{-1}\sigma(i+1,i)_s .$$

If $\varrho(i+1,i)_{*s}$ is the homomorphism of the homology groups induced by the chain mapping $\varrho(i+1,i)_*$, then

$$p(i+1,i)_{*s}\varrho(i+1,i)_{*s} = \mathrm{id} .$$

5. Let M be a compact metric space and ν_i, $i = 1, 2, \ldots$, a fundamental system of finite open covering of the space M such that

a) every element of the covering ν_i has a diameter less than 2^{-i},

b) the covering ν_{i+1} is a star refinement of the covering ν_i for $i = 1, 2, \ldots$

The ordered $(k+1)$-tuple $\sigma^k = (x_0, \ldots, x_k)$ of points of the space M is called a *k-dimensional Vietoris simplex of the space M*. We denote by $M(\nu_i)$ the simplicial complex of all ν_i-Vietoris simplexes of the space M. If A is a subset of the space M, then by $M(\nu_i) \cap A$ we shall denote the closed subcomplex $M(\nu_i)$ consisting of all simplexes of $M(\nu_i)$ with vertices belonging to the set A.

We denote by $V_*(M, \nu_i)$ the chain complex with rational coefficients of the simplicial complex $M(\nu_i)$. By $|\sigma|$ we denote the closed simplicial complex of all ν_i-Vietoris simplexes of the chain σ for a given $\sigma \in V_*(M, \nu_i)$.

Let

$$\tilde{V}_*(M) = \prod \{V_*(M, \nu_i)| \ i = 1, \ldots\} .$$

The elements of the group $\tilde{V}_*(M)$ are called the Vietoris chains of the space M.

Let $a = \{a_i| \ i = 1, \ldots\}$ be the Vietoris chain of the space M. The chain a is called a *convergent Vietoris chain* if the chain a_i is homologous to the chain a_{i-1} in $V_*(M, \nu_{i-1})$ for $i = 2, 3, \ldots$

By $V_*(M)$ we shall denote the linear space over the rationals with the basis consisting of all convergent Vietoris chain of the space M.

Let M and $\overline{M}$ be compact metric spaces and

$$\nu = \{\nu_i| \ i = 1, 2, \ldots\}, \quad \bar{\nu} = \{\bar{\nu}_i| \ i = 1, 2, \ldots\}$$

fundamental systems of finite open covering satisfying a) and b) in M and $\overline{M}$.

Let $f\colon M \to \overline{M}$ be a continuous mapping. We shall define the chain mapping $f_*\colon V_*(M) \to V_*(\overline{M})$.

Let us consider the covering $\bar{\nu}_i, f^{-1}\bar{\nu}_i$ is an open, finite covering of the space M. Let ν_{i_0} be a covering which is a refinement of the covering $f^{-1}\bar{\nu}_i$ and $\nu_{i_0} \in \nu$. Then the image of the complex $M(\nu_{i_0})$ under the mapping f belongs to the complex $\overline{M}(\bar{\nu}_i)$. So for

$$c = \{c_i| \ i = 1, \ldots\} \in V_*(M)$$

we have

$$f_*(c) = \{f(c_i)| \ i = 1, \ldots\} .$$

By $N(\omega)$ we shall denote the nerve of the covering ω for a given finite, open covering of the space M.

Let ω', ω'' be a finite, open covering of the space M

$$\omega' = \{U_1', \ldots, U_s'\}, \quad \omega'' = \{U_1'', \ldots, U_t''\}$$

and the covering ω' is a refinement of the covering ω''. Then we have the simplicial mapping $i\colon N(\omega') \to N(\omega'')$ given by $i(U_k') = U_m''$ for $U_k' \in \omega'$, and $U_m'' \in \omega''$ such that $U_k' \subset U_m''$.

