# Adding a random or a Cohen real: topological consequences and the effect on Martin's axiom

by

**J. Roitman** (Wellesley, Mass.)

**Abstract.** Let $M$ be a model of set theory, $B$ either the Cohen real or random real algebra. Then in $M^B$ the product of ccc spaces need not be ccc, and there is a space all of whose finite products are $L$-spaces; thus $\mathrm{MA}+\neg\,\mathrm{CH}$ fails in $M^B$. On the other hand, if $M \models \mathrm{MA}_{\Sigma\text{-linked}}$, so does $M^B$.

**§ 0. Introduction.** This paper studies the effect that adding a Cohen or a random real to a model of set theory has on certain topological and combinatorial properties. In particular, it is shown that $\mathrm{MA} = \neg\,\mathrm{CH}$ fails, and that a weak form of it is preserved.

Since the main results are proved using forcing techniques, and since many of them are essentially of topological interest, I have included a section which states explicitly the forcing facts that will be used, and tries to give enough of an overview so that the skeptical topologist with little background in forcing has at least some chance of following the arguments. This is Section 3. The reader who wants a better reference on forcing is referred to [6] and the forthcoming [1]; the set theorist curious about the topology is referred to [7] and [13].

Thanks are due to Kenneth Kunen for much helpful and stimulating conversation, and for permission to include Theorem 1.2; thanks also to Mary Ellen Rudin for her generous hospitality in Madison, where many of these results were proved.

**§ 1. Definitions and statement of results.** Set theory means Zermelo–Fraenkel + + choice; CH is the continuum hypothesis; MA is Martin's axiom (defined in Section 5). Models of set theory are assumed transitive. An ordinal is the set of its predecessors. $|A|$ is the cardinality of $A$. $\neg$ means "not".

A real is considered to be either a function from $\omega$ into 2 or from $\omega$ into $\omega$; the choice will be clear from the context. We say, respectively, $r \in 2^\omega$ or $r \in \omega^\omega$.

All spaces are assumed Hausdorff. A space is ccc iff every pairwise disjoint family of open sets is countable. A space is *hereditarily Lindelöf* — abbreviated hL (resp. *hereditarily separable* — abbreviated hs) iff every subspace is Lindelöf (resp. separable). An $L$-space is regular hL not hs; an $S$-space is regular hs not hL. A family of countable sets is *almost disjoint* iff its pairwise intersections are finite.

Consider the following sentences:

C: The product of two ccc spaces is ccc.

L: $\exists X$ (every finite product of $X$ is an $L$-space).

Ad: If $\{A_\alpha\colon \alpha<\omega_1\}$ is an almost disjoint family, then for some $A\subset\bigcup_{\alpha<\omega_1}A_\alpha$, $|A|=|\bigcup_{\alpha<\omega_1}A_\alpha|$ and $\forall\alpha<\omega_1$ $(|A\cap A_\alpha|<\omega)$.

$\mathrm{Ad}_\omega$: If $\{A_\alpha\colon \alpha<\omega_1\}$ is an almost disjoint family subsets of $\omega$, and $B\subset\omega_1$, then for some infinite $A\subset\bigcup_{\alpha<\omega_1}A_\alpha$ $\forall\alpha<\omega_1$ $(|A\cap A_\alpha|<\omega\leftrightarrow\alpha\in B)$.

These sentences have the property that each of them is true under CH iff it is false under $\mathrm{MA}+\neg\mathrm{CH}$. Precisely:

I. (Laver–Galvin) $\mathrm{CH}\to\neg\mathrm{C}$,
(Kunen) $\mathrm{MA}+\neg\mathrm{CH}\to\mathrm{C}$.

II. (Juhasz–Hajnal) $\mathrm{CH}\to\mathrm{L}$,
(Kunen) $\mathrm{MA}+\neg\mathrm{CH}\to\neg\mathrm{L}$.

III. $\mathrm{CH}\to\neg\mathrm{Ad}$,
(Wage) $\mathrm{MA}+\neg\mathrm{CH}\to\mathrm{Ad}$.

IV. $\mathrm{CH}\to\neg\mathrm{Ad}_\omega$,
(Solovay) $\mathrm{MA}+\neg\mathrm{CH}\to\mathrm{Ad}_\omega$.

It is the fate of these sentences with which we are concerned.

Given a model $M$ of set theory, and a Boolean algebra $\boldsymbol{B}$ complete in $M$, $M^{\boldsymbol{B}}$ is the Boolean extension of $M$. (The reader who is uncomfortable with this notation is invited to substitute $M[r]$ for $M^{\boldsymbol{B}}$ below, where $r$ is respectively a Cohen or a random real, $M[r]$ the smallest model of set theory containing $M$ as a class and $r$ as a set.)

THEOREM 1.1. *Given a model $M$ of set theory, and $\boldsymbol{B}$ the Cohen or random real algebra in $M$*

(a) *$L$ is true in $M^{\boldsymbol{B}}$,*

(b) *$C$ is false in $M^{\boldsymbol{B}}$.*

The result (a) above vis-a-vis random reals is due to Kunen.

As an instant corollary we have that $\mathrm{MA}+\neg\mathrm{CH}$ is false in $M^{\boldsymbol{B}}$.

THEOREM 1.2 (Kunen). *If $M$ is a model of set theory, and $\boldsymbol{B}$ the Cohen real algebra in $M$, then* Ad *is false in $M^{\boldsymbol{B}}$.*

Lest this lull us into believing that $M^{\boldsymbol{B}}$ slavishly agrees with CH in these matters, we have

THEOREM 1.3. *Let $M$ be a model of set theory. Then*

(a) *If $B$ is either the Cohen or random real algebra in $M$, and* $\mathrm{MA}_{\Sigma\text{-linked}}$ *is true in $M$, the* $\mathrm{MA}_{\Sigma\text{-linked}}$ *is true in $M^B$ ($\Sigma$-linked will be defined in Section 5).*

(b) *If $\boldsymbol{B}$ is the Cohen real algebra in $M$, and* $\mathrm{MA}_{\Sigma\text{-centered}}$ *is true in $M$, the* $\mathrm{MA}_{\Sigma\text{-centered}}$ *is true in $M^B$ ($\Sigma$-centered will be defined in Section 5).*

(c) *In particular, under either of the above hypotheses,* $\mathrm{Ad}_\omega$ *is true in $M^B$.*



Again, (a) vis-a-vis a random real is due to Kunen.

These theorems have further topological consequences. Since a theorem of Zenor tells us that the statement $L$ is equivalent to $\exists X\forall n<\omega$ ($X^n$ and $S$-space), from 1.1 we have the existence in $M^B$ of a space all of whose finite products are $S$-spaces. The consequences of 1.3 are quite numerous, and only a few important ones are listed below. Credit is given to the discoverer of the fact from $\mathrm{MA}+\neg\mathrm{CH}$. Topological definitions and many of the proofs can be found in [11].

COROLLARY 1.4. *Let $M$, $\boldsymbol{B}$ be as in any part of* 1.3. *Then in $M^{\boldsymbol{B}}$ the following statements are true:*

(a) (Silver) $\exists$ *a $Q$-set (= an uncountable subspace of the reals with every subset a relative $G_\delta$).*

(b) (Tall) $\exists$ *a normal non-metrizable Moore space.*

(c) (Solovay) *No uncountable subset of the reals is a Luzin space (= every first category subset is countable) or Sierpiński (= every measure zero set is countable).*

(d) (Przymusiński) *The square of a $\varkappa$-Sorgenfrey line is normal, for $\varkappa<C$.*

Finally, some questions:

(1) In $M^B$ is there a Suslin tree? A Luzin space? (Note that Kunen proved that $\mathrm{MA}_{\Sigma\text{-centered}}\Rightarrow$ ($\exists$ a Luzin space $\leftrightarrow\exists$ a Suslin tree), so under the hypotheses of 1.3 these are the same question; it is well-known that if a Suslin tree exists in $M$, it does in $M^B$.)

