THEOREM 3. *If $X$ is a compact metric continuum and $H_0$ is a subcollection of $H$ such that $X$ can be mapped onto every member of $H_0$, then $H_0$ is countable.*

Proof. This is a consequence of Theorem 2 of this paper and Theorem 3 of [5].

THEOREM 4. *Uncountably many members of $H$ are not a continuous image of the pseudo-arc.*

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# The theory of Archimedean real closed fields in logics with Ramsey quantifiers

by

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**Abstract.** The theory of Archimedean real closed fields is shown to be complete, decidable, and model complete in a class of logics, due to Malitz and Magidor, which extend the logic with the cardinal quantifier, "There exist infinitely many...".

It is assumed that the reader is familiar with the model theory of first order logic as set forth in the book [1] of Chang and Keisler. In particular, the reader should be acquainted with the definitions of completeness, model completeness, and decidability in reference to first order logic and to certain of its extensions described below.

It is well known that there is no first order theory of Archimedean real closed fields as distinct from the theory of real closed fields. In fact, using the method of elimination of quantifiers [6] or [2], it follows that the first order theory of real closed fields is complete, decidable, and model complete. The situation is not altered when a new cardinal quantifier $Q$, with the $\aleph_0$-interpretation [5], is added to the logic: Add a formation rule to those of first order logic; if $\varphi$ is a formula, then so is $Qx\varphi$; and $\mathfrak{A}$ is a model of $Qx\varphi$ just in case there are infinitely many elements $x$ in the domain of $\mathfrak{A}$ which satisfy $\varphi$. In the case of real closed fields, the method of elimination of quantifiers can be extended to the cardinal quantifier [3] or [7], showing that the theory of these fields in the extended logic is complete, decidable, and model complete. Thus, as in the case of first order logic, there is no theory of Archimedean real closed fields in the logic with the quantifier "There exist infinitely many..." which is distinct from the theory of all real closed fields. In contrast with the above, the situation is different in logics, described below, due to Malitz and Magidor [4], which are generalizations of the logic with the cardinal quantifier. These logics have enough expressive power to distinguish Archimedean from non-Archimedean fields. It will be shown by the method of elimination of quantifiers that the theory of Archimedean real closed fields in these logics of Malitz and Magidor is complete, decidable, and model complete.

**Logics with Ramsey quantifiers.** For each positive integer $n$ the logic $Q^n$ is obtained by adding a new quantifier $Q^n$ which binds $n$ variables and a new formation rule to those of first order logic: If $\varphi$ is a formula and the variables $x_1, \ldots, x_n$ are distinct, then $Q^n x_1 \ldots x_n \varphi$ is also a formula. The logic $Q^{<\omega}$ is obtained from first order logic by adding all the quantifiers $Q^n$ together with the corresponding formation rules.

The intended interpretation of $Q^n x_1 \ldots x_n \varphi$ is, "There is an infinite set $I$ such that whenever the variables $x_1, \ldots, x_n$ are interpreted by distinct elements $a_1, \ldots, a_n$ of $I$, then $\varphi$ holds". Therefore, writing $\mathfrak{A} \models \varphi[\underset{\sim}{a}]$ where $\underset{\sim}{a}$ is an interpretation, in the structure $\mathfrak{A}$, of the free variables, which satisfies the formula $\varphi$; for each positive integer $n$ a new clause must be added to the usual recursive definition of satisfaction for first order logic: $\mathfrak{A} \models Q^n x_1 \ldots x_n \varphi[\underset{\sim}{a}]$ just in case there is an infinite subset $I$ of the domain of $\mathfrak{A}$ such that whenever $a_1, \ldots, a_n$ are distinct elements of $I$, then $\mathfrak{A} \models \varphi[a_1/x_1, \ldots, a_n/x_n, \underset{\sim}{a}]$. Here the notation indicates how each of the variables $x_1, \ldots, x_n$ is to be interpreted in $\mathfrak{A}$.

The logic $Q^1$ coincides with the logic with the cardinal quantifier, "There exist infinitely many...". It seems appropriate to refer to the logics $Q^n$, for $n \geqslant 2$, as logics with Ramsey quantifiers because of the similarity between their semantics and the well-known statement of Ramsey's theorem. (For a statement of Ramsey's theorem, see [1], p. 145.)

**The theory of Archimedean real closed fields.** Recall that an *ordered field* is a linearly ordered structure, satisfying the field axioms, such that multiplication, by positive elements, and addition, by all elements, is compatible with the ordering. An ordered field is an *Archimedean field* iff each member of the field is bounded above by some positive integer. An ordered field is a *real closed field* just in case each positive element has a square root and Weierstrass' Nullstellensatz holds for polynomials of a single variable with coefficients from the field, i.e., if $p(x)$ is such a polynomial, $a$ and $b$ elements of the field such that $a<b$ and $p(a)<0$ and $p(b)>0$, then for some $c$ between $a$ and $b$, $p(c) = 0$.

Let $L = \{+, \cdot, -, 0, 1, <\}$ be the language appropriate for ordered fields, i.e., $+$ and $\cdot$ are binary function symbols denoting addition and multiplication, 0 and 1 are constant symbols denoting the additive and multiplicative units, $-$ is a unary function symbol denoting the additive inverse, and $<$ is a binary relation symbol for the ordering relation. It is well known that there is a set, denoted hereafter by RCF, of first order sentences of the language $L$ such that $\mathfrak{A} \models$ RCF iff $\mathfrak{A}$ is a real closed field; for example, Weirestrass' Nullstellensatz for polynomials is translated into first order logic by an infinite set of sentences: For each positive integer $n$

$$\forall x_0 \ldots \forall x_n \forall y_1 \forall y_2 [y_1 < y_2 \wedge x_0 + x_1 y_1 + \ldots + x_n y_1^n < 0 \wedge$$
$$\wedge x_0 + x_1 y_2 + \ldots + x_n y_2^n > 0 \rightarrow \exists z (y_1 < z \wedge z < y_2 \wedge x_0 + x_1 z + \ldots + x_n z^n = 0)] .$$

PROPOSITION. *There is a set, denoted hereafter by* ARCF, *of $Q^2$-sentences of the language $L$ such that $\mathfrak{A} \models$ ARCF iff $\mathfrak{A}$ is an Archimedean real closed field.*