The covering ν_{i+1} is a star refinement of the covering ν_i. Let us define the chain mapping

$$\zeta\colon V_*(M, \nu_{i+1}) \to C_*(N(\nu_i)) .$$

Suppose $\sigma^k = (x_0, \ldots, x_k)$ is a ν_{i+1}-Vietoris simplex of the space M. The vertex x_s belongs to some element V_s of the covering ν_{i+1}. Let W_s be an element of the covering ν_i such that

$$W_s \supset \mathrm{St}(V_s, \nu_{i+1}) .$$

Then we put

$$\zeta(x_0, \ldots, x_k) = 0$$

if the simplex $(W_0, \ldots, W_s)$ degenerates and

$$\zeta(x_0, \ldots, x_k) = (W_0, \ldots, W_k)$$

otherwise.

Here $\mathrm{St}(V_s, \nu_{i+1})$ is the star of the set V_s in the covering ν_{i+1}.

Also we have the chain mapping

$$\varphi\colon C_*(N(\nu_{i+1})) \to V_*(M, \nu_i) .$$

This mapping is defined as follows. For a given vertex U of the simplicial complex $N(\nu_{i+1})$, $\varphi(U)$ is a point belonging to U.

We shall use the mapping φ in the following case:

$$\varphi\colon C_*(A_{i+1}, \mu_{i+1}) \to V_*(X, \omega_i) .$$

In this case we put

$$\varphi(\mathrm{St}(e_k^{i+1}, \mu_{i+1})) = a_k^{i+1} .$$

We also know that if $a \in \tilde{V}_*(M)$ and $\zeta(a)$ is a Čech chain, then $a \in V_*(X)$, [1].

6. We denote by $\omega_i^{(s_i)}$ the covering $\{p_i^{-1}(\mathrm{St}(a, \mu_i^{(s_i)}))|\ a$ is a vertex of the triangulation $\mu_i^{(s_i)}\}$. If the triangulation μ_i is the nerve of the covering ω_i, then $\mu_i^{(s_i)}$ is a nerve of the covering $\omega_i^{(s_i)}$. For a given vertex a of the triangulation $\mu_i^{(s_i)}$ we have

$$p_i^{-1}(\mathrm{St}(a, \mu_i^{(s_i)})) = \bigcup \{p_{i+1}^{-1}(\mathrm{St}(b, \mu_{i+1}))|\ p(i+1, i)(b) = a\};$$

also

$$\mathrm{St}(b', \mu_{i+i}) \cap \mathrm{St}(b'', \mu_{i+1}) = \emptyset$$

for

$$p(i+1, i)(b') = p(i+1, i)(b'') \quad \text{and} \quad b' \neq b''.$$

It follows that the mapping $p(i+1, i)$ is a canonical simplicial mapping of the nerve of the covering ω_{i+1} in the nerve of the covering $\omega_i^{(s_i)}$, i.e., the image of the set $p_{i+1}^{-1}(\mathrm{St}(b, \mu_{i+1}))$ under the mapping $p(i+1, i)$ is the set $p_i^{-1}(\mathrm{St}(a, \mu_i^{(s_i)}))$, where $p(i+1, i)(b) = a$ (here a is a vertex of the triangulation $\mu_i^{(s_i)}$, and b is a vertex of the triangulation μ_{i+1}).

It follows that

$$(11) \qquad (a_{i_0}^i, \ldots, a_{i_s}^i) \subset O(p_i^{-1}(a_{i_0}^i), 2^{-i+1});$$

we use the following notation: for a given set A in the space X, and a positive real δ

$$O(A, \delta) = \{x \in X|\ \sup(\|x-y\|\ |\ y \in A) < \delta\}.$$

Let K be a finite simplicial complex with a triangulation τ, and let $\tau^{(k)}$ be the k-barycentric subdivision of τ.

We shall use the chain mapping

$$\chi(\tau, k)_* = \{\chi(\tau, k)_s|\ s = 0, 1, \ldots\}$$

"k-barycentric subdivision of chains"

$$\chi(\tau, k)_*\colon C_*(K, \tau) \to C_*(K, \tau^{(k)}).$$

Let us consider the simplex $\delta_i^s = (a_{i_0}^i, \ldots, a_{i_s}^i)$. We denote by Δ_{i+l}^s the chain

$$\varrho(i+l, i+l-1)_s \chi(\mu_{i+ \ -1}, s_{i+l-1})_s \ldots \varrho(i+1, i)_s \chi(\mu_i, s_i)_s(\delta_i^s)$$

for $l > 1$ and δ_i^s for $l = 0$.