(2) (an old chestnut) Is there an absolute example of an $L$-space? An $S$-space? Are these the same question?

**§ 2. The basic space.** In this section we give our basic construction, prove that it is canonical, and mention the basic criteria for $L$ and $S$ spaces.

Throughout this paper we will be working with subspaces of $2^\omega$, $2^{\omega_1}$, $\omega^{\omega_1}$ and $\omega^\omega$, so we recall the convention that the standard basis for the space $\beta^\varkappa$ is given by the collection of $N_\sigma$'s, where $\sigma$ is a finite function with domain $\subset\varkappa$ and range $\subset\beta$; $N_\sigma=\{f\in\beta^\varkappa\colon f\restriction\operatorname{dom}\sigma=\sigma\}$; each $N_\sigma$ clopen.

DEFINITION 2.1 (the basic construction). Let $\mathscr{F}=\{f_\alpha\colon \omega<\alpha<\omega_1\}$ be a family of functions such that $f_\alpha\colon \alpha\xrightarrow{1\text{-}1}\omega$. Note that $f_\alpha$ need not be onto. Suppose further that $r\in 2^\omega$. For $\omega<\alpha<\omega_1$, $\beta<\omega_1$, define

$$g_\alpha(\beta)\overset{!}{=}\begin{cases}r(f_\alpha(\beta)) & \text{if } \beta<\alpha,\\ 0 & \text{otherwise}\end{cases}$$

and define $X_{\mathscr{F},r}=\{g_\alpha\colon \omega<\alpha<\omega_1\}$.

$X_{\mathscr{F},r}$ — given the right $\mathscr{F}$ and $r$ — will become the $L$-space of 1.1. (a); a twist of it according to ideas of Galvin will give us 1.1 (b); and a modification of it will give us 1.2.

$X_{\mathscr{F},r}$ is quite canonical, as will be shown below. First, however, we define a dual familiar to many who work with $S$ and $L$ spaces.

DEFINITION 2.2. Let $X = \{g_\alpha: \alpha<\omega_1\} \subset 2^{\omega_1}$. The *dual* $X^*$ is the set of functions $h_\alpha$, $\alpha<\omega_1$, where $h_\alpha(\beta) = g_\beta(\alpha)$ for all $\alpha, \beta<\omega_1$.

We note that using the following characterization of hL and hs, it is easy to prove that, if $X$ is as in 2.2, $\forall n<\omega$ ($X^n$ is hL) $\leftrightarrow \forall n<\omega$ ($X^{*n}$ is hs), thus giving an alternate proof of the theorem of Zenor quoted in Section 1.

FACT 2.3. *A space $X$ is* hL (*resp.* hs) *iff $\forall$ uncountable $Y \subset X \forall$ uncountable $U$ a basic open cover of $Y$ $\exists u \in U$ such that $\{y \in Y: y \in u\}$ is uncountable* (*resp. $\exists y \in Y$ such that $\{u \in U: y \in u\}$ is uncountable*).

Fact 2.3 will be used in the proof of 1.1. a.

DEFINITION 2.4. A subset of $2^{\omega_1}$ is *left* iff it consists of a family of functions with countable support; it is *right* iff it equals some $X^*$, where $X$ is left.

PROPOSITION 2.5. *Every uncountable left set contains a subspace which is a subspace of an $X_{\mathcal{F},r}$.*

Proof. Let $X$ be an uncountable left set, $X = \{g_\alpha: \alpha<\omega_1\}$. By induction construct a sequence $\beta_\alpha \omega_1$ where support $g_\alpha \beta_\alpha$. Let $\beta_k^\alpha$: $k<\omega$ enumerate the ordinals below $\beta_\alpha$. Define $f_\alpha$: $\beta_\alpha^{1-1}\omega$ by

$$f_\alpha(\gamma) = 2n+i \leftrightarrow \beta_k^\alpha = \gamma, \quad g_\alpha(\gamma) = i \quad \text{and} \quad |\{j<k: g_\alpha(\beta_j^\alpha) = i\}| = n-1.$$

Let $r$ be the characteristic function of the even integers. By definition,

$$f_\alpha(\gamma) = f_{\alpha'}(\gamma') \Rightarrow g_\alpha(\gamma) = g_{\alpha'}(\gamma') = r(f_\alpha(\gamma)).$$

Expanding $\{f_\alpha: \alpha<\omega_1\}$ to a full set of collapsing function $\mathcal{F}$, we have $X \subset X_{\mathcal{F},r}$.

COROLLARY 2.6. *Every 0-dimensional L-space contains an L-subspace under a possibly weaker topology which is homeomorphic to a subspace of an $X_{\mathcal{F},r}$; and every S-space contains an S-subspace homeomorphic to some $X^*$, where $X$ is a subspace of some $X_{\mathcal{F},r}$.*

Proof. As is well known, every $S$-space has an $S$-subspace homeomorphic to a right space, and every 0-dimensional $L$-space has an $L$-subspace which under a possibly weaker $L$-space topology is homeomorphic to a left space.

A parenthetic comment: While $\neg$CH $\to$ every $L$-space has a 0-dimensional $L$-subspace, and CH $\to \exists$ a 0-dimensional $L$-space, it is not known whether ($\exists$ an $L$-space with no 0-dimensional $L$-subspace) is consistent.

## § 3. Some facts about forcing and a touch of combinatorics.

In this section we identify the well-known set-theoretic facts that will be used. First, the general setting.

Let $M$ be a model of set theory, and in $M$ let $\boldsymbol{B}$ be a complete Boolean algebra (this means that $M$ thinks $\boldsymbol{B}$ is a complete Boolean algebra). Then the Boolean-valued model $M^{\boldsymbol{B}}$ (not to be confused with the functions from $\boldsymbol{B}$ to $M$) is defined in $M$ (for a definition, see e.g. [5]). The elements of $M^{\boldsymbol{B}}$ are called terms, and a term is best thought of as a set of possible elements together with their probabilities, the probabilities being elements of $\boldsymbol{B}$. Smallness in $\boldsymbol{B}$ is considered a virtue: if $b \leqslant b'$,



then $b$ tells us more (the less probable an event, the more it can be distinguished from other events).

Let $\varphi$ be a sentence with parameters in $M^{\boldsymbol{B}}$. If $b \in \boldsymbol{B}$, we say that $b$ forces $\varphi$ (abb. $b \Vdash \varphi$) iff $\forall b' \leqslant b$, $b'$ not $\Vdash \neg\varphi$, and we say that $\varphi$ holds in $M^{\boldsymbol{B}}$ (abb. $M^{\boldsymbol{B}} \vDash \varphi$) iff $\forall b \in \boldsymbol{B}(b \Vdash \varphi)$ iff $\{b: b \Vdash \varphi\}$ is dense in $\boldsymbol{B}$ (where $D$ dense in $\boldsymbol{B}$ iff $\forall b \in \boldsymbol{B} \exists b' \in D$ $(b' \leqslant b)$). This is neither as circular nor as vague as it sounds, although it is beyond the scope of this paper to go beyond this formalism; checking that $\varphi$ holds in $M^{\boldsymbol{B}}$ usually reduces to checking how big a chunk of a particular function can be decided by a single condition.

