Approximate continuity of functions of two variables

by

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Abstract. In this work we shall consider the (well known) approximately continuous functions of two variables and we shall introduce the notions of approximate continuity on the product, approximate continuity on the hyperbolas, Δ_{xy}-approximate continuity, Δ_{xy}-approximate continuity on the product and Δ_{xy}-approximate continuity on the hyperbolas. In the first part of the paper we shall investigate the relations between the introduced classes of approximately continuous functions. Next we shall generalize the representation theorems of Bögel and Tasche for the case of approximately continuous functions. At the end of the work we shall give the necessary and sufficient conditions under which the function $f \circ g$ is approximately continuous in some sense for every function f, which is approximately continuous in the same sense, where g is a homeomorphism transforming the plane onto itself.

K. Bögel in [1] studied the relation between the continuity of the real function f of two real variables and the continuity of the rectangle function generated by f. This work deals with a similar question concerning approximate continuity. We shall introduce six kinds of approximate continuity of functions of two variables. In three definitions we shall use the ordinary increment of functions and in the other — the so called two-dimensional increment. In the first part of the paper we shall study in detail various relations between the introduced classes of approximately continuous functions. Next we shall prove, for the case of approximately continuous functions, some theorems which are similar to the theorems proved by Bögel in [1] and Tasche in [2] for continuous functions. The last part of the paper includes some conditions under which the superposition $f \circ g$ is approximately continuous when f is an approximately continuous function and g is a homeomorphism transforming the plane onto itself.

We shall use the following notation:

$\{[a, b]; c\}$ — an interval (parallel to the Ox-axis) with the end-points (a, c) and (b, c).

$\{a; [b, c]\}$ — an interval (parallel to the Oy-axis) with the end-points (a, b) and (a, c).

$[a, c; b, d]$ — a rectangle $\min(a, b) \leqslant x \leqslant \max(a, b)$, $\min(c, d) \leqslant y \leqslant \max(c, d)$.

$\Delta_x f\{[x_1, x_2]; c\} = f(x_2, c) - f(x_1, c)$,

$\Delta_y f\{a; [y_1, y_2]\} = f(a, y_2) - f(a, y_1)$,

$\Delta_{xy} f[x_1, y_1; x_2, y_2] = f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)$.

Δ_x, Δ_y, and Δ_{xy} are additive rectangle functions and the following equalities hold (see [1], p. 50):

(1) $$\Delta_{xy} f[x_1, y_1; x_2, y_2] = \Delta_x f\{[x_1, x_2]; y_2\} - \Delta_x f[\{x_1, x_2]; y_1\} = \Delta_y f\{x_2; [y_1, y_2]\} - \Delta_y f\{x_1; [y_1, y_2]\},$$

(2) $$f(x_2, y_2) - f(x_1, y_1) = \Delta_x f\{[x_1, x_2]; y_1\} + \Delta_y f\{x_1; [y_1, y_2]\} + \Delta_{xy} f[x_1, y_1; x_2, y_2],$$

(3) $$\Delta[\sum_{i=1}^{n} f_i] = \sum_{i=1}^{n} \Delta f_i,$$

where Δ stands for Δ_x, Δ_y, Δ_{xy}.

It is not difficult to see that every additive rectangle function can be represented as the Δ_{xy}-function for some function of two variables (see [1], p. 50).

We suppose that all the sets under consideration are measurable.

Let $|\cdot|_1$ and $|\cdot|_2$ denote the linear and the two-dimensional Lebesgue measure, respectively. Recall that the number $\varrho_{x_0} A = \lim_{h\to 0^+} |A \cap [x_0 - h, x_0 + h]|_1 / 2h$ (when $A \subset R$ and $x_0 \in R$) is termed the density of A at x_0 (if the limit exists). The definitions of the right-hand and left-hand, upper and lower densities are similar with obvious modifications. If $A \subset R^2$, and $(x_0, y_0) \in R^2$, then the number

$$\varrho_{(x_0,y_0)} A = \lim_{h\to 0^+} |A \cap [x_0 - h, y_0 - h; x_0 + h, y_0 + h]|_2 / 4h^2$$

is termed the density of A at (x_0, y_0).

We shall make use of the following properties of density:

(4) if $\varrho_{x_0} A' = \varrho_{x_0} A'' = 1$, then $\varrho_{x_0}(A' \cap A'') = 1$,

if $\varrho_{(x_0,y_0)} A' = \varrho_{(x_0,y_0)} A'' = 1$, then $\varrho_{(x_0,y_0)}(A' \cap A'') = 1$,

(5) if $\varrho_{x_0} A_1 = \varrho_{y_0} A_2 = 1$, then $\varrho_{(x_0,y_0)}(A_1 \times A_2) = 1$.