Let us put

$$\gamma_{i+l}^s = \varphi(\Delta_{i+l}^s) \quad \text{for} \quad l = 0, 1, \ldots$$

It follows that

$$\Delta_{i+l}^s \in C_s(A_{i+l}, \mu_{i+l})$$

and

$$\gamma_{i+l}^s \in V_s(X, \omega_{i+l-1}).$$

It follows from (II) that

$$(12) \qquad |\gamma_{i+l}^s| \subset O(p_i^{-1}(a_{i_0}^i), 2^{-i+1}).$$

Let us consider the chains

$$\gamma = \{\gamma_{i+l}^s|\ l = 0, 1, \ldots\}$$

and

$$\Delta = \{\Delta_{i+l}^s|\ l = 0, 1, \ldots\}.$$

We shall consider the chains Δ_{i+l}^s as chains of the nerves of the coverings ω_{i+l}. The chain γ_{i+l}^s is a ω_{i+l-1}-Vietoris chain of the space X. Also the image of the chain Δ_{i+l}^s under the chain mapping

$$\varphi\colon C_*(A_{i+l}, \mu_{i+l}) \to V_*(X, \omega_{i+l-1})$$

is the chain γ_{i+l}^s.

The chain γ is a convergent Vietoris chain of the space X. Indeed, we know that the mapping $p(i+l, i+l-1)$ is a canonical simplicial mapping of the nerve of the covering ω_{i+l} in the nerve of the covering $\omega_{i+l-1}^{(s_l+l-1)}$.

Let $j(s_{i+l-1})$ be the canonical simplicial mapping of the nerve of the covering $\omega_{i+l-1}^{(s_l+l-1)}$ in the nerve of the covering ω_{i+l-1}.

It follows from (10) that

$$(13) \qquad p(i+l, i+l-1)_s(\Delta_{i+l}^s) = \chi(\mu_{i+l-1}, s_{i+l-1})_s(\Delta_{i+l-1}^s).$$

We have from (13)

$$(14) \qquad j(s_{i+l-1})p(i+l, i+l-1)_s(\Delta_{i+l}^s) = j(s_{i+l-1})\chi(\mu_{i+l-1}, s_{i+l-1})_s(\Delta_{i+l-1}^s).$$

It is well known that the chain Δ_{i+l-1}^s is homologous to the chain

$$j(s_{i+l-1})\chi(\mu_{i+l-1}, s_{i+l-1})_s(\Delta_{i+l-1}^s)$$

in the simplicial complex $N(\omega_{i+l-1})$ (the nerve of the covering ω_{i+l-1}). Therefore γ is a convergent Vietoris chain of the space X, [1].

So we have the chain mapping

$$w_i^* = \{w_i^s|\ s = 0, 1, \ldots\},$$

where

$$w_i^s\colon C_s(A_i, \mu_i) \to V_s(X)$$

and

$$w_i^s(\delta_i^s) = \{\gamma_{i+l}^s|\ l = 0, 1, \ldots\}.$$

This chain mapping induces the homomorphism

$$\tilde{w}_i^*\colon H_*(A_i) \to H_*(X).$$

It follows from (14) that the homomorphism w_i^* is the inverse homomorphism of the homomorphism p_{i*}, which is induced by the mapping $p_i\colon X \to A_i$.

7. Let us consider the sequence of coverings $\{\omega_i|\ i = 1, 2, \ldots\}$ in the space X. We know that the covering ω_{i+1} is a star refinement of the covering ω_i.

We have the Vietoris mapping $p\colon Y\to X$ of the metric compact space Y onto the space X.

Let us recall one definition which belongs to E. Begle, [1].