$M^{\boldsymbol{B}}$ is also a model of ZFC — that is, if $\varphi$ is a theorem of ZFC, $M^{\boldsymbol{B}} \vDash \varphi$ — and all the usual laws of logic hold. While $M^{\boldsymbol{B}}$ is itself a definable subclass in $M$, $M^{\boldsymbol{B}}$ has terms for all of $M$'s elements, and $M$ is definably an inner model of $M^{\boldsymbol{B}}$. Thus in either model we can talk about the other one.

We will not observe the usual typographical distinction between elements of $M$ and elements of $M^{\boldsymbol{B}}$; thus if $A \in M$, we will use the same symbol for its term in $M^{\boldsymbol{B}}$. We note that $M^{\boldsymbol{B}}$ adds no new finite sets of ordinals, and that given a set in $M$ there are certain things that are true of it in $M$ iff true of it in $M^{\boldsymbol{B}}$ — most importantly for our discussion, the properties of being an ordinal, being finite (hence being infinite), being an almost disjoint family, being pairwise disjoint, and, for the particular algebras we will be using, being countable and hence being uncountable. We then have the following principle: if $\varphi$ is a hypothesis in which the only properties ascribed to elements of $M$ are as above, and $\varphi \Rightarrow M^{\boldsymbol{B}} \vDash \psi$, then $M^{\boldsymbol{B}} \vDash \varphi \to \psi$.

A final convention: in proofs involving both $M$ and $M^{\boldsymbol{B}}$, unless stated otherwise, the model to which an object belongs or in which a statement is true is assumed to be $M$.

DEFINITION 3.1. The Cohen real algebra $\boldsymbol{B}_C$ is the set of equivalence classes of Borel sets of $2^\omega$ modulo sets of first category. (For the proof of 1.2 we shall actually use $\omega^\omega$, but this is irrelevant.) The random real algebra $\boldsymbol{B}_R$ is the set of equivalence classes of Borel sets of $2^\omega$ modulo sets of measure zero. $\mu$ will always mean probability measure.

DEFINITION 3.2. Let $\boldsymbol{B} = \boldsymbol{B}_C$ (respectively $\boldsymbol{B}_R$). Then a term $r \in M^{\boldsymbol{B}}$ is a Cohen real (resp. a random real) over $M$ iff $\forall b \in b \forall u \in b$ $(b \Vdash r \in u)$. We also say that $r$ is $\boldsymbol{B}$-generic over $M$.

If $u \in b$, we write $b = [u]$ and note that in either algebra $[N_\sigma] = \{\sup b: b \Vdash r \in N_\sigma\}$ where $N_\sigma$ as in Section 2. Using this we prove:

FACT 3.3. *Let $r$ be Cohen or random over $M$, $\boldsymbol{B}$ the appropriate algebra. Suppose $b \in \boldsymbol{B}$ and $\{\sigma_n: n<\omega\} \in M$, each $\sigma_n$ a finite function from $\omega$ into 2 (or, where the underlying space is $\omega^\omega$, into $\omega$) and the domains of the $\sigma_n$'s are pairwise disjoint. Then $M^{\boldsymbol{B}_C} \vDash r \in \bigcup_{n<\omega} N_{\sigma_n}$; and if $\exists k \forall n$ $(|\mathrm{dom}\,\sigma_n| = k)$ then $M^{\boldsymbol{B}_R} \vDash r \in \bigcup_{n<\omega} N_{\sigma_n}$.*

Proof. Suppose not. Then for some $b \in \boldsymbol{B}$, $\{\sigma_n: n<\omega\}$ as above, $b \Vdash r \notin \bigcup_{n<\omega} N_{\sigma_n}$. But then $b \leqslant [\bigcap_{n<\omega} (N_{\sigma_n})^c]$ which, since the domains of the $\sigma_n$'s are disjoint, is a first category and, under the additional assumption, a measure zero set, which contradicts the appropriate definition of $\boldsymbol{B}$.

Fact 3.3 will be used to prove the key lemma in the next section.

Both $\boldsymbol{B}_C$ and $\boldsymbol{B}_R$ have particularly nice pictures attached. For $\boldsymbol{B}_C$, we note that $\boldsymbol{P}_C =$ the standard neighborhood basis for $2^\omega$ (respectively $\omega^\omega$) is a countable dense set. Hence if $\{\varphi_\alpha\colon \alpha<\omega_1\} \in M$ is an uncountable set of sentences with parameters in $M^{\boldsymbol{B}_C}$, and, for each $\alpha<\omega_1$, $b_\alpha \in \boldsymbol{B}_C$ decides $\varphi_\alpha$ (that is, either $b_\alpha \Vdash \varphi_\alpha$ or $b_\alpha \Vdash \varphi_\alpha$), by density of $\boldsymbol{P}_C$ in $\boldsymbol{B}_C$, wlog each $b_\alpha \in \boldsymbol{P}_C$, and hence some single $b$ decides uncountably many $\varphi_\alpha$. Since $\forall b \in \boldsymbol{B}_C \{\varphi\colon b \Vdash \varphi\} \in M$, we can roughly conclude that $M$ knows an uncountable piece of what $M^{\boldsymbol{B}_C}$ knows.

While $\boldsymbol{B}_R$ has no equally pleasant dense partial order, there is a separable metric space associated with it that is quite useful: given $[u]$, $[u'] \in \boldsymbol{B}_R$, define

$$d([u], [u'] = \mu(u \Delta u'))\,,$$

where $\Delta$ is symmetric difference. So given $\{\varphi_\alpha\colon \alpha<\omega_1\}$, $\{b_\alpha\colon \alpha<\omega_1\}$, such that each $b_\alpha \in \boldsymbol{B}_R$, $\varphi_\alpha$ has all parameters in $M^{\boldsymbol{B}_R}$, $b_\alpha$ decides $\varphi_\alpha$: either $\{b_\alpha\colon \alpha<\omega_1\}$ is really countable, and we can reach the same conclusion we did before, or uncountably many $b_\alpha$'s cluster via the metric around some $b$, and thus infinitely many of them have a common non-empty inf, $b^*$. Since $\{\varphi : b^* \Vdash \varphi\} \in M$, and since $b_\alpha \geqslant b^* \Rightarrow b^* \Vdash \varphi_\alpha$, we can roughly conclude that $M$ knows an infinite piece of what $M^{\boldsymbol{B}_R}$ knows.

Letting $A$ be a term, and $\varphi_\alpha$ the sentence "$\alpha \in A$" the above argument sketches the proof of:

FACT 3.4. *Suppose $M^{\boldsymbol{B}} \models A$ an uncountable set of ordinals, where $\boldsymbol{B} = \boldsymbol{B}_C$ or $\boldsymbol{B}_R$. Then the set $W_A$ is dense in $\boldsymbol{B}$, where $W_A$ is the set of $b \in \boldsymbol{B}$ for which $\exists A_b \in M$ ($b \Vdash A_b$ an infinite subset of $A$). Hence $M^{\boldsymbol{B}} \models \exists A' \in M$ ($A'$ infinite $\subset A$). In fact, if $\boldsymbol{B} = \boldsymbol{B}_C$ we can require that each $A_b$ is uncountable.*

Fact 3.4 will be used to turn terms in $M^{\boldsymbol{B}}$ into sets in $M$ to which we can then apply the key lemma derived from 3.3. We will actually apply 3.4 to terms for families of finite sets of ordinals, or finite functions, but by canonical 1-1 maps into ordinals, this is irrelevant.