DEFINITION 1. We shall say that the function f: $R^2 \to R$ is *approximately continuous at* (x_0, y_0) if and only if there exists a set $A \subset R^2$ such that $\varrho_{(x_0,y_0)} A = 1$ and

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (x,y)\in A}} (f(x,y) - f(x_0,y_0)) = 0.$$

DEFINITION 2. We shall say that the function f: $R^2 \to R$ is *approximately continuous on the product at* (x_0, y_0) if and only if there exist a pair of sets $A_1, A_2 \subset R$ such that $\varrho_{x_0} A_1 = 1$, $\varrho_{y_0} A_2 = 1$ and

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (x,y)\in A_1\times A_2}} (f(x,y) - f(x_0,y_0)) = 0.$$

DEFINITION 3. We shall say that the function f: $R^2 \to R$ is *approximately continuous on the hyperbolas at* (x_0, y_0) if and only if there exists a set $A \subset R^+$ (R^+ denotes the set of all positive numbers) such that $\varrho_0^+ A = 1$ (ϱ^+ denotes the right-hand density) and

$$\lim_{\substack{(x-x_0)(y-y_0)\to 0\\ |(x-x_0)(y-y_0)|\in A}} (f(x,y) - f(x_0,y_0)) = 0.$$

If, in the above definitions, we put the two-dimensional increment $\Delta_{xy} f[x_0, y_0; x, y]$ in the place of the ordinary increment, we shall obtain the definitions of the Δ_{xy}-approximately continuous function (the Δ_{xy}-approximately continuous function on the product, the Δ_{xy}-approximately continuous function on the hyperbolas, respectively) at (x_0, y_0).

Finally we shall say that the function f: $R^2 \to R$ is Δ_x (Δ_y)*-approximately continuous at* (x_0, y_0) if and only if the function $\varphi(x) = f(x, y_0)$ and $\psi(y) = f(x_0, y)$ are approximately continuous (as a function of one variable) at x_0 and y_0, respectively.

Now we shall study the connections between the introduced classes of functions.

THEOREM 1. *If a function f: $R^2 \to R$ is approximately continuous on the product at (x_0, y_0), then it is approximately continuous at (x_0, y_0). If f is Δ_{xy}-approximately continuous on the product at (x_0, y_0), then it is Δ_{xy}-approximately continuous at (x_0, y_0).*

Proof. The theorem follows immediately from (5).

Remark 1. The converse theorem is not true. For example, the characteristic function of the set $\{(x, y)\colon y > x^2 \text{ or } y < -x^2 \text{ or } y = 0\}$ is approximately continuous and Δ_{xy}-approximately continuous at (0,0), but is neither approximately continuous on the product nor Δ_{xy}-approximately continuous on the product at this point.

LEMMA 1. *If $A \subset R^+$, $\varrho_0^+ A = 1$ and $B = \{(x, y)\colon |xy| \in A\}$, then $\varrho_{(0,0)} B = 1$.*

Proof. It is easy to see that B is a measurable set. Let $T(h)$ be a triangle with vertices $(0,0)$, $(0,h)$, (h,h). B is symmetric with respect to the following axes: $x = 0$, $y = 0$, $y = x$, $y = -x$; thus it suffices to prove that

(6) $$\lim_{h\to 0^+} \frac{|T(h) \cap B|_2}{|T(h)|_2} = 1.$$

Put $aC = \{at\colon t \in C\}$ for $C \subset R$, $a \in R$ and $D_x = \{y\colon (x, y) \in D\}$ for $D \subset R^2$, $x \in R$. In virtue of the theorem of Fubini we have

$$|T(h) \cap B|_2 = \int_0^h |(T(h) \cap B)_x|_1 \, dx,$$

$$|T(h)|_2 = \int_0^h |(T(h)_x|_1 \, dx = \int_0^h x \, dx.$$

Observe that for $0<x<h$ we have $(T(h)\cap B)_x = [0, x]\cap(x^{-1})A$. It is well known that $|cA|_1 = |c|\cdot|A|_1$ (see, for example [4], p. 64). Hence

$$\frac{|(T(h)\cap B)_x|_1}{|(T(h))_x|_1} = \frac{|[0, x]\cap(x^{-1})A|_1}{|[0, x]|_1} = \frac{|[0, x^2]\cap A|_1}{|[0, x^2]|_1}.$$

From the assumption it follows that for every $\varepsilon>0$ there exists a $\delta>0$ such that, for every real x, if $0<x<\delta$, then

$$\frac{|A\cap[0, x]|_1}{|[0, x]|_1} > 1-\varepsilon.$$

Obviously we can suppose that $\delta<1$. Then for $0<x<\delta$ we have $x^2<x$; hence

$$\frac{|[0, x^2]\cap A|_1}{|[0, x^2]|_1} > 1-\varepsilon.$$

Thus for every x such that $0<x<\delta$ and for each $h>x$ we have

$$|(T(h)\cap B)_x|_1 > (1-\varepsilon)|(T(h))_x|_1.$$

If $h<\delta$, then

$$\int_0^h |(T(h)\cap B)_x|_1\,dx > (1-\varepsilon)\int_0^h |(T(h))_x|_1\,dx,$$

and so (6) holds.

COROLLARY. *If* $A\subset R^+$, $\varrho_0^+ A = 1$ *and* $B = \{(x, y)\colon |(x-x_0)(y-y_0)|\in A\}$, *then* $\varrho_{(x_0,y_0)}B = 1$.