Proof. Let ARCF be RCF $\cup \{\neg \sigma\}$ where $\sigma$ is

$$\exists x Q^2 yz (0<y \wedge y<x \wedge 0<z \wedge z<x \wedge |y-z|>1)$$

and $|y-z|>1$ is the formula $y-z>1 \vee z-y>1$. It is enough to show that every ordered field is non-Archimedean just in case it is a model of $\sigma$: If $\mathfrak{A}$ is a non-Archimedean ordered field, then there is an $x$ in the domain of $\mathfrak{A}$ such that every positive integer is less than $x$. Let $I$ be the set of all even positive integers; then $I$ is an infinite set and if $y$ and $z$ are distinct elements of $I$, then $|y-z|>1$, $0<z<x$, and $0<y<x$. Therefore $\mathfrak{A} \models \sigma$. On the other hand, if $\mathfrak{A}$ is an ordered field and $\mathfrak{A} \models \sigma$, then there is an $x$ in the domain of $\mathfrak{A}$ and an infinite subset $I$ of the domain of $\mathfrak{A}$ such that if $y$ and $z$ are distinct elements of $I$, then $|y-z|>1$, $0<y<x$, and $0<z<x$. For each positive integer $n$ let $I_n = \{a \in I | \; n-1<a \leqslant n\}$. Then for each positive integer $n$, $\mathrm{Card}(I_n) \leqslant 1$; for if $a$ and $b$ are in $I_n$ with $a<b$, then $b-a>1$ so that $b>a+1$. But $n-1<a$, so that $n<a+1<b$ which is contrary to $b \leqslant n$. Since $\mathrm{Card}(I_1 \cup \ldots \cup I_n) \leqslant n$ and $I$ is infinite, for each $n$ there is a $c$ in $I-(I_1 \cup \ldots \cup I_n)$. This $c$ must be larger than $n$, so that for each positive integer $n$ there is a $c$ in $I$ such that $n<c<x$.

COROLLARY. *There is a set* ARCF *of $Q^{<\omega}$-sentences such that $\mathfrak{A} \models$ ARCF iff $\mathfrak{A}$ is an Archimedean real closed field.*

COROLLARY. *For $n>2$ there is a set, also denoted by* ARCF, *of $Q^n$-sentences such that $\mathfrak{A} \models$ ARCF iff $\mathfrak{A}$ is an Archimedean real closed field.*

Proof. In $\sigma$ replace $Q^2 yz$ by $Q^n yz z_3 \ldots z_n$ where $y, z, z_3, \ldots, z_n$ are distinct variables.

It follows from the proposition and its corollaries that for each $n \geqslant 2$, the $Q^n$-theory, as well as the $Q^{<\omega}$-theory, of RCF is neither complete nor model complete. The rest of the paper is devoted to showing that the above-mentioned theories of ARCF are complete, model complete, and decidable. The starting point and key to accomplishing this task is the following well-known theorem due to A. Tarski. (For a proof see [6] or [2].)

THEOREM (Tarski). *Every first order formula $\varphi$ of the language $L$ is equivalent, in all models of* RCF, *to a quantifier free formula $\psi$ whose free variables form a subset of those of $\varphi$.*

The immediate goal is to prove the theorem whose statement is obtained from the theorem of Tarski by replacing "first order" by "$Q^n$" and "RCF" by "ARCF".

LEMMA. *For each term $t$ of the language L there is a polynomial $p$ with integer coefficients, in the variables $x_1, \ldots, x_n$ which appear in $t$, such that for every model $\mathfrak{A}$ of* ARCF *and every interpretation $\underset{\sim}{a}$ of the variables $\mathfrak{A} \models p = t[\underset{\sim}{a}]$.*

LEMMA. *Each atomic formula of the language L is equivalent, in all models of* ARCF, *to a polynomial equality or inequality, i.e., to a formula of the form $p = 0$ or of the form $p>0$ where $p$ is a polynomial in several variables with integer coefficients.*

LEMMA. *Each quantifier free formula of the language L is equivalent, in all models of* ARCF, *to a disjunction of formulas of the form*

$$p_1 = 0 \wedge \ldots \wedge p_k = 0 \wedge q_1 > 0 \wedge \ldots \wedge q_n > 0$$

*where the $p_i$ and $q_j$ are polynomials with integer coefficients.*

As a consequence of these lemmas and Tarski's theorem, only formulas of the form $Q^n x_1 \ldots x_n(\theta_1 \vee \ldots \vee \theta_k)$, where each $\theta_i$ is a conjunction of polynomial equalities or inequalities, need be the focus of consideration. It would greatly simplify matters if the quantifier $Q^n$ distributed over disjunctions, but unfortunately as the formula $Q^2 xy(x<y \vee y<x)$ illustrates, this is not necessarily the case. However, a modified distributive law for $Q^n$ over disjunction is the content of the next lemma.

Let $\mathfrak{S}(n)$ be the symmetric group on the set $\{1, \ldots, n\}$ and for each integer $k \geqslant 2$ let $k^n$ be the set of functions from $\mathfrak{S}(n)$ into the set $\{1, \ldots, k\}$. If $\theta$ is a quantifier free formula and $\sigma \in \mathfrak{S}(n)$, then $\theta^\sigma$ is the formula obtained from $\theta$ by replacing each of the variables $x_i$, for $i \in \{1, \ldots, n\}$, by $x_{\sigma(i)}$.

LEMMA (distributive law for $Q^n$ over disjunction). *Each formula of the form $Q^n x_1 \ldots x_n(\theta_1 \vee \ldots \vee \theta_k)$, where each of the $\theta_i$ is a conjunction of polynomial equalities or inequalities, is equivalent, in all models of* ARCF, *to*

$$\bigvee_{f \in k^n} Q^n x_1 \ldots x_n(x_1 < \ldots < x_n \rightarrow \bigwedge_{\sigma \in \mathfrak{S}(n)} \theta^\sigma_{f(\sigma)})$$

Proof. If for some $f \in k^n$,

$$\mathfrak{A} \models Q^n x_1 \ldots x_n(x_1 < \ldots < x_n \rightarrow \bigwedge_{\sigma \in \mathfrak{S}(n)} \theta^\sigma_{f(\sigma)}[\underset{\sim}{a}],$$

then there is an infinite subset $I$ of the domain $A$ of $\mathfrak{A}$ such that whenever $a_1, \ldots, a_n$ are distinct elements of $I$,

$$\mathfrak{A} \models (x_1 < \ldots < x_n \rightarrow \bigwedge_{\sigma \in \mathfrak{S}(n)} \theta^\sigma_{f(\sigma)})[a_1/x_1, \ldots, a_n/x_n, a].$$