If $b \in \boldsymbol{B}_R$, $b = [u]$, we write $\mu(b) = \mu(u)$. Using this notation, to prove 1.3 for the random real extension, we shall need

FACT 3.5. *Let $\varphi$ be such that $\forall x \in (0, 1)\ \exists b \in \boldsymbol{B}_R$ ($b \Vdash \varphi$ and $\mu(b)>x$). Then $M^{\boldsymbol{B}_R} \vdash \varphi$.*

Proof. Otherwise some $b \Vdash \neg\varphi$ and $\mu(b) = x \in (0, 1)$. But then $b' \Vdash \varphi \Rightarrow \mu(b') \leqslant 1-x \in (0, 1)$. Contradiction.

Finally, we close this section with a combinatorial lemma due to Marczewski that has nothing to do with forcing.

FACT 3.6. *Let $A$ be an uncountable collection of finite sets. Then $\exists A'$ an uncountable subset of $A$, and $F$ a finite set, such that $a, a' \in A \Rightarrow a \cap a' = F$ and $|a| = |a'|$. $A'$ is called a $\Delta$-system, and $F$ its root. (Use of fact 3.6 will be called a $\Delta$-system argument.)*



**§ 4. The constructive proofs; destroying** MA. In this section we construct the spaces of 1.1 and the family of sets promised in 1.2, thus demonstrating that adding the most innocent sort of generic real destroys MA+$\neg$CH. In the next section it will be shown that the destruction is not total.

Recalling construction 2.1 let $\Gamma$ be the hypothesis

($\Gamma$) $X_{\mathscr{F},r} = \{g_\alpha\colon \omega<\alpha<\omega_1\} \in M^{\boldsymbol{B}}$ as in 2.1, $\mathscr{F} \in M$, and $r$ $\boldsymbol{B}$-generic over $M$.

Let the hypothesis $\Gamma'$ be: $\Gamma+\{\text{range} f_\alpha\colon \omega<\alpha<\omega_1\}$ is an almost disjoint family.

PROPOSITION 4.1. *Assume $\Gamma$, $\boldsymbol{B} = \boldsymbol{B}_C$ or $\boldsymbol{B}_R$; and fix infinite $\alpha<\omega_1$. Then $M^{\boldsymbol{B}} \models \exists\gamma<\alpha\exists\beta>\alpha(g_\alpha(\beta) = 1-g_\alpha(\gamma))$. Furthermore, if $\alpha$ is a limit, then $M^{\boldsymbol{B}} \models \forall\gamma<\alpha\,\exists\beta>\gamma(g_\alpha(\beta) = 1)$.*

Proof. Let $A = \text{range } f_\alpha \in M$. Then by 3.3 $M^{\boldsymbol{B}} \models r \in \bigcup_{n\in A} N_{\langle n, 0\rangle} \cap \bigcup_{n\in A} N_{\langle n, 1\rangle}$, which by two applications of 3.3 and by definition of $g_\alpha$ proves the first part. For the second part, let $\alpha_j \nearrow \alpha$. Then $M^{\boldsymbol{B}} \models \forall n(r \in \bigcup_{j>n} N_{\langle f_\alpha(\alpha_j), 1\rangle})$ which proves the second part.

We now compress all the dirty work of this section into one lemma, which roughly says that given a countable union $\bigcup_{j<\omega} N_{\sigma_j}$ in $M$ of clopen neighborhoods of $(2^{\omega_1})^n$ with the right properties, and a term $\vec{g}$ in $(X_{\mathscr{F},r})^n$ with all indices high enough, $\vec{g}$ is forced to be in $\bigcup_{j<\omega} N_{\sigma_j}$. Since the lemma has to serve different theorems, it is stated as a common hypothesis (with explanations) followed by slightly different hypotheses all leading to substantially the same conclusions.

LEMMA 4.2. *Let $\boldsymbol{B} = \boldsymbol{B}_C$ or $\boldsymbol{B}_R$. Let $A \in M$ be a countable collection of disjoint finite sets of ordinals ($A$ will roughly become the domains of the $\sigma_j$'s above). Let $F^*$ be a finite set of ordinals with $\inf F^* > \sup\{\sup F\colon F \in A\}$ ($F^*$ is the set of indices for functions in $\vec{g}$ above). Let $\vec{g}_{F^*}$ be the term for a tuple of functions enumerating $\{g_\alpha\colon \alpha \in F^*\} \in M^{\boldsymbol{B}}$ ($\vec{g}_{F^*}$ is our $\vec{g}$ as promised). Suppose each $F = \bigcup_{\alpha\in F^*} F_\alpha$, for $F \in A$, and suppose $\sigma_{\alpha,F}$ is a finite function into 2 with domain $F_\alpha$ ($N_{\sigma_{\alpha,F}}$ is the neighborhood $g_\alpha$ hopes to belong to, for $\alpha \in F^*$). Let $\Phi$ be the sentence: $\vec{g}_{F^*} \in \bigcup_{F\in A} \prod_{\alpha\in F^*} N_{\sigma_{\alpha,F}}$. Then*

(a) *Assume $\Gamma$. If $\exists i$ with range $\sigma_{\alpha,F} = \{i\}$ for all $F \in A$, $\alpha \in F^*$, and all elements of $A$ have the same cardinality, then $M^{\boldsymbol{B}} \models \Phi$.*

(b) *Assume $\Gamma'$. If $\{\langle\sigma_{\alpha,F}\colon \alpha \in F^*\rangle\colon F \in A\} \in M$ (implicit in the hypothesis of* (a)), *then $M^{\boldsymbol{B}_C} \models \Phi$. Furthermore, if all elements of $A$ have the same cardinality, $M^{\boldsymbol{B}_R} \models \Phi$.*

Proof. (a) For $F \in A$ let $a_F = \bigcup_{\alpha\in F^*} \text{range } (f_\alpha/\!\!\wedge F_\alpha)$. Each $a_F$ finite of cardinality $\leqslant |F|$, hence wlog they all have the same cardinality, hence since each $f_\alpha$ is 1-1, wlog they are all disjoint. Let $\sigma_F$ have domain $a_F$, range $\sigma_F = \{i\}$. Then by 3.3, $M^{\boldsymbol{B}} \models r \in \bigcup_{F\in A} \sigma_F$, which by construction of $X_{\mathscr{F},f}$ proves $M^{\boldsymbol{B}} \models \Phi$.

(b) Let $c = \{\gamma\colon \exists\alpha \neq a' \in F^* \exists\gamma'(f_\alpha(\gamma) = f_{\alpha'}(\gamma'))\}$. By the hypothesis, $c$ is finite, hence wlog $c \cap F = \emptyset$ for $F \in A$. Let $a_{\alpha,F} = \text{range } (f_\alpha/\!\!\wedge F_\alpha)$. The $a_{\alpha,F}$'s are disjoint since $c \cap F = \emptyset$. Let $\sigma_F$ have domain $\bigcup_{\alpha \in F^*} a_{\alpha,F}$, $\sigma_F(m) = \sigma_{\alpha,F}(f_\alpha^{-1}(m))$ for the unique $\alpha$ for which $m \in a_{\alpha,F}$. (For $\boldsymbol{B} = \boldsymbol{B}_R$, the additional hypothesis ensures the $\bigcup_{\alpha \in F^*} a_{\alpha,F}$'s wlog have the same cardinality.) Under the appropriate hypothesis for the appropriate algebra, again by 3.3 $M^{\boldsymbol{B}} \models r \in \bigcup_{F \in A} \sigma_F$, and we are done.