THEOREM 2. *If a function* $f\colon R^2\to R$ *is approximately continuous on the hyperbolas at* (x_0, y_0), *then it is approximately continuous at* (x_0, y_0). *If* f *is* Δ_{xy}*-approximately continuous on the hyperbolas at* (x_0, y_0), *then it is* Δ_{xy}*-approximately continuous at* (x_0, y_0).

Proof. The theorem easily follows from the above corollary.

Remark 2. The function from Remark 1 shows that the converse theorem is again not true.

Remark 3. The approximate continuity (Δ_{xy}-approximate continuity) on the product at (x_0, y_0) does not imply the approximate continuity (Δ_{xy}-approximate continuity) on the hyperbolas at that point. As an example we can use (for $(x_0, y_0) = (0, 0)$) the following function:

$$f(x, y) = \begin{cases} 1 & \text{for } x = n^{-1},\ n = 1, 2, \ldots, y\neq 0, \\ 0 & \text{for remaining } (x, y). \end{cases}$$

Remark 4. The approximate continuity (Δ_{xy}-approximate continuity) on the hyperbolas at (x_0, y_0) does not imply the approximate continuity (Δ_{xy}-approximate continuity) on the product at (x_0, y_0). Let $A\subset R^+$ be a set such that $\varrho_0^+ A = 1$ and 0 is the point of accumulation of $R^+ - A$. If f is a characteristic function of the set $\{(x, y)\colon |xy|\in A\cup\{0\}\}$, then it is easy to see that f is approximately continuous on the hyperbolas at $(0, 0)$ and Δ_{xy}-approximately continuous on the hyperbolas at that point. We shall prove that f is neither approximately continuous on the product nor Δ_{xy}-approximately continuous on the product at $(0, 0)$.

Let $A_1, A_2\subset R$ be such sets that $A_1\times A_2\subset\{(x, y)\colon |xy|\in A\}$. There exists a sequence $\{a_n\}$ such that $a_n\searrow 0$ and $a_n\notin A$ for $n = 1, 2, \ldots$ It is easy to see that for every $x\in R$ we have $x\notin A_1$ or $a_n x^{-1}\notin A_2$. Consider the intervals $[\sqrt{a_n}, 2\sqrt{a_n}]$ on the Ox-axis and $[2^{-1}\sqrt{a_n}, \sqrt{a_n}]$ on the Oy-axis. Let

$$A_1^{(n)} = [\sqrt{a_n}, 2\sqrt{a_n}]-A_1, \quad A_2^{(n)} = [2^{-1}\sqrt{a_n}, \sqrt{a_n}]-A_2.$$

If $f_n(x) = a_n x^{-1}$; then we have $[\sqrt{a_n}, 2\sqrt{a_n}] = A_1^{(n)}\cup f_n^{-1}(A_2^{(n)})$. Hence $|A_1^{(n)}|_1\geqslant 2^{-1}\sqrt{a_n}$ or $|f_n^{-1}(A_2^{(n)})|_1\geqslant 2^{-1}\sqrt{a_n}$. In the first case we have

$$\frac{|(R-A_1)\cap[0, 2\sqrt{a_n}]|_1}{2\sqrt{a_n}} > \frac{|A_1^{(n)}|_1}{2\sqrt{a_n}} \geqslant \frac{1}{4}, \tag{7}$$

because $R-A_1\supset A_1^{(n)}$. In the second case we shall use the following inequality:

$$|f_n^{-1}(A_2^{(n)})|_1 \leqslant \max_{x\in[2^{-1}\sqrt{a_n}, \sqrt{a_n}]} |(f_n^{-1})'(x)|\cdot|A_2^{(n)}|_1,$$

(see [5], p. 227). It is not difficult to compute that $\max_{x\in[2^{-1}\sqrt{a_n}, \sqrt{a_n}]} |(f_n^{-1})'(x)| = 4$; hence

$$\frac{|(R-A_2)\cap[0, \sqrt{a_n}]|_1}{\sqrt{a_n}} \geqslant \frac{|A_2^{(n)}|_1}{\sqrt{a_n}} \geqslant \frac{1}{4}\cdot\frac{|f_n^{-1}(A_2^{(n)})|_1}{\sqrt{a_n}} \geqslant \frac{1}{8}, \tag{8}$$

because $R-A_2\supset A_2^{(n)}$.

For every natural n one of the inequalities (7) or (8) holds, and so it is impossible to have $\varrho_0 A_1 = 1$ and $\varrho_0 A_2 = 1$ simultaneously. Then f is not approximately continuous (Δ_{xy}-approximately continuous) on the product at $(0, 0)$.

THEOREM 3. *If a function* $f\colon R^2\to R$ *is* Δ_x*-approximately and* Δ_y*-approximately continuous and* Δ_{xy}*-approximately continuous* (Δ_{xy}*-approximately continuous on the product*) *at* (x_0, y_0), *then it is approximately continuous* (*approximately continuous on the product*) *at that point.*

Proof. We shall prove the theorem in the case of approximate continuity on the product. The proof in the remaining case is quite similar.