Let $c_1, \ldots, c_n$ be distinct elements $I$ and let $\tau$ and $\varrho$ be elements of $\mathfrak{S}(n)$ such that $c_{\tau(1)} < \ldots < c_{\tau(n)}$ and $\varrho = \tau^{-1}$. Then $\mathfrak{S} \models \theta^\varrho_{f(\varrho)}[c_{\tau(1)}/x_1, \ldots, c_{\tau(n)}/x_n, \underset{\sim}{a}]$; so

$$\mathfrak{A} \models \theta^\varrho_{f(\varrho)}[c_1/x_{\varrho(1)}, \ldots, c_n/x_{\varrho(n)}, a];$$

so that $\mathfrak{A} \models \theta_{f(\varrho)}[c_1/x_1, \ldots, c_n/x_n, \underset{\sim}{a}]$. Thus $\mathfrak{A} \models Q^n x_1 \ldots x_n(\theta_1 \vee \ldots \vee \theta_k)[\underset{\sim}{a}]$.

Now suppose that $\mathfrak{A} \models Q^n x_1 \ldots x_n(\theta_1 \vee \ldots \vee \theta_k)[\underset{\sim}{a}]$. Then there is an infinite subset $J_0$ of $A$ such that whenever $a_1, \ldots, a_n$ are distinct elements of $J_0$, $\mathfrak{A} \models \theta_1 \vee \ldots \vee \theta_k[a_1/x_1, \ldots, a_n/x_n, \underset{\sim}{a}]$. Let $\{\sigma_1, \ldots, \sigma_{n!}\}$ be an enumeration of $\mathfrak{S}(n)$ and for a set $J$, let $[J]^n$ denote the set of all subsets $K$ of $J$ such that $\mathrm{Card}(K) = n$. For each $i \in \{1, \ldots, n!\}$, define an infinite subset $J_i$ of $J_0$ such that for each $j$, $J_j \subset J_{j-1}$ and at the same time define a function $g$ from $\mathfrak{S}(n)$ into $\{1, \ldots, k\}$ by specifying the value of $g(\sigma_i)$ as follows: For $i \in \{1, \ldots, n!\}$ and $j \in \{1, \ldots, k\}$, assuming that $J_{i-1}$ has already been defined, let $H^i_j$ be the set of all those $\{a_1, \ldots, a_n\} \in [J_{i-1}]^n$ such that if $\{a_1, \ldots, a_n\} = \{c_1, \ldots, c_n\}$ and $c_1 < \ldots < c_n$, then $\mathfrak{A} \models \theta^{\sigma_i}_j[c_1/x_1, \ldots, c_n/x_n, \underset{\sim}{a}]$



Then $H^i_1 \cup \ldots \cup H^i_k = [J_{i-1}]^n$ because $\{c_1, \ldots, c_n\} \in [J_{i-1}]^n \subseteq [J_0]^n$, and for some $h$, $\mathfrak{A} \models \theta_h[c_{\varrho(1)}/x_1, \ldots, c_{\varrho(n)}/x_n, \underset{\sim}{a}]$ where $\varrho = \sigma_i^{-1}$; so that $\mathfrak{A} \models \theta^{\sigma_i}_h[c_1/x_1, \ldots, c_n/x_n, \underset{\sim}{a}]$. By Ramsey's theorem, there is an $m$, which is taken to be the value of $g(\sigma_i)$, and an infinite subset, which is taken to be $J_i$, of $J_{i-1}$ such that $[J_i]^n \subseteq H^i_m$. Now let $a_1, \ldots, a_n$ be distinct elements of the infinite subset $J_{n!}$ of $A$ such that $a_1 < \ldots < a_n$ and let $\sigma = \sigma_i$ be an element of $\mathfrak{S}(n)$. Since $J_{n!} \subseteq J_i$, then $\{a_1, \ldots, a_n\} \in [J_i]^n \subseteq H^i_{g(\sigma)}$; so that $\mathfrak{A} \models \theta^\sigma_{g(\sigma)}[a_1/x_1, \ldots, a_n/x_n, \underset{\sim}{a}]$. Therefore

$$\mathfrak{A} \models \bigvee_{f \in k^n} Q^n x_1 \ldots x_n(x_1 < \ldots < x_n \rightarrow \bigwedge_{\sigma \in \mathfrak{S}(n)} \theta^\sigma_{f(\sigma)})[\underset{\sim}{a}].$$

This lemma allows us to concentrate on formulas of the form

$$Q^n x_1 \ldots x_n(x_1 < \ldots < x_n \rightarrow \theta)$$

where $\theta$ is a conjunction of polynomial equalities or inequalities.

Let $\theta$ be a conjunction of polynomial equalities or inequalities. Let $u$, $v$, $w$, and $z$ be variables which do not appear in $\theta$. Let $\Phi_i(\theta)$, for $1 \leqslant i \leqslant 4$, be the following first order formulas:

$\Phi_1(\theta)$ is $\forall z \exists x \exists v \forall y[z<x \wedge x<v \wedge (v<y \rightarrow \theta)]$,

$\Phi_2(\theta)$ is $\forall z \exists y \exists u \forall x[y<z \wedge u<y \wedge (x<u \rightarrow \theta)]$,

$\Phi_3(\theta)$ is $\exists w \forall z \exists x \exists v \forall y[z<w \rightarrow z<x \wedge x<v \wedge v<w \wedge (v<y \wedge y<w \rightarrow \theta)]$,

and

$\Phi_4(\theta)$ is $\exists w \forall z \exists y \exists u \forall x[w<z \rightarrow y<z \wedge u<u \wedge w<u \wedge (x<u \wedge w<x \rightarrow \theta)]$.

Let $\varphi$ be a formula of $Q^{<\omega}$, let $\mathfrak{A}$ be a structure linearly ordered by $<$, and suppose $\mathfrak{A} \models Q^n x_1 \ldots x_n \varphi[\underset{\sim}{a}]$. Denote by $I(\varphi[\underset{\sim}{a}])$, a countably infinite subset $I$ of the domain of $\mathfrak{A}$, together with an enumeration $\langle c_i \rangle_{i=0}^\infty$ of $I$ in either increasing (with respect to $<$) or decreasing order, such that whenever $a_1, \ldots, a_n$ are distinct elements of $I$, then $\mathfrak{A} \models \varphi[a_1/x_1, \ldots, a_n/x_n, \underset{\sim}{a}]$.

