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Accepté par la Rédaction le 4. 10. 1976

Hereditarily indecomposable tree-like continua

by

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Abstract. In this paper it is shown that there is an uncountable collection of mutually exclusive hereditarily indecomposable tree-like continua in the plane such that if M is a compact metric continuum then M cannot be mapped onto every member of the collection.

1. Introduction. In 1951 Bing [2] asked if each non-degenerate hereditarily indecomposable plane continuum which does not separate the plane is a pseudo-arc. In an abstract in 1951 Anderson [1] stated that there exist hereditarily indecomposable tree-like continua which are not homeomorphic to the pseudo-arc. It is the purpose of this paper to demonstrate that there is an hereditarily indecomposable tree-like continuum which is not a continuous image of the pseudo-arc.

In [5] we demonstrated the existence of a collection G of atriodic tree-like continua with the property that if M is a compact metric continuum then M cannot be mapped onto every member of G. In this paper we construct a collection H of mutually exclusive hereditarily indecomposable tree-like continua in the plane such that if g is in G then g is a continuous image of some member of H. Thus, H also has the property that if M is a compact metric continuum then M cannot be mapped onto every member of H.

For notation (including T, f, and g) and conventions used in this paper the reader is referred to [3], [4], and [5].

2. We present in this section the main working lemma of the paper.

LEMMA. *Suppose k is a mapping of T onto T such that* (1) *if t is a point of $\left\{O, A, B, C, \frac{A}{2}\right\}$ then $k(t) = t$ and $k^{-1}(t) = \{t\}$ and* (2) *if t is a point of $T-\left\{O, \frac{A}{2}\right\}$ and α is the component of $T-\left\{O, \frac{A}{2}\right\}$, containing t then $k(t)$ is in α. Then there is a mapping h of T onto T such that* (1) *if t is a point of $\left\{O, A, B, C, \frac{A}{2}\right\}$ then*

$h(t) = t$ *and* $h^{-1}(t) = \{t\}$, (2) *if* t *is a point of* $T-\left\{O, \frac{A}{2}\right\}$ *and* α *is the component of* $T-\left\{O, \frac{A}{2}\right\}$ *containing* t *then* $h(t)$ *is in* α, *and* (3) $kf = fh$.

Proof. Let $h(O) = O$, $h\left(\frac{B}{2}\right) = \frac{B}{2}$, and $h\left(\frac{A}{2}\right) = \frac{A}{2}$. Suppose t is in $T-\left\{O, \frac{A}{2}, \frac{B}{2}\right\}$ and L is the component of $T-\left\{O, \frac{A}{2}, \frac{B}{2}\right\}$ containing t. Then, let $h(t) = L \cap f^{-1}(kf(t))$.

It is easy to see that h is a function and $kf = fh$.

Let $S = \left\{O, \frac{A}{4}, \frac{A}{2}, \frac{3A}{4}, \frac{B}{3}, \frac{B}{2}, \frac{2B}{3}, \frac{C}{2}\right\}$. Note that if t is a point of $S \cup \{A, B, C\}$ then $h(t) = t$ and $h^{-1}(t) = \{t\}$. Moreover, if t is in $T-S$ and α is the component of $T-S$ containing t then $h(t)$ is in α. Conclusions (1) and (2) are consequences of these observations. To complete the proof of the lemma we must show that h is continuous.

Suppose t is a point of $T-S$ and U is an open set containing $h(t)$. There is an open set V containing $h(t)$ such that V is a subset of U, $f(V)$ is open, and if K is the component of $T-S$ containing t then V is a subset of K. In fact, $V = K \cap (f^{-1}f(V))$. Since kf is continuous, there is a connected open subset W of $T-S$ containing t such that $kf(W)$ is a subset of $f(V)$. Then $h(W)$ is a subset of V. Therefore h is continuous at each point of $T-S$.

Suppose t is a point of $\left\{O, \frac{A}{2}\right\}$ and U is an open set containing $h(t)$. There is an open set V containing $h(t)$ such that V is a subset of U, $f(V)$ is open, and if K is the component of $(T-S) \cup \{t\}$ containing t then V is a subset of K. In fact, $V = K \cap (f^{-1}f(V))$. Since kf is continuous there is a connected open subset W of $(T-S) \cup \{t\}$ containing t such that $kf(W)$ is a subset of $f(V)$. Then, $h(W)$ is a subset of V. Therefore, h is continuous at each point of $\left\{O, \frac{A}{2}\right\}$.

Suppose t is a point of $\left\{\frac{A}{4}, \frac{3A}{4}, \frac{B}{3}, \frac{2B}{3}, \frac{C}{2}, \frac{B}{2}\right\}$ and U is open set containing $h(t)$. There is an open set V containing $h(t)$ such that V is a subset of U and if K is the component of $(T-S) \cup \{t\}$ containing t then V is a subset of K. In fact, $V = K \cap (f^{-1}f(V))$ but in this case $f(V)$ is not open. However, there is a connected open subset W of $(T-S) \cup \{t\}$ containing t such that $kf(W)$ is a subset of $f(V)$. Then, $h(W)$ is a subset of V. Therefore, h is continuous at each point of $\left\{\frac{A}{4}, \frac{3A}{4}, \frac{B}{3}, \frac{2B}{3}, \frac{C}{2}, \frac{B}{2}\right\}$.

This completes the proof.

Remark. If k is a mapping satisfying the hypothesis of the preceding lemma, there is a mapping h satisfying the conclusion of the lemma with condition (3) replaced by $kg = gh$.