Let $A_1', A_2'\subset R$ be the sets connected with the Δ_x-approximate and Δ_y-approximate continuity of f at (x_0, y_0) and let $A_1'', A_2''\subset R$ be the sets connected with the Δ_{xy}-approximate continuity on the product of f at (x_0, y_0). Put $A_1 = A_1'\cap A_1''$ and $A_2 = A_2'\cap A_2''$. In virtue of (4) and (5) we have

$$\varrho_{(x_0,y_0)}(A_1\times A_2) = 1. \tag{9}$$

For every $\varepsilon>0$ there exists a $\delta>0$ such that, if $(x,y)\in A_1\times A_2$, $|x-x_0|<\delta$, $|y-y_0|<\delta$, then

(10) $$|\Delta_x f\{[x_0,x];y_0\}|<\tfrac{1}{3}\varepsilon\,,$$

(11) $$|\Delta_y f\{x_0;[y_0,y]\}|<\tfrac{1}{3}\varepsilon\,,$$

(12) $$|\Delta_{xy} f[x_0,y_0;x,y]|<\tfrac{1}{3}\varepsilon\,.$$

Hence in virtue of (2) we have for such points (x,y)

$$|f(x,y)-f(x_0,y_0)|<\varepsilon\,,$$

and f is approximately continuous on the product at (x_0,y_0).

Remark 5. A similar theorem for the approximate continuity on the hyperbolas does not hold. As an example we can use (for $(x_0,y_0)=(0,0)$) the following function:

$$f(x,y)=\begin{cases}1 & \text{for}\quad x=n^{-1},\ n=1,2,\dots,\ y\text{ — an arbitrary number}\,,\\ 0 & \text{for the remaining } (x,y)\,.\end{cases}$$

Remark 6. Bögel proved in [1], p. 51, that from the continuity of f at (x,y) it follows that f is Δ_x-continuous, Δ_y-continuous and Δ_{xy}-continuous at that point. It is easy to see that in the case of approximate continuity a similar theorem is not true.

Using the method of proof of Theorem 3, one can prove without difficulty the following theorems:

THEOREM 4. *If a function $f\colon R^2\to R$ is approximately continuous, Δ_x (Δ_y)-approximately continuous and Δ_{xy}-approximately continuous at (x_0,y_0), then it is Δ_y (Δ_x)-approximately continuous at that point. If f is Δ_x (Δ_y)-approximately continuous and approximately continuous at (x_0,y_0), then it is Δ_{xy}-approximately continuous at that point.*

THEOREM 5. *If a function $f\colon R^2\to R$ is approximately continuous on the product, Δ_x (Δ_y)-approximately continuous and Δ_{xy}-approximately continuous on the product at (x_0,y_0), then it is Δ_y (Δ_x)-approximately continuous at that point. If f is Δ_x-, Δ_y-approximately continuous and approximately continuous on the product at (x_0,y_0), then it is Δ_{xy}-approximately continuous on the product at that point.*

THEOREM 6. *If a function $f\colon R^2\to R$ is approximately continuous on the hyperbolas, Δ_{xy}-approximately continuous on the hyperbolas and Δ_x (Δ_y)-approximately continuous at (x_0,y_0), then it is Δ_y (Δ_x)-approximately continuous at that point.*

The following theorem also has its analogue in Bögel [1]:

THEOREM 7. *A function $f\colon R^2\to R$ is Δ_{xy}-approximately continuous at (x_0,y_0) if and only if it can be represented in the form*

(13) $$f(x,y)=g(x,y)+\varphi(x)+\psi(y)\,,$$

where g is approximately continuous, Δ_x-approximately and Δ_y-approximately continuous at (x_0,y_0).

Proof. 1. Suppose that (13) is fulfilled. It is easy to see that $\Delta_{xy}(\varphi+\psi)\equiv 0$; so for every (x,y) we have in virtue of (3)

(14) $$\Delta_{xy} f[x_0,y_0;x,y]=\Delta_{xy} g[x_0,y_0;x,y]\,.$$

From Theorem 4 we conclude that g is Δ_{xy}-approximately continuous at (x_0,y_0). From (14) it immediately follows that f is also Δ_{xy}-approximately continuous at (x_0,y_0).

2. Suppose now that f is Δ_{xy}-approximately continuous at (x_0,y_0). Then the function $g(x,y)=f(x_0,y_0)+\Delta_{xy} f[x_0,y_0;x,y]$ is approximately continuous at (x_0,y_0) and Δ_x-approximately and Δ_y-approximately continuous at that point, because it is not difficult to verify that $\Delta_x g\{[x_0,x];y_0\}\equiv 0$ and $\Delta_y g\{x_0;[y_0,y]\}\equiv 0$. If we put $\varphi(x)=\Delta_x f\{[x_0,x];y_0\}$, $\psi(y)=\Delta_y f\{x_0;[y_0,y]\}$, then g, φ and ψ fulfil (13).

THEOREM 8. *A function $f\colon R^2\to R$ is Δ_{xy}-approximately continuous on the product at (x_0,y_0) if and only if it can be represented in the form*

$$f(x,y)=g(x,y)+\varphi(x)+\psi(y)\,,$$

where g is approximately continuous on the product at (x_0,y_0), Δ_x-approximately and Δ_y-approximately continuous at (x_0,y_0).