LEMMA A. *Let $\mathfrak{A}$ be a model of* ARCF; *let $\theta$ be a conjunction of polynomial equalities or inequalities in which the variables $u$, $v$, $w$, and $z$ do not appear, and let $\bar{\theta}^1 = \bar{\theta}$ be the formula $x<y \rightarrow \theta$. If, for $i \in \{1, \ldots, 4\}$, $\mathfrak{A} \models \Phi_i(\theta)[\underset{\sim}{a}]$, then $\mathfrak{A} \models Qxy\bar{\theta}[\underset{\sim}{a}]$, and if $i = 1$, then $I(\bar{\theta}[\underset{\sim}{a}])$ can be taken to be a set unbounded from above and enumerated in increasing order; if $i = 2$, then $I(\bar{\theta}[\underset{\sim}{a}])$ can be taken to be a set unbounded from below and enumerated in decreasing order; if $i = 3$, then $I(\bar{\theta}[\underset{\sim}{a}])$ can be taken to be a set enumerated in increasing order and bounded above; and if $i = 4$, then $I(\bar{\theta}[\underset{\sim}{a}])$ can be taken to be a set enumerated in decreasing order and bounded below.*

Proof. For example consider the cases when $i = 1$ and $i = 4$; the cases when $i = 2$ and $i = 3$ are similar.

When $i = 1$, the formula $\Phi_1(\theta)$ is used to choose sequences $\langle z_i \rangle_{i=0}^\infty$, $\langle v_i \rangle_{i=0}^\infty$, and $\langle x_i \rangle_{i=0}^\infty$: First arbitrarily pick $z_0 = 0$. Then assuming $z_i$ has been chosen, pick $x_i$ and $v_i$ so that

$$\mathfrak{A} \models z<x \wedge x<v \wedge \forall y(v<y \rightarrow \theta)[z_i/z, x_i/x, v_i/v, \underset{\sim}{a}].$$

The formula $\Phi_1(\theta)$ asserts that such an $x_1$ and $v_1$ exist. Finally, let $z_{i+1} = v_i+1$. Note that $\langle x_i\rangle_{i=0}^{\infty}$ is an increasing sequence which is not bounded above. Let $x_i$ and $x_j$ be members of the sequence $\langle x_i\rangle_{i=1}^{\infty}$ with $i<j$. Then $x_i<v_i<z_{i+1}<x_{i+1}\leqslant x_j$ and $\mathfrak{A}\models\theta[x_i/x, x_j/y, \underset{\sim}{a}]$; so that $\mathfrak{A}\models Q^2xy\bar{\theta}$.

When $i = 4$, the formula $\Phi_4(\theta)$ is used to choose sequences $\langle z_i\rangle_{i=0}^{\infty}$, $\langle y_i\rangle_{i=0}^{\infty}$, and $\langle u_i\rangle_{i=0}^{\infty}$: First choose $w_0$ so that

$$\mathfrak{A}\models\forall z\exists y\exists u\forall x[w<z\to y<z\wedge u<y\wedge w<u\wedge(x<u\wedge w<x\to\theta)][w_0/w, \underset{\sim}{a}].$$

Then choose $z_0 = w_0+1$; assuming $z_i$ has been chosen such that $w_0<z_i$, pick $y_i$ and $u_i$ so that

$$\mathfrak{A}\models y<z\wedge u<y\wedge w<u\wedge\forall x(x<u\wedge w<\to\theta)[w_0/w, z_i/z, y_i/y, u_i/u, \underset{\sim}{a}].$$

Finally, let $z_{i+1} = u_i$. Note that $\langle y_i\rangle_{i=0}^{\infty}$ is a decreasing sequence bounded below by $w_0$. Let $y_i$ and $y_j$ be members of the sequence $\langle y_i\rangle_{i=0}^{\infty}$ such that $i<j$. Then $y_i>u_i = z_{i+1}>y_{i+1}\geqslant y_j>w_0$ and $\mathfrak{A}\models\theta[y_i/y, y_j/x, \underset{\sim}{a}]$; so that $\mathfrak{A}\models Q^2xy\bar{\theta}$.

LEMMA B. *Let $\mathfrak{R}$ be the ordered field of all real numbers; let $\theta$ and $\bar{\theta}$ be as in the previous lemma, and suppose $\mathfrak{R}\models Qxy\bar{\theta}[\underset{\sim}{a}]$. If $I(\bar{\theta}[\underset{\sim}{a}])$ can be chosen to be a set unbounded from above and enumerated in increasing order, then $\mathfrak{R}\models\Phi_1(\theta)[\underset{\sim}{a}]$; if $I(\bar{\theta}[\underset{\sim}{a}])$ can be chosen to be a set unbounded from below and enumerated in decreasing order, then $\mathfrak{R}\models\Phi_2(\theta)[\underset{\sim}{a}]$; if $I(\bar{\theta}[\underset{\sim}{a}])$ can be chosen to be a set enumerated in increasing order and bounded above, then $\mathfrak{R}\models\Phi_3(\theta)[\underset{\sim}{a}]$; and if $I(\bar{\theta}[\underset{\sim}{a}])$ can be chosen to be a set enumerated in decreasing order and bounded below, then $\mathfrak{R}\models\Phi_4(\theta)[\underset{\sim}{a}]$.*

Proof. For example, consider the cases involving $\Phi_2(\theta)$ and $\Phi_3(\theta)$; the cases involving $\Phi_1(\theta)$ and $\Phi_4(\theta)$ are similar. Let $\theta$ be $p_1 = 0\wedge\ldots\wedge p_n = 0\wedge q_1>0\wedge\ldots\wedge q_k>0$ where the $p_i$ and the $q_j$ are polynomials in several variables with integer coefficients.