For the proof of 1.1 our goal is to prove that $\Gamma' \Rightarrow M^{\boldsymbol{B}} \models \forall n < \omega (X^n_{\mathscr{F},r}$ is an $L$-space), where $\boldsymbol{B} = \boldsymbol{B}_C$ or $\boldsymbol{B}_R$. Regularity is clear since $M^{\boldsymbol{B}} \models X_{\mathscr{F},r} \subset 2^{\omega_1}$; since $M^{\boldsymbol{B}} \models X_{\mathscr{F},r}$ is a left space, by the well-known fact the sup of the supports of a left-space is $\omega_1$ iff the space is not separable, we need

$$M^{\boldsymbol{B}} \models \sup\{\beta\colon \exists g \in X_{\mathscr{F},r}(g(\beta) = 1)\} = \omega .$$

This follows from 4.1. The remaining step in the proof of 1.1 (a) is

PROPOSITION 4.3. *Assume* $\Gamma'$, $\boldsymbol{B} = \boldsymbol{B}_C$ *or* $\boldsymbol{B}_R$. *Then* $M^{\boldsymbol{B}} \models \forall n < \omega (X^n_{\mathscr{F},r}$ *is* hL).

Proof. Fix $n$, and for $U$, $Y$ terms in $M^{\boldsymbol{B}}$, let $M^{\boldsymbol{B}} \models *$, where $*$ is

(*) $U = \{\prod_{j<n} N_{\sigma_{\alpha,j}}\colon \alpha < \omega_1\}$ is a cover of $Y$ an uncountable subset of $(X_{\mathscr{F},r})^n$.

The idea of the proof is to reduce $U$ and $Y$ so that any infinite subset of $U$ in $M$ meets the hypothesis of 4.2 (b), and covers all but countably many elements of $Y$. This will mean that at least one element of $U$ is uncountable relative to $Y$, and by 2.3 we will be done.

The proof proceeds, then, by a sequence of reductions using $\Delta$-system and counting arguments in $M^{\boldsymbol{B}}$.

Since $M^{\boldsymbol{B}} \models *$, wlog the conjunction of the following sentences is true in $M^{\boldsymbol{B}}$:

(a) $\forall j < n \{\operatorname{dom} \sigma_{\alpha,j}\colon \alpha < \omega_1\}$ is a $\Delta$-system with each set of size $k_j$ and root $F_j$.

(b) $\forall j < n \forall \gamma \in F_j \forall \alpha, \alpha' (\sigma_{\alpha,j}(\gamma) = \sigma_{\alpha',j}(\gamma))$.

(c) $\{\{\alpha_j\colon j<n\}\colon \langle g_{\alpha_j}\colon j<d\rangle \in Y\}$ is a $\Delta$-system with root $F$.

(d) If $\langle g_{\alpha_j}\colon j<n\rangle$ and $\langle g_{\alpha_j}\colon j<n\rangle \in Y$, then $\alpha_j = \alpha_{j'} \Leftrightarrow \alpha'_j = \alpha'_{j'}$.

In other words, everything that can be a $\Delta$-system is; functions agree on the common root of their domains, and repetitions of functions in the $n$-tuples of $Y$ are uniform.

Letting $\sigma_j = \sigma_{\alpha,j} \upharpoonright F_j$ (well-defined by (b)) we have $M^{\boldsymbol{B}} \models Y \subset \prod_{j<n} N_{\sigma_j}$; by a further counting argument we can use (c) to conclude that in $M^{\boldsymbol{B}}$ the following is true:

(e) If $\langle g_{\alpha_j}\colon j<n\rangle$, $\langle g_{\alpha'_j}\colon j<n\rangle \in Y$ and $\alpha_j = \alpha'_{j'}$, then $j = j'$.

And finally, using (e) and what has gone before, wlog the following is true in $M^{\boldsymbol{B}}$.

(a′) $\{\operatorname{dom} \sigma_{\alpha,j}\colon \alpha < \omega_1\}$ is a pairwise disjoint family for $j<n$.

(c′) $\{\{\alpha_j\colon j<n\}\colon \langle g_{\alpha_j}\colon j<n\rangle \in Y\}$ is a pairwise disjoint family.

Note that (c′) might reduce the size of $n$, but this is irrelevant.



For notational convenience, write $\vec{\sigma}_\alpha = \prod_{j<n} \sigma_{\alpha,j}$, $N_\alpha = \prod_{j<n} N_{\sigma_{\alpha,j}}$, and $\vec{g}_\alpha = \langle g_{\alpha_j}\colon j<n\rangle$.

By 3.4, $M^{\boldsymbol{B}} \models \exists$ infinite countable $A \in M$ $(\{N_\alpha\colon \vec{\sigma}_\alpha \in A\} \subset U)$. So by 4.2 and the principle mentioned before 3.1, $M^{\boldsymbol{B}} \models \exists$ infinite countable $A \in M$ such that $Y - \bigcup_{\vec{\sigma}_\alpha \in A} N_\alpha$ is countable (because by (c′) at most countably many functions in $Y$ do not meet the hypothesis of 4.2 for a given $A \in M$). But then $M^{\boldsymbol{B}} \models \exists \vec{\sigma}_\alpha \in A$ $(Y \cap N_{\vec{\sigma}_\alpha}$ uncountable); so by 2.3 we are done.

The reader will note that, for $n = 1$, we have proved that $X_{\mathscr{F},r}$ has a property suspiciously like that of an $L$-space of Juhász and Hajnal constructed from CH. In their space, any nicely separated countably infinite uniform family of neighborhoods covers all but a countable piece of the space; the construction easily adapts to give every finite product $L$. That we look only at families in $M$ is important, because a corollary of 1.3 due to Silver is that under the appropriate hypotheses on $M$ the Juhász–Hajnal space does not exist in $M$.

Now for the proof of 1.1 (b), which closely parallels the construction of Galvin in his proof that CH$\rightarrow\neg$C.

DEFINITION 4.4. Let $E$ be a set of unordered pairs from $\omega_1$, $\bigcup E = \omega_1 - \omega$. The set $H \subset \omega_1$ is *homogeneous for* $E$ iff $\forall \alpha, \beta \in H (\alpha \neq \beta \Rightarrow \{\beta, \alpha\} \in E)$. $X$ is the *space for* $E$ iff $X = \{H\colon H$ maximal homogeneous for $E\}$ under the clopen basis consisting of $\{N_F\colon F$ finite homogeneous$\}$, where $N_F = \{H\colon H \supseteq F\}$. We note that points of $X$ are subsets of $\omega_1$.

PROPOSITION 4.5 (Galvin). *Let both* $E$ *and* $E'$ *satisfy the hypothesis of* 4.4, $E \cap E' = \emptyset$, $X$ *the space for* $E$, $X'$ *the space for* $E'$. *Then* $X * X'$ *is not* ccc.

Proof. For $\omega < \alpha < \omega_1$, let $u_\alpha = \{\langle H, H'\rangle\colon \alpha \in H \cap H'\}$. $u_\alpha \neq \emptyset$, but if $\beta \neq \alpha$ then $\langle H, H'\rangle \in u_\alpha \cap u_\beta$ implies $\{\beta, \alpha\} \subset H \cap H'$ hence $\{\beta, \alpha\} \in E \cap E'$, which is impossible. So each $u_\alpha \cap u_\beta = \emptyset$, $\alpha \neq \beta$.

Now to find disjoint $E$, $E'$ whose associated spaces are ccc.

PROPOSITION 4.6. *Assume* $\Gamma$, $\boldsymbol{B} = \boldsymbol{B}_C$ *or* $\boldsymbol{B}_R$, *and define* $E_0$, $E_1$ *in* $M^{\boldsymbol{B}}$ *by* $E_i = \{\{\beta, \alpha\}\colon \beta < \alpha$ *and* $g_\alpha(\beta) = i\}$. *Then if* $X_i$ *is the associated space for* $E_i$, $M^{\boldsymbol{B}} \models X_i$ *is* ccc.