The proof is quite similar and will be omitted.

Remark 7. The above theorems are of the local kind, as the following example shows:

Let $A_1\subset R$ be a set such that $\bar{\varrho}_0 A_1>0$ and $\bar{\varrho}_0(R-A_1)>0$ (here $\bar{\varrho}$ means the upper density). Put

$$f(x,y)=\begin{cases}x & \text{if}\quad y\in A_1\,,\\ x^2+1 & \text{if}\quad y\notin A_1\,.\end{cases}$$

This function is Δ_{xy}-continuous at every point, because it is not difficult to see that $|\Delta_{xy} f[x_0,y_0;x,y]|\leqslant|x^2-x_0^2|+|x-x_0|$ for every (x_0,y_0) and (x,y); so f is Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product) at every point. We shall show that it is impossible to find a function g which is approximately continuous, Δ_x-approximately and Δ_y-approximately continuous at every point and two functions φ and ψ such that (13) is fulfilled. Suppose that such functions do exist. Hence

(15) $$f(x,y)-\psi(y)=g(x,y)+\varphi(x)\,.$$

From the Δ_y-approximate continuity of g it follows that for every x there exists a set $A_x\subset R$ such that $\varrho_0 A_x=1$ and that the following limit exists:

(16) $$\lim_{\substack{y\to 0\\ y\in A_x}}[g(x,y)+\varphi(x)]=g(x,0)+\varphi(x)\,.$$

It is easy to see that 0 is a point of accumulation of $A_x \cap A_1$ and of $A_x \cap (R-A_1)$. Hence

$$\limsup_{\substack{y\to 0\\ y\in A_x}} f(x,y) - \liminf_{\substack{y\to 0\\ y\in A_x}} f(x,y) = |x^2+1-x| > 0\,, \tag{17}$$

and from the fact that the right side of (15) has a limit we conclude that the left-hand side of (15) must also to have a limit, and so

$$\limsup_{\substack{y\to 0\\ y\in A_x}} \psi(y) - \liminf_{\substack{y\to 0\\ y\in A_x}} \psi(y) = |x^2+1-x|\,.$$

This is impossible because ψ does not depend on x.

M. Tasche in [6] proved the global theorem on the representation of a function which is Δ_{xy}-continuous on the hyperbolas at every point in the form (13), where g is continuous at every point. We shall prove a similar theorem for the Δ_{xy}-approximate continuity on the hyperbolas.

THEOREM 9. *If a function $f\colon R^2 \to R$ is Δ_{xy}-approximately continuous on the hyperbolas at every point, then it can be represented in the form*

$$f(x,y) = g(x,y) + g_1(x) + g_2(y)\,,$$

where g is approximately continuous at every point.

Proof. Put

$$g(x,y) = \Delta_{xy} f[a,c;x,y]\,,$$
$$g_1(x) = f(x,c) - f(a,c)\,,$$
$$g_2(y) = f(a,y)\,,$$

where (a,c) is an arbitrary fixed point.

It is easy to verify that the required equality is fulfilled.

A function g is approximately continuous on the hyperbolas at (a,c), and so it is approximately continuous at (a,c). Let $(x_0,y_0) \neq (a,c)$. We shall prove that g is approximately continuous at (x_0,y_0). Suppose that $a \neq x_0$ and $c \neq y_0$ (in the case where $a = x_0$ or $c = y_0$ the proof is simpler). A function f is Δ_{xy}-approximately continuous on the hyperbolas at (x_0,y_0), and so there exists a set $A \subset R^+$ such that $\varrho_0^+ A = 1$ and

$$\lim_{\substack{(x-x_0)(y-y_0)\to 0\\ |(x-x_0)(y-y_0)|\in A}} \Delta_{xy} f[x_0,y_0;x,y] = 0\,.$$

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$|x-x_0|\cdot|c-y_0| < \delta \quad\text{and}\quad |x-x_0|\cdot|c-y_0| \in A\,,$$
$$|a-x_0|\cdot|y-y_0| < \delta \quad\text{and}\quad |a-x_0|\cdot|y-y_0| \in A\,,$$
$$|x-x_0|\cdot|y-y_0| < \delta \quad\text{and}\quad |x-x_0|\cdot|y-y_0| \in A\,,$$

then the following inequalities hold:

$$|\Delta_{xy} f[x,c;x_0,y_0]| < \tfrac{1}{3}\varepsilon\,,$$
$$|\Delta_{xy} f[a,y;x_0,y_0]| < \tfrac{1}{3}\varepsilon\,,$$
$$|\Delta_{xy} f[x,y;x_0,y_0]| < \tfrac{1}{3}\varepsilon\,.$$