Suppose $I(\bar{\theta}[\underset{\sim}{a}])$ is unbounded from below and enumerated in decreasing order. Then, for every real number $r$ there is a member $b$ of $I(\bar{\theta}[\underset{\sim}{a}])$ such that $b<r$ and the set $\{c\in I(\bar{\theta}[\underset{\sim}{a}])|\ c<b\} = I_b$ is infinite and unbounded from below. For each $c\in I_b$ and each $i$ and $j$, $\mathfrak{R}\models p_i = 0[c/x, b/y, \underset{\sim}{a}]$ and $\mathfrak{R}\models q_j>0[c/x, b/y, \underset{\sim}{a}]$. Since the only polynomial, with real coefficients, in one variable, and with infinitely many zeros, is the zero-polynomial; then $\mathfrak{R}\models\forall xp_i = 0[b/y, \underset{\sim}{a}]$. Since non-zero polynomials in one variable and with real coefficients have only finitely many zeros, by Weierstrass' Nullstellensatz for polynomials, there exist real numbers $d_j$ such that $d_j<b$ and

$$\mathfrak{R}\models\forall x(x<u\to q_j>0)[d_j/u, b/y, a].$$

Let $d = \min\{d_j|\ 1\leqslant j\leqslant k\}$. Then

$$\mathfrak{R}\models\forall x(x<u\to\theta)[d/u, b/y, \underset{\sim}{a}].$$

Thus $\mathfrak{R}\models\Phi_2(\theta)[\underset{\sim}{a}]$.

Now suppose $I(\bar{\theta})[\underset{\sim}{a}]$ is enumerated in increasing order and bounded above. Let $s$ be the least upper bound of $I(\bar{\theta})[\underset{\sim}{a}]$). Then for every real number $r$ which is less than $s$, there is a member $b$ of $I(\bar{\theta}[\underset{\sim}{a}])$ such that the $\{c\in I(\bar{\theta}[\underset{\sim}{a}])|\ b<c\} = I_b$ is infinite. For each $c\in I_b$ and each $i$ and $j$, $\mathfrak{R}\models p_i = 0[b/x, c/y, \underset{\sim}{a}]$ and $\mathfrak{R}\models q_j>0[b/x, c/y, \underset{\sim}{a}]$. Then $\mathfrak{R}\models\forall y(p_i = 0)[b/x, \underset{\sim}{a}]$ and there exist real numbers $d_j$ such that $b<d_j<s$ and



$$\mathfrak{R}\models\forall y(v<y\wedge y<w\to q_j>0)[s/w, b/x, d_j/v, \underset{\sim}{a}].$$

Let $d = \max\{d_j|\ 1\leqslant i\leqslant k\}$. Then

$$\mathfrak{R}\models\forall y(v<y\wedge y<w\to\theta)[s/w, b/x, d/v, \underset{\sim}{a}].$$

Therefore $\mathfrak{R}\models\Phi_3(\theta)[\underset{\sim}{a}]$.

Once more, let $\theta$ be a conjunction of polynomial equalities or inequalities. Let $u$, $v$, and $w$ be variables which do not appear in $\theta$. Let $\Psi_i^n(\theta)$ for $1\leqslant i\leqslant 4$, be the following first order formulas:

$\Psi_1^n(\theta)$ is $\exists v\forall y[x<x_2<\ldots<x_n\to x_n<v\wedge(v<y\to\theta)]$,

$\Psi_2^n(\theta)$ is $\exists u\forall x[x_2<\ldots<x_n<y\to u<x_2\wedge(x<u\to\theta)]$,

$\Psi_3^n(\theta)$ is $\exists v\forall y\big[x<w\wedge x_2<w\wedge\ldots\wedge x_n<w\wedge \wedge(x<x_2<\ldots<x_n\to x_n<v<w\wedge(v<y<w\to\theta))\big]$,

and

$\Psi_4^n(\theta)$ is $\exists u\forall x\big[w<x_2\wedge\ldots\wedge w<x_n\wedge w<y\wedge \wedge(x_2<\ldots<x_n<y\to w<u<x_2\wedge(w<x<u\to\theta))\big]$.

LEMMA a. *Let $\mathfrak{A}$ be a model of ARCF; let $\theta$ be a conjunction of polynomial equalities or inequalities in which the variables $u$, $v$, and $w$ do not appear, and, for $n\geqslant 2$, let $\bar{\theta}^n$ be the formula $x<x_2<\ldots<x_n<y\to\theta$. If $\mathfrak{A}\models Q^nxx_2\ldots x_n\Psi_1^n(\theta)[\underset{\sim}{a}]$ and $I(\Psi_1^n(\theta)[\underset{\sim}{a}])$ can be taken to be a set unbounded from above and enumerated in increasing order, then $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set also unbounded from above and enumerated in increasing order; if $\mathfrak{A}\models Q^nx_2\ldots x_ny\Psi_2^n(\theta)[\underset{\sim}{a}]$ and $I(\Psi_2^n(\theta)[\underset{\sim}{a}])$ can be taken to be a set unbounded from below and enumerated in decreasing order, then $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set also unbounded from below and enumerated in decreasing order; if there is an element $c$ of the domain of $\mathfrak{A}$ such that $\mathfrak{A}\models Qxx_2\ldots x_n\Psi_3^n(\theta)[c/w, \underset{\sim}{a}]$ and $I(\Psi_3^n(\theta)[c/w, \underset{\sim}{a}])$ can be taken to a set enumerated in increasing order and bounded above (by $c$), then $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set also enumerated in increasing order and bounded above (by $c$); if there is an element $c$ in the domain of $\mathfrak{A}$ such that $\mathfrak{A}\models Q^nx_2\ldots x_ny\Psi_4^n(\theta)[c/w, \underset{\sim}{a}]$ and $I(\Psi_4^n(\theta)[c/w, \underset{\sim}{a}])$ can be taken to be a set enumerated in decreasing order and bounded below (by $c$), then $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}]$ can be chosen to be a set also enumerated in decreasing order and bounded below (by $c$).*

Proof. For example, consider the cases involving $\Psi_1^n(\theta)$ and $\Psi_4^n(\theta)$; the cases involving $\Psi_2^n(\theta)$ and $\Psi_3^n(\theta)$ are similar.