Let $f\colon A\to B$ be a map of the compact metric space A onto the space B. The mapping f is called an *n-Vietoris mapping* if for every open covering ω of the space A and for every point $y\in B$ there exists an open and finite covering $\pi=\pi(\omega,y)$ of the space A, such that

a) π is a refinement of the covering ω,

b) every k dimensional cycle in the complex $A(\pi)\cap f^{-1}(y)$ is homologous to zero in the complex $A(\omega)\cap f^{-1}(y)$ for $q\leqslant k\leqslant n$.

It is proved in [1], § 6 that if we consider a Vietoris homology with rational coefficients, then every Vietoris mapping is an n-Vietoris mapping for $m=0,1,\ldots$

Therefore the mapping $p\colon Y\to X$ is an n-Vietoris mapping for $m=0,1,\ldots$

8. Now we shall prove Theorem 1′. It is sufficient to prove that there exists a point z_i in the space Y such that

(15) $$\|p(z_i)-q(z_i)\|<2^{-i+3}\quad \text{for every } i=1,2,\ldots$$

Indeed, let us have a sequence $\{z_i|\ i=1,2,\ldots\}$ in the space Y and let every point z_i of the sequence satisfy (15).

If the space Y is a metric compact space, we can suppose without loss of generality that this sequence is convergent. Let z_0 be the limit of the sequence $\{z_i|\ i=1,2,\ldots\}$. It follows from (15) that $\|p(z_0)-q(z_0)\|<2^{-i+3}$.

Now we shall construct such a sequence.

Let us consider the covering ω_i of the space X, and the covering $q^{-1}(\omega_{i+1})$ of the space Y. The mapping p is an n-Vietoris mapping. Here $n=\dim X=\dim X_i$ for $i=1,2,\ldots$ Lemma 2, § 4, [1] implies the following

LEMMA. *There exists an integer l and a chain mapping T of $(n+1)$-skeleton of $V_*(X,\omega_{i+l})$ in $V_*(Y,q^{-1}(\omega_{i+1}))$ such that*

a) *the chain $pT(\sigma^k)$ is a barycentric subdivision of the chain σ^k for every simplex σ^k, $0\leqslant k\leqslant n+1$ in $V_*(X,\omega_{i+1})$,*

b) *$|pT(\sigma^k)|$ belongs to some element of the covering ω_{i+l},*

c) *there exists a point $g(\sigma^k)\in X$ such that*

(16) $$\mathrm{St}\big(g(\sigma^k),\omega_i\big)\supset|\sigma^k|\,,$$

(17) $$\mathrm{St}\big(p^{-1}(g(\sigma^k)),q^{-1}(\omega_{i+1})\big)\supset|T(\sigma^k)|\,.$$

Actually this lemma follows from the proof of Lemma 2, § 4, [1]. By T we denote the same chain mapping constructed in the proof of Lemma 2, §4, [1].

We shall define a chain mapping

$$\alpha_*=\{\alpha_s|\ s=0,1,\ldots\}\colon C_*(A_i,\mu_i)\to C_*(A_i,\mu_i)$$

such that the Lefschetz number

$$\lambda(\alpha_*)=\sum(-1)^i\mathrm{tr}\,\alpha_i$$

of the chain mapping α_* will be equal to the number of coincidence

$$\Lambda(p,q)=\sum(-1)^i\mathrm{tr}\,q_ip_i^{-1}$$

of the mappings p and q.

Let $\delta_i^s=(a_{i_0}^i,\ldots,a_{i_s}^i)$ be a simplex of the triangulation μ_i and

$$w_i^s(\delta_i^s)=\{\gamma_{i+k}^s|\ k=0,1,\ldots\}\,.$$

We have

$$\gamma_{i+k}^s\in V_s(X,\omega_{i+k-1})$$

and

$$|\gamma_{i+k}^s|\subset O\big(p_i^{-1}(b_{i_0}^i),2^{-i+1}\big)\,.$$

Then by definition

$$\alpha_s(\delta_i^s)=\zeta qT(\gamma_{i+l+1}^s)\,.$$

It follows from Lemma 1, [1] that

$$\tilde\alpha_*=p_{i*}q_*p_*^{-1}p_{i*}^{-1}\,.$$

Here $\tilde\alpha_*$ is the homomorphism of the homology groups induced by the chain mapping α_*.