Put $\delta_1 = \min(\delta|c-y_0|^{-1}, \delta|a-x_0|^{-1}, \sqrt{\delta})$. For (x,y) such that

$$|x-x_0| \in |c-y_0|^{-1}A\,, \quad |y-y_0| \in |a-x_0|^{-1}A\,, \quad |(x-x_0)(y-y_0)| \in A$$

and

$$|x-x_0| < \delta_1\,, \quad |y-y_0| < \delta_1$$

we have

$$|g(x,y) - g(x_0,y_0)|$$
$$\leqslant |\Delta_{xy} f[x,c;x_0,y_0]| + |\Delta_{xy} f[a,y;x_0,y_0]| + |\Delta_{xy} f[x,y;x_0,y_0]| < \varepsilon\,.$$

We shall use the additional notation $C+x = \{t+x\colon t\in C\}$ for $C\subset R$. Obviously $|x-x_0| \in |c-y_0|^{-1}A$ if and only if $x \in (|c-y_0|^{-1}(A \cup (-1\cdot A)) + x_0) = A_1$ and similarly $|y-y_0| \in |a-x_0|^{-1}A$ if and only if $y \in (|a-x_0|^{-1}(A\cup(-1\cdot A)) + y_0) = A_2$. From the fact that $\varrho_0^+ A = 1$ it follows that $\varrho_{x_0} A_1 = 1$ and $\varrho_{y_0} A_2 = 1$. From Lemma 1 we have $\varrho_{(x_0,y_0)} B = 1$, where $B = \{(x,y)\colon |(x-x_0)(y-y_0)| \in A\}$. If we put $A' = (A_1 \times A_2) \cap B$, then $\varrho_{(x_0,y_0)} A' = 1$ and in virtue of the above consideration

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (x,y)\in A'}} (g(x,y) - g(x_0,y_0)) = 0\,,$$

and so g is approximately continuous at (x_0,y_0).

Now we shall study homeomorphisms $g\colon R^2 \xrightarrow[\text{onto}]{} R^2$ with the property that for every approximately continuous function $f\colon R^2\to R$ (Δ_{xy}-approximately continuous function and so on) the superposition $f\circ g$ is approximately continuous (Δ_{xy}-approximately continuous) of the same type.

Observe that the case of approximate continuity on the hyperbolas is trivial, because if a function $f\colon R^2\to R$ is approximately continuous on hyperbolas at every point, then it is a constant function. So every homeomorphism is good enough to preserve this kind of approximate continuity. We shall deal with the remaining kinds of approximate continuity.

The one-dimensional case was studied by Bruckner in [2]. Recall the following definition, introduced by Bruckner:

DEFINITION 4. We say that a homeomorphism $h\colon R_n \xrightarrow[\text{onto}]{} R_n$ (where n is a natural number) preserves density points if and only if, for every measurable set $A\subset R_n$ and for every point $a\in R_n$, if $\varrho_a A = 1$, then $\varrho_{h(a)} h(A) = 1$.

THEOREM 10. *Let $g\colon R^2 \xrightarrow[\text{onto}]{} R^2$ be a homeomorphism. For every approximately continuous function f the function $f\circ g$ is approximately continuous if and only if the homeomorphism $h = g^{-1}$ preserves density points.*

Proof. Observe that a function $f\colon R^2 \to R$ is approximately continuous at every point if and only if for every real number r the sets $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ are d-open (that is, measurable and consisting only of points of density) (see for example [3]).

1. Suppose that h preserves density points. Let f be an arbitrary approximately continuous function and r an arbitrary real number. We have

$$(f\circ g)^{-1}((-\infty, r)) = g^{-1}(f^{-1}((-\infty, r))) = h(f^{-1}((-\infty, r)))$$

and similarly for $(r, +\infty)$. The sets $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ are d-open and h preserves density points; so $h(f^{-1}((-\infty, r)))$ and $h(f^{-1}((r, +\infty)))$ are also d-open. Hence $f\circ g$ is approximately continuous.

2. Suppose that h does not preserve density points. Let (x_0, y_0) be a point of density of a measurable set $S\subset R^2$ such that $h(x_0, y_0)$ is not a point of density of $h(S)$. Let D be a d-open set of type F_σ such that $(x_0, y_0)\in D$, $D\subset S$ and $|S-D|_2 = 0$. Using the method of proof in [7], p. 26, and the n-dimensional version of the Lusin–Menchoff theorem (see [3]), one can prove without difficulty that there exists a function $f\colon R^2\to R$ which is approximately continuous and fulfils the following conditions: $0<f(x, y)<1$ for $(x, y)\in D$ and $f(x, y) = 0$ for $(x, y)\notin D$. Then it is easy to see that the set $(f\circ g)^{-1}((0, +\infty)) = h(D)$ is not d-open (a point $h(x_0, y_0)$ is a "bad" point), and so $f\circ g$ is not approximately continuous.