Suppose $\mathfrak{A}\models Q^nxx_2\ldots x_n\Psi_1^n(\theta)[\underset{\sim}{a}]$ and $I = I(\Psi^n(\theta)[\underset{\sim}{a}])$ is unbounded from above and enumerated in increasing order. The formula $\Psi_1^n(\theta)$ is used to choose sequences $\langle v_i\rangle_{i=1}^{\infty}$ and $\langle a_i\rangle_{i=1}^{\infty}$, the sequence $\langle a_i\rangle_{i=1}^{\infty}$ being a subsequence of the enumeration of $I$: First arbitrarily pick $a_1<\ldots<a_n\in I$. Then, assuming $a_{n+k}(k\geqslant 0)$ has been chosen, for each subsequence $s=\langle c_i\rangle_{i=1}^{n}$ of length $n$ of $\langle a_i\rangle_{i=1}^{n+k}$, pick $v_s$ so that

$$\mathfrak{A}\models\forall y[x<x_2<\ldots<x_n\to x_n<v\wedge(v<y\to\theta)][c_1/x,c_2/x_2,\ldots,c_n/x_n,v_s/v,\underset{\sim}{a}].$$

The formula $\Psi_1^n(\theta)$ asserts that such a $v_s$ exists. Let $v_{k+1}$ be the largest such $v_s$ and pick $a_{n+k+1}$ such that $a_{n+k+1}\in I$ and $a_{n+k+1}>\max\{v_{k+1},1+a_{n+k}\}$. Note that $\langle a_i\rangle_{i=1}^{\infty}$ is an increasing sequence which is not bounded above. Let $\langle c_i\rangle_{i=1}^{n+1}$ be a subsequence of $\langle a_i\rangle_{i=1}^{\infty}$ with $c_n=a_j$ $(j\geqslant n)$ and $s=\langle c_i\rangle_{i=1}^{n}$. Then $c_{n+1}\geqslant a_{j+1}>v_{j+1}\geqslant v_s$ so that

$$\mathfrak{A}\models\theta[c_1/x,c_2/x_2,\ldots,c_n/x_n,c_{n+1}/y,\underset{\sim}{a}];$$

thus $\mathfrak{A}\models Qxx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$.

Suppose $\mathfrak{A}\models Q^nx_2\ldots x_ny\Psi_4^n(\theta)[c/w,\underset{\sim}{a}]$ and $I=I(\Psi_4^n(\theta)[c/w,\underset{\sim}{a}])$ is enumerated in decreasing order and bounded below by $c$. The formula $\Psi_4^n(\theta)$ is used to choose sequences $\langle u_i\rangle_{i=1}^{\infty}$ and $\langle a_i\rangle_{i=1}^{\infty}$, the sequence $\langle a_i\rangle_{i=1}^{\infty}$ being a subsequence of the enumeration of $I$: First arbitrarily pick $a_1>\ldots>a_n\in I$. Then, assuming $a_{n+k}(k\geqslant 0)$ has been chosen, for each subsequence $s=\langle c_i\rangle_{i=1}^{n}$ of length $n$ of $\langle a_i\rangle_{i=1}^{n+k}$, pick $u_s$ so that

$$\begin{aligned}\mathfrak{A}\models\forall x\big[&w<x_2\wedge\ldots\wedge w<x_n\wedge w<y\wedge\\&\wedge(x_2<\ldots<x_n<y\to w<u<x_2\wedge(w<x<u\to\theta))\big]\\&[c_n/x_2,\ldots,c_2/x_n,c_1/y,c/w,u_s/u,\underset{\sim}{a}].\end{aligned}$$

Let $u_{k+1}$ be the smallest such $u_s$ and pick $a_{n+k+1}$ such that $a_{n+k+1}\in I$ and $a_{n+k+1}<u_{k+1}$. Note that $\langle a_i\rangle_{i=1}^{\infty}$ is a decreasing sequence bounded below by $c$. Let $\langle c_i\rangle_{i=1}^{n+1}$ be a subsequence of $\langle a_i\rangle_{i=1}^{\infty}$ with $c_n=a_j(j\geqslant n)$ and $s=\langle c_i\rangle_{i=1}^{n}$. Then $c<c_{n+1}\leqslant a_{j+1}<u_{j+1}\leqslant u_s$ so that

$$\mathfrak{A}\models\theta[c_{n+1}/x,c_n/x_2,\ldots,c_2/x_n,c_1/y,\underset{\sim}{a}];$$

thus $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}[\underset{\sim}{a}]$.

Lemma b. *Let $\theta$ and $\bar{\theta}^n$ be as in the previous lemma and suppose*

$$\mathfrak{R}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}].$$

*If $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set unbounded from above and enumerated in increasing order, then $\mathfrak{R}\models Q^nxx_2\ldots x_n\Psi_1^n(\theta)[\underset{\sim}{a}]$ and $I(\Psi_1^n(\theta)[\underset{\sim}{a}])$ can be taken to a set also unbounded from above and enumerated in increasing order; if $I(\bar{\theta}^n)[\underset{\sim}{a}])$ can be chosen to a set unbounded from below and enumerated in decreasing order, then*

$$\mathfrak{R}\models Q^nx_2\ldots x_ny\Psi_2^n(\theta)[\underset{\sim}{a}]$$

*and $I(\Psi_2^n(\theta)[\underset{\sim}{a}])$ can be taken to be a set also unbounded from below and enumerated in decreasing order; if $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set enumerated in increasing order and bounded above, then there is a real number $c$ such that $\mathfrak{R}\models Qxx_2\ldots x_n\Psi_3^n(\theta)[c/w,\underset{\sim}{a}]$ and $I(\Psi_3^n(\theta)[c/w,\underset{\sim}{a}]$ can be taken to be a set also enumerated in increasing order and bounded above; if $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be chosen to be a set enumerated in decreasing order and bounded below, then there is a real number $c$ such that $\mathfrak{R}\models Qx_2\ldots x_ny\Psi_4^n(\theta)[c/w,\underset{\sim}{a}]$ and $I(\Psi_4^n(\theta)[c/w,\underset{\sim}{a}])$ can be taken for set also enumerated in decreasing order and bounded below.*



Proof. For example, consider the cases involving $\Psi_2^n(\theta)$ and $\Psi_3^n(\theta)$; the cases involving $\Psi_1^n(\theta)$ and $\Psi_4^n(\theta)$ are similar. Let $\theta$ be $p_1=0\wedge\ldots\wedge p_h=0\wedge q_1>0\wedge\ldots\wedge q_k>0$ where the $p_i$ and $q_j$ are polynomials in several variables with integer coefficients.