(Since $E_0 \cap E_1 = \emptyset$, and since by 4.1 their respective unions are large enough, 4.6 is all that is needed to prove 1.1 (b). We also note that the Galvin proof from CH essentially reconstructs the modified Juhász–Hajnal space mentioned after 4.3; which can be used analogously to our use of $X_{\mathscr{F},r}$ in the CH proof.)

Proof of 4.6. We give the proof for $i = 0$. Suppose $U$ is a term such that $M^{\boldsymbol{B}} \models U$ is an uncountable family of open subsets of $X_0$. We need to show that $U$ is not pairwise disjoint in $M^{\boldsymbol{B}}$.

We work entirely in $M^{\boldsymbol{B}}$. Suppose $U = \{N_\alpha\colon \alpha < \omega_1\}$. We let $F_\alpha$ be the finite subset of $\omega_1$ for which $N_\alpha = N_{F_\alpha}$. Then, since $U$ is uncountable, wlog $\{F_\alpha\colon N_{F_\alpha} \in U\}$ is a $\Delta$-system with root $F^*$, and by 3.4 $\exists A \in M$ ($A$ is countably infinite and $F \in A \Rightarrow N_F \in U$). Given $N_F, N_{F'} \in U$, with $\inf(F - F^*) > \sup(F' - F^*)$, how can they be disjoint? Only if $\exists\alpha\in F-F^*$ and $\exists\beta\in F'-F^*$ with $g_\alpha(\beta)=1$; i.e. only if $\vec{g}_F=\langle g_\alpha:\ \alpha\in F-F^*\rangle\notin(N_{\sigma_{F'}})^{|F-F^*|}$ where $\operatorname{dom}\sigma_{F'}=F'-F^*$ and $\sigma_{F'}$ is identically 0. But if $Y=\{\vec{g}_F:\ F\in A\}$, by 4.2 $Y-\bigcup_{F\in A}N_F$ is countable (remember that the sets of indices for distinct members of $Y$ are disjoint). And $\vec{g}_F\in N_{\sigma_{F'}}\leftrightarrow N_F\cap N_{F'}\neq\emptyset$, for $\inf(F-F^*)>\sup(F'-F^*)$. So $U$ is not pairwise disjoint.

Note. Because of their origins in an $L$-space, it is easy to see that a maximal homogeneous subset of an $E_i$ is countable, and hence that the spaces $X_i$ have the further Suslin-tree-like property that given an uncountable collection of basic open sets, at least two are mutually disjoint. This is true of the Galvin space as well.

This section closes with the proof of 1.2, due to Kunen. Again, the construction used is an adaptation of $X_{\mathcal{F},r}$.

Definition 4.9. Let $\mathcal{F}=\{f_\alpha:\ \omega<\alpha<\omega_1\}$, $f_\alpha:\ \alpha\xrightarrow{1-1}\omega$. Letting $B$ be the set of countable limits of limit ordinals, fix, for $\alpha\in B$, a sequence of limit ordinals $\beta_k^\alpha\nearrow\alpha$, $k<\omega$. Let $h_k^\alpha$ list $f_\alpha^{-1}([\beta_k^\alpha,\ \beta_{k+1}^\alpha))$ in increasing order. Let $r\in\omega^\omega$, and let

$$a_k^\alpha=\{h_k^\alpha(j):\ j\leqslant r(k)\}.$$

Define, for $\alpha\in B$, the set $A_\alpha=\bigcup_{k<\omega}a_k^\alpha$.

(What $A_\alpha$ does is look at each interval $[\beta_k^\alpha,\beta_{k+1}^\alpha)$; take each such interval as a chunk of the domain of $f_\alpha$, and using $h_k^\alpha$ as a guide, throw in the first $r(k)$ things listed.)

Proposition 4.10. *$\{A_\alpha:\ \alpha\in B\}$ as in 4.9 is an almost disjoint family of countable sets.*

Proof. Each $A_\alpha$ is clearly countable. Given $\alpha\in B$, $\beta_k^\alpha\nearrow\alpha$ and $\alpha>\alpha'\in B$, $\exists k(\beta_k^\alpha>\alpha')$; hence $A_\alpha\cap A_{\alpha'}\subset\bigcup_{j<k}a_j^\alpha$ which is finite.

Under the right hypotheses, it is these $A_\alpha$'s which will become the family contradicting Ad.

Proposition 4.11. *Let $r$ be $B_C$-generic over $M$, and let $\mathcal{F}$, $\{\beta_k^\alpha:\ k<\omega,\ \alpha\in B\}\in M$. (hence $\{h_k^\alpha:\ k<\omega,\ \alpha\in B\}\in M$) be as in 4.9. Then where $A_\alpha$ is defined as in 4.9, $M^{B_C}\models$ every uncountable subset of $\omega_1$ intersects some $A_\alpha$ infinitely often.*

Proof. By 4.10, $M^{B_C}\models\{A_\alpha:\ \alpha\in B\}$ is an almost disjoint family of countable sets. The fact that $\boldsymbol{B}_C$ is now defined with reference to $\omega^\omega$ is no problem, since the relevant generalizations of facts in Section 3 all go through. What we will not be able to do, however, is put a bound on the ranges of the finite functions we consider, which is why the proof we not work for $\boldsymbol{B}_R$.

Again, work entirely in $M^{B_C}$. Suppose $A$ is an uncountable subset of $\omega_1$; wlog between any two ordinals in $A$ there is a limit ordinal. By 3.4 $\exists$ infinite countable $A'\subset A$, $A'\in M$; and by our second assumption on $A$ $\exists\alpha\exists\{\delta_k:\ k<\omega\}\in M$, each $\delta_k\in A'$, $\delta_k\nearrow\alpha\in B$. Wlog we have a sequence in $M$



$$\beta_{n_0}^\alpha\leqslant\delta_0<\beta_{n_0+1}^\alpha\leqslant\beta_{n_1}^\alpha\leqslant\delta_1<\beta_{n_1+1}^\alpha\ldots$$

and we let $m_k=f_\alpha^{-1}(\delta_k)$.

Let $\xi$ be the characteristic function of $A_\alpha$. Then a similar proof to that of 4.2 gives us $\xi\in\bigcup_{n<k<\omega}N_{\sigma_k}$ where $\sigma_k$ is identically 0 on its domain,

$$\operatorname{dom}\sigma_k=[\beta_{n_k}^\alpha,\beta_{n_k+1}^\alpha)\cap\{f_\alpha^{-1}(j):\ j\leqslant m_k\},$$

for all $k<\omega$. But this implies $A_\alpha\cap A$ infinite, and we are done.

**§ 5. A salvage operation.** In Section 4, by results I, II and III quoted in Section 1, MA+$\neg$CH was destroyed in three ways. Here some of it is saved.

Recall the definition of Martin's axiom: If $\boldsymbol{P}$ is a ccc partial order (i.e. no countable antichains) and $\{D_\alpha:\ \alpha<\varkappa<c\}$ is a collection of dense subsets of $\boldsymbol{P}$, then there is a filter $G\subset P$ which intersects each $D_\alpha$. $G$ is called generic for the $D_\alpha$'s.

Note that CH$\Rightarrow$MA (Rasiowa–Sikorski).

If $\Phi$ is a property of partial orders, we define $\mathrm{MA}_\Phi$: If $\boldsymbol{P}$ is a ccc partial order with property $\Phi$ and $\{D_\alpha:\ \alpha<\varkappa<c\}$ is a collection of dense subsets of $\boldsymbol{P}$, then there is a filter $G\subset\boldsymbol{P}$ which is generic for the $D_\alpha$'s.

