A characterization of certain branched coverings as group actions

by

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Abstract. The question of when a light-open mapping is the orbit of a group action is in general unsolved. We give a partial answer for a large class of light-open mappings which implies, for example, that if $f\colon M \Rightarrow N$ is any finite-to-one proper open map between separable connected manifolds without boundary, then f is the orbit map of a group action if and only if $f|_{M-f^{-1}(f(B_f))}$ is a regular covering, where B_f is the set of points in M at which f fails to be a local homeomorphism.

1. Introduction. It is well known that if a totally disconnected group G acts on a space X, then the orbit map $\pi\colon X \Rightarrow X/G$ is a light-open mapping. In [5], McAuley asks how to determine when a given light-open mapping is the orbit map of a group action. The question in general remains unsolved. Recently, however, Edmonds [3] provided a solution for finite-to-one PL open maps between compact normal n-circuits (pseudo-manifolds) using some interesting but elaborate techniques involving dual cells. In this note we obtain a more general result using elementary techniques and containing Edmonds result as a special case. Also, in [6], McAuley shows that a special type of light-open mapping on S^2 is the orbit map of a group action by using a characterization of light-open mappings in terms of coverings due to McAuley and Robinson.

2. Definitions and results. A map (i.e. continuous function) $f\colon X \to Y$ is an *open map* (*closed map*) if and only if whenever H is an open (closed) set in X, $f(H)$ is open (closed) in $f(X)$. The map is *light* if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$. We will indicate an onto mapping with a double arrow: $f\colon X \Rightarrow Y$. A finite-to-one open map is called a *branched covering*. A subset A of a space X is said to *separate* X *locally* at a point x if and only if there is a neighborhood U of x such that if V is any connected open set containing x with $V \subseteq U$, then $V-A$ is not connected. A subset P of a space X is *thin* if and only if the interior of P is empty and there is no point in X at which P separates X locally. If $f\colon X \Rightarrow Y$ is a branched co-

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vering, the set B_f is the set of points in X where f fails to be a local homeomorphism. The set B_f is a closed subset of X (see [2], Remark 1.1). Furthermore, by slightly modifying the proof of Theorem 3.2 in [7], we have:

THEOREM 2.1. *If X is a locally compact metric space and $f\colon X\Rightarrow Y$ is a branched covering, then B_f, $f(B_f)$, and $f^{-1}(f(B_f))$ all have empty interior.*

It is well known and easy to prove that if X and Y are locally compact metric spaces then a finite-to-one map $f\colon X\Rightarrow Y$ is a proper map (the inverse image of a compact set is compact) if and only if f is a closed map. It is also easy to prove that if P is a thin subset of a locally connected space X and R is any open connected subset of X, then $R-P$ is connected.

DEFINITION 2.2. A branched covering $f\colon X\Rightarrow Y$ is called a *special branched covering* if and only if 1) X and Y are both locally compact, locally connected, connected, metric spaces; 2) f is a proper map: and 3) $f^{-1}(f(B_f))$ is thin.

It easily follows from the above remarks that both B_f and $f(B_f)$ are thin subsets of X and Y respectively. Furthermore, $X-f^{-1}(f(B_f))$ is an open connected subset of X and, since f is proper, $f|_{X-f^{-1}(f(B_f))}$ is a covering map.

We now state the main result of this paper.

THEOREM 2.3. *Let $f\colon X\Rightarrow Y$ be a special branched covering. Then, f is the orbit map of a group action if and only if $f|_{X-f^{-1}(f(B_f))}$ is a regular covering.*

Proof. Suppose first that f is the orbit map of a group action of G on X. Then, there is an induced action of G on $X-f^{-1}(f(B_f))$. That $f|_{X-f^{-1}(f(B_f))}$ is a regular covering follows from Proposition 8.2 in [4].

Conversely, assume $f|_{X-f^{-1}(f(B_f))}$ is a regular covering. Then $f|_{X-f^{-1}(f(B_f))}$ is the orbit map of an action of K/H on $X-f^{-1}(f(B_f))$ where $H = f_*(\pi_1(X-f^{-1}(f(B_f))))$ and $K=\pi(Y-f(B_f))$. Furthermore, K/H may be realized as a finite group of homeomorphisms: the deck transformations of $X-f^{-1}(f(B_f))$. Let $h\in K/H$. We claim that h is uniformly continuous on $C-f^{-1}(f(B_f))$, where C is any compact subset of X. For if not, there exist sequences $\langle a_i\rangle$ and $\langle b_i\rangle$ of points in $X-f^{-1}(f(B_f))$ and $\varepsilon_0>0$ such that $a_i\to x$ and $b_i\to x$ for some $x\in X$, but $d(h(a_i), h(b_i))\geqslant\varepsilon_0$. Let $\{R_n\}$ be a nested collection of open connected subsets of X containing x such that the diameter of $f(R_n)<1/n$. Since f is proper, Y is locally compact, and each of $h(R_n)-f^{-1}(f(B_f))$, $f^{-1}(f(R_n))$, and $R_n-f^{-1}(f(B_f))$ is connected, it follows that some subsequence of $\langle h(R_n-f^{-1}(f(B_f)))\rangle$ converges to a connected subset L of $f^{-1}(f(x))$. Since each $R_n-f^{-1}(f(B_f))$ contains a pair $\{a_{i_n}, b_{i_n}\}$, we conclude that the diameter of $L\geqslant\varepsilon_0$. This contradicts the fact that if is light. Hence, h extends to a map of $\mathrm{cl}(X-f^{-1}(f(B_f)))=X$ onto $\mathrm{cl}(X-f^{-1}(f(B_f)))=X$. One can employ a similar argument to show that k is one-to-one. Furthermore, since f is proper, k is an open map and, hence, a homeomorphism. It is easy to see that k covers the identity. To see that the group of extension acts transitively on the fibres of X, let $y\in Y$ and x_1 and x_2 be elements of $f^{-1}(y)$. We can find sequences $\langle u_n\rangle$, $\langle v_n\rangle$ in $X-f^{-1}(f(B_f))$ such that $f(u_n)=f(v_n)$, $u_n\to x_1$ and $v_n\to x_2$. Since K/H acts transitively on fibres of $X-f^{-1}(f(B_f))$, for each n there is an $h_n\in K/H$ such that $h_n(u_n) = v_n$. Since K/H is finite, there is an $h'\in K/H$ and a sequence $\langle n_i\rangle$ such that $h'(u_{n_i}) = v_{n_i}$. If k' is the extension of h', then $k'(x_i) = x_2$. Q.E.D.

COROLLARY 2.4. *A proper branched covering $f\colon M^n\Rightarrow N^n$, where M^n and N^n are connected, separable, n-dimensional manifolds without boundary, is the orbit map of a group action if and only if $f|_{X-f^{-1}(f(B_f))}$ is a regular covering.*

Proof. Černavskiĭ [1] and Väisälä [7] show that for such an f, $\dim f^{-1}(f(B_f))\leqslant n-2$. Hence, $f^{-1}(f(B_f))$ is thin.

COROLLARY 2.5. *If $f\colon X\Rightarrow Y$ is any proper light-open mapping where X and Y are connected, separable two dimensional manifolds without boundary, then f is the orbit map of a group action if and only if $f|_{X-f^{-1}(f(B_f))}$ is a regular covering.*

Proof. It follows from Theorem 5.1 in [8] that f will be a special branched covering.

Added in proof. *In branched covering and orbit maps*, Michig. Math. J. 23 (1976), pp. 289–301, Edmonds recently and independently generalized the results in [3].

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