Suppose $I(\bar{\theta}^n[\underset{\sim}{a}])=I$ is unbounded from below and enumerated in decreasing order. Let $c_1<\ldots<c_n\in I$. Then the set $\{c\in I|\ c<c_1\}=I_1$ is infinite and unbounded from below. For each $c\in I_1$ and each $i$ and $j$,

$$\mathfrak{R}\models p_i=0[c/x,c_1/x_2,\ldots,c_n/y,\underset{\sim}{a}]\quad\text{and}\quad\mathfrak{R}\models q_j>0[c/x,c_1/x_2,\ldots,c_n/y,\underset{\sim}{a}].$$

Then $\mathfrak{R}\models\forall x(p_i=0)[c_1/x_2,\ldots,c_n/y,\underset{\sim}{a}]$ and there exist real numbers $d_j$ such that $d_j<c_1$ and

$$\mathfrak{R}\models\forall x(x<u\to q_j>0)[d_j/u,c_1/x_2,\ldots,c_n/y,\underset{\sim}{a}].$$

Let $d=\min\{d_j|\ 1\leqslant j\leqslant k\}$. Then

$$\mathfrak{R}\models\forall x(x<u\to\theta)[d/u,c_1/x_2,\ldots,c_n/y,\underset{\sim}{a}].$$

Thus $\mathfrak{R}\models Qx_2\ldots x_ny\Psi_2^n(\theta)[a]$ and $I(\Psi_2^n(\theta)[\underset{\sim}{a}])$ can be taken to be $I=I(\bar{\theta}^n[a])$.

Now suppose $I(\bar{\theta}^n[a])=I$ is enumerated in increasing order and bounded above. Let $c$ be the least upper bound of $I$ and let $a_1<\ldots<a_n\in I$. Then the set $\{a\in I|\ a>a_n\}=I_1$ is infinite. For each $a\in I_1$ and each $i$ and $j$,

$$\mathfrak{R}\models p_i=0[a_1/x,\ldots,a_n/x_n,a/y,\underset{\sim}{a}]\quad\text{and}\quad\mathfrak{R}\models q_j>0[a_1/x,\ldots,a_n/x_n,a/y,\underset{\sim}{a}].$$

Then $\mathfrak{R}\models\forall x(p_i=0)[a_1/x,\ldots,a_n/x_n,\underset{\sim}{a}]$ and there exist real numbers $d_j$ such that $a_n<d_j<c$ and

$$\mathfrak{R}\models\forall y(v<y<w\to q_j>0)[c/w,d_j/v,a_1/x,\ldots,a_n/x_n,\underset{\sim}{a}].$$

Let $d=\max\{d_j|\ 1\leqslant i\leqslant k\}$. Then

$$\mathfrak{R}\models\forall y(v<y<w\to\theta)[c/w,d/v,a_1/x,\ldots,a_n/x_n,\underset{\sim}{a}].$$

Therefore $\mathfrak{R}\models Q^nxx_2\ldots x_n\Psi_3^n(\theta)[c/w,a]$ and $I(\Psi_3^n(\theta)[c/w,\underset{\sim}{a}])$ can be taken to be $I=I(\bar{\theta}^n[\underset{\sim}{a}])$.

Lemma. *Let $\theta$, $\bar{\theta}^1$, and $\bar{\theta}^n$ (for $n\geqslant 2$) be as in Lemmas A and a, and let $\mathfrak{A}$ be a model of ARCF. For $n\geqslant 1$ there are quantifier free formulas $\psi_i^n(\theta)$ $(1\leqslant i\leqslant 4)$ whose free variables form a subset of those of $Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n$ ($Q^2xy\bar{\theta}^1$ for $n=1$) such that $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be taken to be a set unbounded from above and enumerated in increasing order iff $\mathfrak{A}\models\psi_1^n(\theta)[\underset{\sim}{a}]$; $\mathfrak{A}\models Q^{n+1}xx_2\ldots x_ny\bar{\theta}^n[\underset{\sim}{a}]$ and*

*$I(\bar{\theta}^n[\underset{\sim}{a}])$ can be taken to be a set unbounded from below and enumerated in decreasing order iff $\mathfrak{A} \models \psi_2^n(\theta)[\underset{\sim}{a}]$; $\mathfrak{A} \models Q^{n+1}xx_2 \ldots x_n y\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be taken to be a set enumerated in increasing order and bounded above iff $\mathfrak{A} \models \psi_3^n(\theta)[\underset{\sim}{a}]$; and $\mathfrak{A} \models Q^{n+1}xx_2 \ldots x_n y\bar{\theta}^n[\underset{\sim}{a}]$ and $I(\bar{\theta}^n[\underset{\sim}{a}])$ can be taken to be a set enumerated in decreasing order and bounded below iff $\mathfrak{A} \models \psi_4^n(\theta)[\underset{\sim}{a}]$.*

Proof. Let $\mathfrak{A}$ be a model of ARCF. It can be assumed that $\mathfrak{A}$ is a substructure of $\mathfrak{R}$, the field of all real numbers ([8], p. 241). Proceed by induction on $n$: For $n = 1$ consider the case when $i = 1$; the cases when $i = 2, 3$, and 4 are similar. By Lemma A, if $\mathfrak{A} \models \Phi_1(\theta)[\underset{\sim}{a}]$, then $\mathfrak{A} \models Qxy\bar{\theta}^1[\underset{\sim}{a}]$ and $I(\bar{\theta}^1[\underset{\sim}{a}])$ can be taken to be a set unbounded from above and enumerated in increasing order. If $\mathfrak{A} \models Qxy\bar{\theta}^1[\underset{\sim}{a}]$ and $I(\bar{\theta}^1[\underset{\sim}{a}])$ is a set unbounded from above and enumerated in increasing order, then, since $\mathfrak{A}$ is a substructure of $\mathfrak{R}$, $\mathfrak{R} \models Qxy\bar{\theta}^1[\underset{\sim}{a}]$ and $I(\bar{\theta}^1[\underset{\sim}{a}])$ remains a set unbounded from above and enumerated in increasing order. By Lemma B, $\mathfrak{R} \models \Phi_1(\theta)[\underset{\sim}{a}]$. By Tarski's theorem, $\Phi_1(\theta)[\underset{\sim}{a}]$ is equivalent, in all models of RCF, to a quantifier free formula $\psi_1^1(\theta)$. Hence $\mathfrak{R} \models \psi_1^1(\theta)[\underset{\sim}{a}]$, and so $\mathfrak{A} \models \psi_1^1(\theta)[\underset{\sim}{a}]$.