Definition 5.1. $\boldsymbol{P}$ is *$\Sigma$-linked* iff $\boldsymbol{P}=\bigcup_{n<\omega}P_n$ where any two elements of a given $P_n$ are compatible (we say $P_n$ is linked). $\boldsymbol{P}$ is *$\Sigma$-centered* iff $\boldsymbol{P}=\bigcup_{n<\omega}P_n$ where any finite subset of a given $P_n$ has non-zero inf in $\boldsymbol{P}$ (we say $P_n$ is centered).

Fact 5.2. *$\boldsymbol{B}_C$ is $\Sigma$-centered and $\boldsymbol{B}_R$ is $\Sigma$-linked.*

Proof. For $\boldsymbol{B}_C$, let $\{b_n:\ n<\omega\}$ enumerate the countable dense set. Then letting $B_n=\{b\in\boldsymbol{B}_C:\ b\geqslant b_n\}$, we are done.

For $\boldsymbol{B}_R$, we let $\{b_n:\ n<\omega\}$ be a dense subset of the associated separable metric space, and if $\mu(b_n)>1/m$ we define $B_{n,m}=\{b:\ \mu(b)>1/m$ and $d(b,b_n)<m/2\}$. Then each $B_{n,m}$ defined is linked, and we are done.

Theorem 5.3. *Let $M$ be a model of set theory, $\boldsymbol{B}=\boldsymbol{B}_C$ or $\boldsymbol{B}_R$. Then $\mathrm{MA}_{\Sigma\text{-linked}}$ is true in $M\to M^{\boldsymbol{B}}\models\mathrm{MA}_{\Sigma\text{-linked}}$; and $\mathrm{MA}_{\Sigma\text{-centered}}$ is true in $M\to M^{B_C}\models\mathrm{MA}_{\Sigma\text{-centered}}$.*

Proof. We give the proof only for $\Sigma$-linked, since the proof for $\Sigma$-centered is exactly parallel. First we give the proof for $\boldsymbol{B}_C$ and then indicate how to modify it for $\boldsymbol{B}_R$.

Let $\boldsymbol{B}=\boldsymbol{B}_C$, $\boldsymbol{P}=\boldsymbol{P}_C$ its countable dense subset. Suppose $\boldsymbol{Q}$ is a term, $M^{\boldsymbol{B}}\models\boldsymbol{Q}$ is $\Sigma$-linked. Then $\exists$ terms $Q_n$, $n<\omega$, such that $M^{\boldsymbol{B}}\models(\boldsymbol{Q}=\bigcup_{n<\omega}Q_n$ and each $Q_n$ is linked). Suppose further that $\{D_\alpha:\ \alpha<\varkappa<c\}$ is a term, $M^{\boldsymbol{B}}\models\forall\alpha<\varkappa$ ($D_\alpha$ is dense in $\boldsymbol{Q}$). We proceed to construct $\boldsymbol{Q}^*$, a $\Sigma$-linked partial order in $M$, which reflects much of the structure of $\boldsymbol{Q}$:

$$\boldsymbol{Q}^*=\{\langle s,\{q_b\}_{b\in s}\rangle:\ s\text{ a maximal finite pairwise incompatible subset of }P,\text{ and }\forall b\in s\exists n(b\Vdash q_b\in Q_n)\}$$

The partial order is defined by:

$$\langle s, \{q_b\}_{b\in s}\rangle \leqslant \langle s', \{q'_b\}_{b\in s'}\rangle \text{ iff } s \text{ refines } s' \text{ and if } b'\in s',\ b\in s,$$
$$\text{then } b\leqslant b' \Rightarrow b \Vdash q_b \leqslant q'_{b'}.$$

$Q^*$ is $\Sigma$-linked: Given any finite $s = \{b_0, \ldots, b_k\} \subset P$ and any $k$-tuple $\langle m_0, \ldots, m_k\rangle$ of integers, $\{\langle s, \{q_b\}_{b\in s}\rangle: b_i \Vdash q_{b_i} \in Q_{m_i}\}$ is linked, and there are only countably many such families defined. So $Q^*$, their union, is $\Sigma$-linked.

For $b \in P$, $\alpha < \varkappa$, let

$$D^*_{b,\alpha} = \{\langle s, \{q_{b'}\}_{b'\in s}\rangle: \exists b'\in s,\ b'\leqslant b,\ b' \Vdash q_{b'} \in D_\alpha\}.$$

Since each $E_\alpha = \{b: \exists q(b \Vdash q \in D_\alpha)\}$ is dense in $P$, each $D^*_{b,\alpha}$ is dense in $Q^*$.

Let $G^*$ be a filter for $Q^*$ in $M$, $G^*$ generic for the $D^*_{b,\alpha}$'s, and define $G \in M^B$ by

$$b \Vdash q \in G \text{ iff } \{b' \leqslant b: \exists \langle s, \{q_b\}\rangle_{b\in s} \in G^*, b'\in s, q = q_{b'}\} \text{ is dense below } b.$$

Then by definition of the partial order on $Q^*$, $M^B \models G$ a filter; and by the $E_\alpha$'s, $M^B \models G \cap D_\alpha \neq \emptyset$ for all $\alpha < \varkappa$. So the Cohen case is done.

For the random real case, we show that $\forall x < 1$ $(b \Vdash \neg\mathrm{MA}_{\Sigma\text{-linked}} \Rightarrow \mu(b) < x)$. By 3.7, this shows $M^{B_R} \models \mathrm{MA}_{\Sigma\text{-linked}}$.

So suppose $b \Vdash \neg\mathrm{MA}_{\Sigma\text{-linked}}$. Then $\exists Q$, $\{D_\alpha: \alpha < \varkappa < c\}$ as in the Cohen case, $b \Vdash \exists$ no filter for $Q$ generic for the $D_\alpha$'s.

Fix $x < 1$, and let $B = B_R$. Proceed as before, changing the definition of $Q^*$ only to remove the requirement that $s$ be maximal, and to add the condition that $\mu(\sup s) > x$.

$Q^*$ is $\Sigma$-linked by a slightly fussier argument: Given the finite tuples $\langle x_0, \ldots, x_k\rangle$, $\langle y_0, \ldots, y_k\rangle$, $\langle\langle n_0, m_0\rangle, \ldots, \langle n_k, m_k\rangle\rangle$, $\langle m'_0, \ldots, m'_k\rangle$ where each $m_i$, $m'_i$, $n_i$ are integers, $x_i$, $y_i$ are rationals, $\sum_{i\leqslant k} x_i - \sum_{i\leqslant k} y_i > x$, and $b_{n_i}$ as in the proof of 5.2, $B_{n_i,m_i}$ defined as in 5.2, then the following set is linked:

$$\{\langle s, \{q_b\}_{b\in s}\rangle: s = \{b_0, \ldots, b_k\},\ \text{each } b_i \in B_{n_i,m_i},\ \mu(b_i) > x_i,$$
$$d(b_i, b_{n_i}) < y_i/2, \text{ and } b_i \Vdash q_{b_i} \in Q_{m'_i}\}.$$

Again, there are only countably many such families.

Now $D^*_\alpha$ is defined to be $\{\langle s, \{q_b\}_{b\in s}\rangle: \forall b\in s(b \Vdash q_b \in D_\alpha)\}$. $D^*_\alpha$ is dense because all we require is $\mu(\sup s) > x$, and given any $q$, $\mu(\sup\{b: b \Vdash q' \in D_\alpha, q' \leqslant q\}) = 1$. $G^*$ and $G$ are defined as before, $G^*$ generic for the $D^*_\alpha$'s and by definition of $D^*_\alpha$ we have that $b \Vdash G \cap D_\alpha = \emptyset \Rightarrow \mu(b) < x$. Hence $\mu(b) < x$ if $b \Vdash \exists$ no filter in $Q$ generic for the $D_\alpha$'s, and we are done.