LEMMA 2. *Let $g\colon R^2\to R^2$ be a homeomorphism and let $h = g^{-1}$. If for every function $f\colon R^2\to R$ Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product, Δ_{xy}-approximately continuous on hyperbolas) the function $f\circ g$ is Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product, Δ_{xy}-approximately continuous on hyperbolas), then the image $h(P)$ of every straight line which is parallel to the Ox-axis or to the Oy-axis is a straight line parallel to the Ox-axis or to the Oy-axis.*

Proof. Suppose that a homeomorphism g does not possess the property described in the lemma. Then there exists a straight line $P\colon y = y_0$ such that $h(P)$ is neither the straight line parallel to the Ox-axis nor the straight line parallel to the Oy-axis, or there exists a straight line $Q\colon x = x_0$ having the image not of the required form. Consider the first case; the proof in the second is quite similar. There exist two points (ξ_1, η_1), $(\xi_2, \eta_2)\in h(P)$ such that $\xi_1\neq\xi_2$, and $\eta_1\neq\eta_2$. Let $(x_1, y_0) = g(\xi_1, \eta_1)$ and $(x_2, y_0) = g(\xi_2, \eta_2)$. Consider the set $A = h(\{[x_1, x_2]; y_0\})$. Obviously it is a compact set. If P_{12} denotes the straight line containing the points (ξ_1, η_1) and (ξ_2, η_2), then there exists a point $(\xi_0, \eta_0)\in A$ such that $(\xi_0, \eta_0)\neq(\xi_1, \eta_1)$, $(\xi_0, \eta_0)\neq(\xi_2, \eta_2)$ and $d((\xi_0, \eta_0), P_{12}) = \sup_{(\xi,\eta)\in A} d((\xi, \eta), P_{12})$, where d is an ordinary distance function in the plane. It is not difficult to see that there exists a circular neighbourhood $K((\xi_0, \eta_0), \varepsilon)$ of (ξ_0, η_0) for which at least one from the following equalities holds:

$$K((\xi_0, \eta_0), \varepsilon)\cap h(P)\cap\{(\xi, \eta)\colon \xi\geqslant\xi_0 \text{ and } \eta\geqslant\eta_0\} = \{(\xi_0, \eta_0)\},$$
$$K((\xi_0, \eta_0), \varepsilon)\cap h(P)\cap\{(\xi, \eta)\colon \xi\geqslant\xi_0 \text{ and } \eta\leqslant\eta_0\} = \{(\xi_0, \eta_0)\},$$
$$K((\xi_0, \eta_0), \varepsilon)\cap h(P)\cap\{(\xi, \eta)\colon \xi\leqslant\xi_0 \text{ and } \eta\geqslant\eta_0\} = \{(\xi_0, \eta_0)\},$$
$$K((\xi_0, \eta_0), \varepsilon)\cap h(P)\cap\{(\xi, \eta)\colon \xi\leqslant\xi_0 \text{ and } \eta\leqslant\eta_0\} = \{(\xi_0, \eta_0)\}.$$

Put

$$f(x, y) = \begin{cases} 1 & \text{for } y = y_0, \\ 0 & \text{for the remaining } (x, y). \end{cases}$$

Obviously f is Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product, Δ_{xy}-approximately continuous on hyperbolas), but $f\circ g$ is not Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product, Δ_{xy}-approximately continuous on hyperbolas) at (ξ_0, η_0), because the set $\{(\xi, \eta)\colon \Delta_{xy}f[\xi_0, \eta_0; \xi, \eta] = 1\}$ has at (ξ_0, η_0) the lower density not less than 4^{-1}.

LEMMA 3. *In the assumptions of the previous lemma the homeomorphism $h = g^{-1}$ is of the form $h(x, y) = (\varphi(x), \psi(y))$ or $h(x, y) = (\psi(y), \varphi(x))$, where $\varphi\colon R\xrightarrow[\text{onto}]{} R$ and $\psi\colon R\xrightarrow[\text{onto}]{} R$ are also homeomorphisms.*

Proof. The lemma easily follows from Lemma 2.

LEMMA 4. *Let $g\colon R^2\xrightarrow[\text{onto}]{} R^2$ be a homeomorphism and let $h = g^{-1}$. If for every function $f\colon R^2\to R$ Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product) the function $f\circ g$ is Δ_{xy}-approximately continuous (Δ_{xy}-approximately continuous on the product), then the functions φ and ψ described in Lemma 3 are homeomorphisms preserving density points.*

Proof. Suppose that φ does not preserve density points (the proof for ψ is similar). By Theorem 3 in [2] there exists a point $x_0\in R$ and two strictly monotonic sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n<b_n$ for every n, $[a_n, b_n]\cap[a_m, b_m] = \emptyset$ for $n\neq m$, $\varrho_{x_0}(\bigcup_{n=1}^{\infty}[a_n, b_n]) = 0$ and $\bar{\varrho}_{\varphi(x_0)}(\varphi(\bigcup_{n=1}^{\infty}[a_n, b_n]))>0$. Suppose that these sequences are decreasing (in the case of increasing sequences the proof is similar). It is not difficult to see that there exists a natural number N such that

$$\varrho_{x_0}\Big(\bigcup_{n=N}^{\infty}[\tfrac{3}{2}a_n-\tfrac{1}{2}b_n, \tfrac{3}{2}b_n-\tfrac{1}{2}a_n]\Big) = 0$$

(the intervals $[\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{3}{2}b_n-\frac{1}{2}a_n]$ are concentric with $[a_n, b_n]$ and are twice as long). For simplicity assume that $N = 1$.