Now assume that the lemma holds for all $n<k$; consider the case when $i = 3$; the cases when $i = 1, 2$, and 4 are similar. By Lemma a, if there is an element $c$ of the domain of $\mathfrak{A}$ such that $\mathfrak{A} \models Q^k xx_2 \ldots x_k \Psi_3^k(\theta)\,[c/w, \underset{\sim}{a}]$ and $I(\Psi_3^k(\theta)[c/w, \underset{\sim}{a}])$ is a set enumerated in increasing order and bounded above, then $\mathfrak{A} \models Q^{k+1}xx_2 \ldots x_k y\bar{\theta}^k[\underset{\sim}{a}]$ and $I(\bar{\theta}^k[\underset{\sim}{a}])$ can be taken to be a set also enumerated in increasing order and bounded above. If $\mathfrak{A} \models Q^{k+1}xx_2 \ldots x_k y\bar{\theta}^k[\underset{\sim}{a}]$ and $I(\bar{\theta}^k[\underset{\sim}{a}])$ is a set enumerated in increasing order and bounded above, then $\mathfrak{R} \models Q^{k+1}xx_2 \ldots x_k y\bar{\theta}^k[\underset{\sim}{a}]$ and $I(\bar{\theta}^k[\underset{\sim}{a}])$ remains a set enumerated in increasing order and bounded above. By Lemma b, there is a real number $r$ such that $\mathfrak{R} \models Q^k xx_2 \ldots x_k \Psi_3^k(\theta)[r/w, \underset{\sim}{a}]$ and $I(\Psi_3^k(\theta)[r/w, \underset{\sim}{a}])$ can be taken to be a set enumerated in increasing order and bounded above. By Tarski's theorem $\Psi_3^k(\theta)$ is equivalent, in all models of RCF, to a quantifier free formula $\psi$. By the distributive law for $Q^k$ over disjunction, there are conjunctions $\varphi_j (1 \leqslant j \leqslant h)$ of polynomial equalities or inequalities such that $Q^k xx_2 \ldots x_k\psi$ is equivalent, in all models of ARCF, to

$$\bigvee_{1 \leqslant j \leqslant h} Q^k xx_2 \ldots x_k(x<x_2<\ldots<x_k \to \varphi_j)\,.$$

Hence, for some $j$,

$$\mathfrak{R} \models Q^k xx_2 \ldots x_k(x<x_2<\ldots<x_k \to \varphi_j)$$

and furthermore, since $I([x<x_2<\ldots<x_k \to \varphi_j][r/w, \underset{\sim}{a}]) = I$ can be taken to be a subset of $I(\psi[r/w, \underset{\sim}{a}]) = I(\Psi_3^k(\theta)[r/w, \underset{\sim}{a}])$, $I$ is a set enumerated in increasing order and bounded above. By the induction hypothesis, there is a quantifier free formula $\sigma$ such that $\mathfrak{R} \models Q^k xx_2 \ldots x_k \psi[r/w, \underset{\sim}{a}]$ and $I(\psi[r/w, \underset{\sim}{a}])$ is a set enumerated in increasing order and bounded above iff for some $j$,

$$\mathfrak{R} \models Q^k xx_2 \ldots x_k(x<x_2<\ldots<x_k \to \varphi_j)[r/w, \underset{\sim}{a}]$$

and $I$ is a set enumerated in increasing order and bounded above iff $\mathfrak{R} \models \sigma[r/w, \underset{\sim}{a}]$. By Tarski's theorem there is a quantifier free formula $\psi_3^k(\theta)$ equivalent, in all models of RCF, to $\exists w\sigma$. Therefore $\mathfrak{R} \models \psi_3^k(\theta)[\underset{\sim}{a}]$. Then $\mathfrak{A} \models \psi_3^k(\theta)[\underset{\sim}{a}]$; $\mathfrak{A} \models \exists w\sigma[\underset{\sim}{a}]$; for some $c$ in the domain of $\mathfrak{A}$, $\mathfrak{A} \models \sigma[c/w, \underset{\sim}{a}]$. Then, for some $j$,

$$\mathfrak{A} \models Q^k xx_2 \ldots x_k(x<x_2<\ldots<x_k \to \varphi_j)$$

and $I$ is a set enumerated in increasing order and bounded above; so that $\mathfrak{A} \models Q^k xx_2 \ldots x_k \psi[c/w, \underset{\sim}{a}]$ and $I(\psi[c/w, \underset{\sim}{a}])$ is enumerated in increasing order and bounded above.



THEOREM. *For each $n \geqslant 2$ every $Q^n$-formula $\varphi$ of the language $L$ is equivalent, in all models of* ARCF, *to a quantifier free formula $\psi$ whose free variables form a subset of those of $\varphi$.*

Proof. Proceed by induction on the formation of the formula $\varphi$. If $\varphi$ is atomic or of one of the forms $\sigma \wedge \tau$ or $\neg\sigma$, then it follows easily from the induction hypothesis that $\varphi$ is equivalent to a quantifier free formula. If $\varphi$ is of the form $\exists x\sigma$, then the induction hypothesis together with Tarski's theorem assures us that such a $\psi$ exists. Consider the case when $\varphi$ is of the form $Q^n x_1 \ldots x_n\sigma$: By the induction hypothesis $\sigma$ can be taken to be a quantifier free formula, and by the distributive law for $Q^n$ over disjunction, only those $\sigma$ need be considered which are of the form $x_1<\ldots<x_n \to \theta$ where $\theta$ is a conjunction of polynomial equalities or inequalities. Let $\mathfrak{A}$ be a model of ARCF. If $\mathfrak{A} \models \bigvee_{1 \leqslant i \leqslant 4} \psi_i^{n-1}(\theta)[\underset{\sim}{a}]$, then, by the lemma, $\mathfrak{A} \models Q^n x_1 \ldots x_n(x_1<\ldots<x_n \to \theta)[\underset{\sim}{a}]$. If $\mathfrak{A} \models Q^n x_1 \ldots x_n(x_1<\ldots<x_n \to \theta)[\underset{\sim}{a}]$, then, since every infinite subset of the domain of $\mathfrak{A}$ contains either an increasing or a decreasing infinite sequence which is either bounded or not bounded, by the lemma, $\mathfrak{A} \models \bigvee_{1 \leqslant i \leqslant 4} \psi_i^{n-1}(\theta)[\underset{\sim}{a}]$.

COROLLARY. *Every $Q^{<\omega}$-formula $\varphi$ of the language $L$ is equivalent, in all models of* ARCF, *to a quantifier free formula $\psi$ whose free variables form a subset of those of $\varphi$.*

It follows from the theorem and its corollary that for $n \geqslant 2$, the $Q^n$-theory as well as the $Q^{<\omega}$-theory of Archimedean real closed fields is both complete and model complete. From the theorem and its corollary, together with the constructive nature of their proofs, it follows that for $n \geqslant 2$, the $Q^n$-theory as well as the $Q^{<\omega}$-theory of Archimedean real closed fields is decidable.

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