3. Main theorems.

THEOREM 1. *Suppose M is the inverse limit of the inverse limit sequence $\{T_i, f_i^j\}$ where for each i $T_i = T$ and f_i^{i+1} is in $\{f, g\}$. Then there exist an hereditarily indecomposable simple triod-like continuum M' and a mapping of M' onto M.*

Proof. We construct M' as the inverse limit of an inverse limit sequence $\{T_i, k_i^j\}$ as follows. For each i let $T_i = T$. If $i = 2j$, let $k_i^{i+1} = f_j^{j+1}$. If $i = 2j-1$, let k_i^{i+1} be a sufficiently crooked map of T_{i+1} onto T_i which also satisfies hypothesis (1) and (2) of the lemma of Section 2. By "sufficiently crooked" here we mean that each non-degenerate proper subcontinuum of the resulting inverse limit should be a pseudo-arc.

That each proper subcontinuum of M' is a pseudo-arc may be seen by observing that if K is a proper subcontinuum of M' there is a positive integer N such that if $i \geqslant N$ then neither O nor $\frac{A}{2}$ is in $\pi_i(K)$. The proof of this is very much like a similar part of the proof of Theorem 1 of [5]. Thus, K is homeomorphic to an inverse limit on intervals with crooked maps and is therefore a pseudo-arc.

Now, we use the lemma of Section 2 to construct a mapping of M' onto M. Let h_1 be the identity mapping on T. Note that $h_1 k_1^2$ satisfies the hypothesis on the mapping k of the lemma. Since f_1^2 is in $\{f, g\}$ and k_2^3 is f_1^2, there is a mapping h_2 of T_3 onto T_2 such that $h_1 k_1^3 = f_1^2 h_2$. Then $h_2 k_3^4$ satisfies the hypothesis on the mapping k of the lemma. This yields a mapping h_3 of T_5 onto T_3 such that $h_2 k_3^5 = f_2^3 h_3$. Continuing this process we obtain the following inverse mapping system:

$$\begin{tikzcd}
T_1 \arrow[d, "h_1"] & T_2 \arrow[l, "k_1^2"'] & T_3 \arrow[l, "f_1^2"'] \arrow[d, "h_2"] & \cdots \arrow[l] & T_{2j-1} \arrow[l] \arrow[d, "h_j"] & T_{2j} \arrow[l, "k_{2j-1}^{2j}"'] & T_{2j+1} \arrow[l, "f_j^{j+1}"'] \arrow[d, "h_{j+1}"] & \cdots \arrow[l] \\
T_1 & & T_2 \arrow[ll, "f_1^2"'] & \cdots \arrow[l] & T_j \arrow[l] & & T_{j+1} \arrow[ll, "f_j^{j+1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Since the diagrams in this system are commutative, the sequence of mappings $h_1, h_2, \ldots$ induce a mapping of a continuum homeomorphic to M' onto M. Thus there is a mapping of M' onto M.

THEOREM 2. *There exists in the plane an uncountable collection H of mutually exclusive hereditarily indecomposable tree-like continua such that if M is a member of G then there exists a member of H which can be mapped onto M.*

Proof. That the theorem is true without the requirement that the members of H be embedded in the plane is a consequence of Theorem 1. To carry out the plane embedding one need observe that the insertion of the crooked maps k_{2j-1}^{2j} in the inverse limit sequences involves changing the construction in Theorem 4 of [4] by alternately inserting appropriately crooked simple tree-chains.

THEOREM 3. *If X is a compact metric continuum and H_0 is a subcollection of H such that X can be mapped onto every member of H_0, then H_0 is countable.*

Proof. This is a consequence of Theorem 2 of this paper and Theorem 3 of [5].

THEOREM 4. *Uncountably many members of H are not a continuous image of the pseudo-arc.*

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Accepté par la Rédaction le 5. 10. 1976

The theory of Archimedean real closed fields in logics with Ramsey quantifiers

by

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Abstract. The theory of Archimedean real closed fields is shown to be complete, decidable, and model complete in a class of logics, due to Malitz and Magidor, which extend the logic with the cardinal quantifier, "There exist infinitely many...".

It is assumed that the reader is familiar with the model theory of first order logic as set forth in the book [1] of Chang and Keisler. In particular, the reader should be acquainted with the definitions of completeness, model completeness, and decidability in reference to first order logic and to certain of its extensions described below.

It is well known that there is no first order theory of Archimedean real closed fields as distinct from the theory of real closed fields. In fact, using the method of elimination of quantifiers [6] or [2], it follows that the first order theory of real closed fields is complete, decidable, and model complete. The situation is not altered when a new cardinal quantifier Q, with the $\aleph_0$-interpretation [5], is added to the logic: Add a formation rule to those of first order logic; if φ is a formula, then so is $Qx\varphi$; and $\mathfrak{A}$ is a model of $Qx\varphi$ just in case there are infinitely many elements x in the domain of $\mathfrak{A}$ which satisfy φ. In the case of real closed fields, the method of elimination of quantifiers can be extended to the cardinal quantifier [3] or [7], showing that the theory of these fields in the extended logic is complete, decidable, and model complete. Thus, as in the case of first order logic, there is no theory of Archimedean real closed fields in the logic with the quantifier "There exist infinitely many..." which is distinct from the theory of all real closed fields. In contrast with the above, the situation is different in logics, described below, due to Malitz and Magidor [4], which are generalizations of the logic with the cardinal quantifier. These logics have enough expressive power to distinguish Archimedean from non-Archimedean fields. It will be shown by the method of elimination of quantifiers that the theory of Archimedean real closed fields in these logics of Malitz and Magidor is complete, decidable, and model complete.