Therefore

(18) $$\lambda(\tilde\alpha_*)=\Lambda(p,q)$$

and

$$\lambda(\alpha_*)=\Lambda(p,q)\,.$$

We have $\Lambda(p,q)\neq 0$, and so $\lambda(\alpha_*)\neq 0$. Therefore there exists a simplex ξ_i^s of the triangulation μ_i such that

$$\xi_i^s\in|\alpha_s(\xi_i^s)|\,.$$

Let $\xi_i^s=(c_{i_0}^i,\ldots,c_{i_s}^i)$. If

$$w_i^s(\xi_i^s)=\{\eta_{i+k}^s|\ k=0,1,\ldots\}\,,$$

then

(19) $$|\eta_{i+l+1}^s|\subset O\big(p_i^{-1}(c_{i_0}^i),2^{-i+1}\big)\,.$$

Let $\beta^s=(x_0,\ldots,x_s)$ be a simplex of the chain η_{i+l+1}^s such that the simplex ξ_i^s belongs to the chain $\zeta qT(\beta^s)$. It follows from (19) that there exists a point y_0 in the set $p_i^{-1}(c_{i_0}^i)$ such that

$$\|x_0-y_0\|<2^{-i+1}$$

Therefore

(20) $$\|x_0-c_{i_0}^i\|<3\cdot 2^{-i}\,.$$

Also, there exists a point $g(\beta)\in X$ such that

(21) $$|\beta^s|\subset \mathrm{St}(g(\beta^s),\omega_i),$$

(22) $$|T(\beta^s)|\subset \mathrm{St}(p^{-1}(g(\beta^s)),q^{-1}(\omega_{i+1})).$$

It follows from (21) that

$$\|x_0-g(\beta^s)\|<2^{-i}.$$

From (20) and the last inequality we obtain

(23) $$\|g(\beta^s)-c^i_{i_0}\|<2^{-i+2}.$$

If $u=(u_0,\ldots,u_s)$ is a Vietoris simplex in $V_*(X,\omega_{i+1})$, then

$$\zeta(u)\in C_*(N(\omega_i)) = C_*(A_i,\mu_i).$$

Here the chain mapping ζ is defined as follows. The vertex u_k is in the set V_k of the covering ω_{i+1}. Let W_k be the element of the covering ω_i containing $\mathrm{St}(V_k,\omega_{i+1})$. Then

$$\zeta(u)=(W_0,\ldots,W_s)$$

when the simplex $(W_0,\ldots,W_s)$ is not degenerate and $\zeta(u)=0$ otherwise.

Let $\gamma=(\gamma_0,\ldots,\gamma_s)$ be such a simplex of the chain $qT(\beta^s)$, that the image of the chain γ under the chain mapping ζ is the simplex $(c^i_{i_0},\ldots,c^i_{i_s})$.

Then

(24) $$\|\gamma_0-c^i_{i_0}\|<2^{-i}.$$

It follows from (22) that

$$\gamma_0\in \mathrm{St}(qp^{-1}(g(\beta^s)),\omega_{i+1}).$$

From (22) and (23) we infer that the distance between the point $g(\beta^s)$ and the set $qp^{-1}(g(\beta^s))$ is less than $11\cdot 2^{-i-1}$. Therefore there exists a point

$$w_i\in qp^{-1}(g(\beta^s))$$

such that

$$\|g(\beta^s)-w_i\|<11\cdot 2^{-i-1}<2^{-i+3}.$$

Let z_i be such a point in the set $p^{-1}(g(\beta^s))$ that $q(z_i)=w_i$. Then

$$\|p(z_i)-q(z_i)\|<2^{-i+3},$$

Theorem 1′ is proved.

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