How far can 5.3 be extended? Not very. A likely candidate is $\mathrm{MA}_{\mathrm{PC}_{\omega_1}}$, where a partial order has $\mathrm{PC}_{\omega_1}$ iff every uncountable subset has an uncountable centered subset. But Kunen has pointed out that Hajnal and Galvin's proof of ZFC$\Rightarrow$($\mathrm{PC}_{\omega_1}$ $\nLeftrightarrow\Sigma$-linked) adapts to the construction of 1.2 to give $M^{B_C}$ does not preserve $\mathrm{MA}_{\mathrm{PC}_{\omega_1}}$.



We do not know what happens in $M^{B_R}$, but probably the property to investigate there, instead of $\mathrm{PC}_{\omega_1}$, is $K$ — every uncountable subset has an uncountable linked subset.

**§ 6. Generalizations to higher cardinals.** Let $\varkappa$ be a regular cardinal. Weak $\varkappa$-CH is the statement that $\alpha < \varkappa \Rightarrow 2^\alpha \leqslant \varkappa$.

Baumgartner's axiom for $\varkappa$ ($\mathrm{BA}(\varkappa)$) says the following: if $P$ is a partial order, any descending chain of $<\varkappa$ elements of $P$ has a non-zero lower bound (we say $P$ is $<\varkappa$-closed, $P = \bigcup_{\alpha<\varkappa} P_\alpha$ where each $P_\alpha$ is linked, and $\{D_\beta: \beta < \lambda < 2^\varkappa\}$ is a family of dense subsets of $P$, then $P$ has a filter generic for the $D_\beta$'s.

We say that a $<\varkappa$-closed partial order is *trivial* iff it contains no embedding of the full binary tree of height $\varkappa$; non-trivial otherwise. $\mathrm{BA}(\varkappa)_{\mathrm{trivial}}$ is trivially true. On the other hand, $\mathrm{BA}(\varkappa) \Rightarrow$(weak $\varkappa$-CH iff $\exists$ non-trivial $P$ satisfying the hypothesis of $\mathrm{BA}(\varkappa)$). Baumgartner has proved that $\mathrm{BA}(\varkappa)$+weak $\varkappa$-CH+$2^\varkappa$ large is consistent (the Rasiowa–Sikorski proof tells us that $2^\varkappa = \varkappa^+ \Rightarrow \mathrm{BA}(\varkappa)$).

Then exactly imitating the proof of 5.3, if $B$ is the algebra for adding a Cohen subset of $\varkappa$ (conditions are functions under reverse inclusion from ordinals $<\varkappa$ into 2), $\mathrm{BA}(\varkappa)$ is true in $M \Rightarrow M^B \models \mathrm{BA}(\varkappa)$.

If $\mathrm{BA}'(\varkappa)$ is $\mathrm{BA}(\varkappa)$ with the word "centered" substituted for "linked", then the entire discussion above holds true with $\mathrm{BA}'(\varkappa)$ substituted for $\mathrm{BA}(\varkappa)$.

On the other hand, if $\varkappa$ is regular and weak $\varkappa$-CH holds in $M$, and $B$ adds a Cohen subset of $\varkappa$ (for (c) below we take functions into $\varkappa$, not 2, analogously with 1.2) then the proofs of Section 4 generalize straight forwardly and we have that in $M^B$ the following results hold (for definitions see [11]).

(a) The products of $\varkappa^+$-cc spaces need not be $\varkappa^+$-cc (note: ccc means $\omega_1$-cc).

(b) $\exists X \forall n < \omega$ ($X^n$ is hereditarily $\varkappa$-Lindelöf but no subspace of cardinality $\varkappa$ is dense).

(c) $\exists A$ an almost disjoint family of $\varkappa^+$ many subsets of $\varkappa^+$, each of cardinality $\varkappa$, such that if $Y \subset \varkappa^+$ and $|Y| = \varkappa^+$ the $Y$ intersects some set in $A$ in a set of cardinality $\varkappa$. (Here almost disjoint means $a, b \in A \Rightarrow |a \cap b| < \varkappa$.)

## References


[1] J. Barwise (editor), *Handbook of Mathematical logic* (to appear).
[2] J. Baumgartner, *A generalization of Martin's axiom*, preprint.
[3] W. W. Comfort, private communication of a talk by Galvin.
[4] F. Galvin, *Chain conditions and products*, preprint.
[5] A. Hajnal and I. Juhász, *On hereditarily α-Lindelöf and α-separable spaces, II*, Fund. Math. 81 (1974), pp. 147–158.
[6] T. Jech, *Lectures in Set Theory*, 1971.
[7] I. Juhász, *Cardinal functions in topology*, 1971.
[8] K. Kunen, *Strong S and L spaces under* MA, preprint.

[9] K. Kunen and F. Tall, *Between Martin's Axiom and Suslin's Hypothesis*, Fund. Math. 102 (1979), pp. 173–181.
[10] E. Marczewski, *Separabilité et multiplication Cartesienne des espaces topologique*, Fund. Math. 34 (1947), pp. 127–143.
[11] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2 (1970), pp. 143–178.
[12] H. Rasiowa and R. Sikorski, *Mathematics of Metamathematics*, Warszawa 1963.
[13] M. E. Rudin, *Lectures on Set-theoretic Topology*, 1975.
[14] M. Wage, *Almost disjoint sets and Martin's axiom* (to appear).


WELLESLEY, MASSACHUSETTS

*Accepté par la Rédaction le 4. 10. 1976*

# Hereditarily indecomposable tree-like continua


by

**W. T. Ingram** (Houston, Tex.)



**Abstract.** In this paper it is shown that there is an uncountable collection of mutually exclusive hereditarily indecomposable tree-like continua in the plane such that if $M$ is a compact metric continuum then $M$ cannot be mapped onto every member of the collection.


**1. Introduction.** In 1951 Bing [2] asked if each non-degenerate hereditarily indecomposable plane continuum which does not separate the plane is a pseudo-arc. In an abstract in 1951 Anderson [1] stated that there exist hereditarily indecomposable tree-like continua which are not homeomorphic to the pseudo-arc. It is the purpose of this paper to demonstrate that there is an hereditarily indecomposable tree-like continuum which is not a continuous image of the pseudo-arc.

In [5] we demonstrated the existence of a collection $G$ of atriodic tree-like continua with the property that if $M$ is a compact metric continuum then $M$ cannot be mapped onto every member of $G$. In this paper we construct a collection $H$ of mutually exclusive hereditarily indecomposable tree-like continua in the plane such that if $g$ is in $G$ then $g$ is a continuous image of some member of $H$. Thus, $H$ also has the property that if $M$ is a compact metric continuum then $M$ cannot be mapped onto every member of $H$.

For notation (including $T$, $f$, and $g$) and conventions used in this paper the reader is referred to [3], [4], and [5].

**2.** We present in this section the main working lemma of the paper.

LEMMA. *Suppose $k$ is a mapping of $T$ onto $T$ such that* (1) *if $t$ is a point of $\left\{O, A, B, C, \frac{A}{2}\right\}$ then $k(t) = t$ and $k^{-1}(t) = \{t\}$ and* (2) *if $t$ is a point of $T-\left\{O, \frac{A}{2}\right\}$ and $\alpha$ is the component of $T-\left\{O, \frac{A}{2}\right\}$, containing $t$ then $k(t)$ is in $\alpha$. Then there is a mapping $h$ of $T$ onto $T$ such that* (1) *if $t$ is a point of $\left\{O, A, B, C, \frac{A}{2}\right\}$ then*