Put

$$f_n(x) = \begin{cases} 1 \text{ for } x = \frac{1}{2}(a_n+b_n), \\ 0 \text{ for } x\leqslant\frac{3}{2}a_n-\frac{1}{2}b_n \text{ or } x\geqslant\frac{3}{2}b_n-\frac{1}{2}a_n, \\ \text{linear and continuous in the intervals } [\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{1}{2}(a_n+b_n)], \\ [\frac{1}{2}(a_n+b_n), \frac{3}{2}b_n-\frac{1}{2}a_n]. \end{cases}$$

Let $f_0(x) = \max_n f_n(x)$ (observe that for every $x \in R$ there are only a finite number of functions f_n which have a positive value at x). It is not difficult to see that f_0 is approximately continuous at every point; it is even continuous for $x \neq x_0$.

Now we shall construct an auxiliary sequence $\{c_n\}$. We have

$$\bar{\varrho}_{\varphi(x_0)}(\varphi(\bigcup_{n=1}^{\infty}[a_n, b_n]))>0 ,$$

and so there exists a number $\eta>0$ and there exists a sequence $\{\xi_k\}$ which is strictly monotonic (assume that it is decreasing: the other case is similar) such that $\xi_k \to \varphi(x_0)$ and

$$\frac{|\varphi(\bigcup_{n=1}^{\infty}[a_n, b_n]) \cap [\varphi(x_0), \xi_k]|_1}{\xi_k - \varphi(x_0)} > \eta \quad \text{for every } k .$$

For every natural n let us choose a number c_n such that

$$\psi(c_n)-\psi(0) = \tfrac{1}{2}\min|\xi_k-\varphi(x_0)| ,$$

where the minimum is taken over such k that

$$|\varphi(a_n)-\varphi(x_0)| \leqslant |\xi_k-\varphi(x_0)| , \qquad |\varphi(b_n)-\varphi(x_0)| \leqslant |\xi_k-\varphi(x_0)| .$$

It is easy to see that $\{c_n\}$ is a monotonic sequence and $c_n \to 0$. Suppose that $c_n>0$ (the other case demands only small changes in the proof).

Put

$$f(x, y) = \begin{cases} f_0(x) & \text{for } x \in [\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{3}{2}b_n-\frac{1}{2}a_n],\ y \geqslant c_n , \\ 0 & \text{for } x \notin \bigcup_{n=1}^{\infty} [\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{3}{2}b_n-\frac{1}{2}a_n] \text{ and arbitrary } y , \\ 0 & \text{for } x \in \bigcup_{n=1}^{\infty} [\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{3}{2}b_n-\frac{1}{2}a_n],\ y \leqslant 0 , \\ \text{linear and continuous in the linear intervals } \{x; [0, c_n]\} & \\ \quad \text{for } x \in [\frac{3}{2}a_n-\frac{1}{2}b_n, \frac{3}{2}b_n-\frac{1}{2}a_n] . & \end{cases}$$

This function is continuous at every point $(x, y) \neq (x_0, 0)$ and it is Δ_{xy}-approximately continuous on the product at $(x_0, 0)$ and so f is Δ_{xy}-approximately continuous on the product at every point. The function $f \circ g$ is not Δ_{xy}-approximately continuous at $(\varphi(x_0), \psi(0))$, because for $\xi \in \varphi[a_n, b_n]$ and $\xi \geqslant \psi(c_n)$ we have $|\Delta_{xy} f \circ g[\varphi(x_0), \psi(0); \xi, \eta]| \geqslant \frac{1}{2}$, and from the construction of $\{c_n\}$ it follows that for every k

$$\frac{|[\xi_k, \psi(0)-|\varphi(x_0)-\xi_k|; 2\varphi(x_0)-\xi, \psi(0)+|\varphi(x_0)-\xi_k|] \cap \bigcup_{n=1}^{\infty} (\varphi([a_n, b_n]) \times [\psi(c_n), +\infty))|_2}{4|\varphi(x_0)-\xi_k|^2} > \frac{1}{8}\eta .$$

So the homeomorphism g does not fulfil the assumptions of the lemma.

THEOREM 11. *Let $g: R^2 \xrightarrow[\text{onto}]{} R^2$ be a homeomorphism and let $h = g^{-1}$. A function $f \circ g$ is Δ_{xy}-approximately continuous on the product for every function f Δ_{xy}-approximately continuous on the product if and only if the homeomorphism h is of the form $h(x, y) = (\varphi(x), \psi(y))$ or $h(x, y) = (\psi(y), \varphi(x))$, where $\varphi: R \xrightarrow[\text{onto}]{} R$ and $\psi: R \xrightarrow[\text{onto}]{} R$ are homeomorphisms preserving density points.*

Proof. The sufficiency is nearly obvious and the necessity follows from Lemmas 2, 3, and 4.

We have found a necessary and sufficient condition for the class of functions approximately continuous and Δ_{xy}-approximately continuous on the product and (a trivial) condition for a class of functions which are approximately continuous on hyperbolas. For the remaining classes we have only a sufficient condition (for functions approximately continuous on the product the condition from Theorem 11 is obviously sufficient) or only a necessary condition (in Lemmas 3 and 4). The problem of finding a necessary and sufficient condition for these classes is open.

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