COROLLARY (6.2). *If (X, a) is a compact n-dimensional metric Z_p-space with one fixed point (or none) and with the action free outside the fixed point set, then (X, a) equivariantly embeds in (R^{2n+1}, β) if n is odd and in (R^{2n+1}, β') if n is even.*

References

[1] R. J. Allen, *Equivariant maps of Z_p-actions into polyhedra*, Fund. Math. 103 (1979), pp. 9–22.

[2] A. H. Copeland, Jr., and J. de Groot, *Linearization of homeomorphisms*, Math. Ann. 144 (1961), pp. 80–92.

[3] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series 4, Princeton, 1948.

[4] J. W. Jaworowski, *Equivariant extensions of maps*, Pacific J. Math. 45 (1) (1973), pp. 229–244.

[5] J. M. Kister and L. N. Mann, *Equivariant imbeddings of compact Abelian Lie groups of transformations*, Math. Ann. 148 (1962), pp. 89–93.

[6] G. D. Mostow, *Equivariant embeddings in euclidean space*, Ann. of Math. 65 (2) (1957), pp. 432–446.

DEPARTMENT OF MATHEMATICS, ST. OLAF COLLEGE
Northfield, Minnesota

Accepté par la Rédaction le 27. 9. 1976

Some remarks concerning the fundamental dimension of the cartesian product of two compacta

by

Sławomir Nowak (Warszawa)

Abstract. It is proved that $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ if Y is an n-dimensional continuum such that $H^n(Y; G) \neq 0$ for every $G \neq 0$ and if X is a compactum and $\mathrm{Fd}(X) \neq 2$ or $\mathrm{Fd}(X) = 2$ and X is not approximatively 2-connected.

We will consider the problem of computing the fundamental dimension of $X\times Y$, where X and Y are compacta. From this point of view the notion of an *$\mathcal{F}$-continuum* will be very convenient. A continuum X with $0 \neq \mathrm{Fd}(X) = n<\infty$ belongs to a class $\mathcal{F}$ (in other words X is an *$\mathcal{F}$-continuum*) iff for every abelian group $G \neq 0$ the n-dimensional Čech cohomology group $H^n(X; G)$ of X with coefficients in G is non-trivial.

Using the universal coefficient theorem and the Künneth formula for homology and cohomology ([13] p. 244 and p. 336), one can check that the class $\mathcal{F}$ contains all connected n-dimensional ANR-sets with the non-trivial n-dimensional Čech homology group $H_n(X)$ over the group Z of integer numbers (in particular, all closed orientable manifolds) and that $X\times Y\in\mathcal{F}$ for all Γ, X, $Y\in\mathcal{F}$.

It is known ([10] p. 74) that there exists a sequence $G_1, G_2, \ldots$ of non-trivial countable abelian groups such that if X is an n-dimensional compactum with $H^n(X; G_k) = 0$ for every $k = 1, 2, \ldots$, then $H^n(X; G) = 0$ for every group G. This fact together with the theorem which states ([6] p. 137) that for every countable group G and its character group G^* the group $H_n(X; G^*)$ is the character group of $H^n(X; G)$ and with the Pontriagin duality ([12] p. 259) imply that if X is an n-dimensional continuum and $H_n(X; G) \neq 0$ for every $G \neq 0$, then $X\in\mathcal{F}$ ($H_n(X; H)$ denotes the Čech homology group of X over H).

It is clear that the class $\mathcal{F}$ is closed with respect to the one-point union.

In [11] it is proved that $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ for every compactum X with $\mathrm{Fd}(X)\geqslant 3$ and every $Y\in\mathcal{F}$.

The purpose of this note is to generalize the last theorem and to show that the assumption that $\mathrm{Fd}(X)\geqslant 3$ may be replaced by the assumption that $\mathrm{Fd}(X) \neq 2$ or $\mathrm{Fd}(X) = 2$ and X is not approximatively 2-connected.

In the final part of the paper we give an algebraic characterization of all compacta X such that $\mathrm{Fd}(X\times Y)=\mathrm{Fd}(X)+\mathrm{Fd}(Y)$ for every $Y\in\mathscr{T}$.

If $f_1\colon X_1\to Y_1$ and $f_2\colon X_2\to Y_2$ are continuous functions, then $f_1\times f_2$ will denote the map from $X_1\times X_2$ to $Y_1\times Y_2$ defined by

$$f_1\times f_2(x_1,x_2)=\big(f_1(x_1),f_2(x_2)\big)\quad\text{for every } (x_1,x_2)\in X_1\times X_2\,.$$

For all CW complexes X, Y and a map $f\colon (X,x_0)\to(Y,y_0)$ $(x_0\in X$ and $y_0\in Y)$ we will denote (respectively) by $\tilde{X}$, $X^{(n)}$, $f_\#$ and f_*^s $(s\geqslant 2)$ the universal covering space of X, the n-skeleton of X, and the homomorphisms $f_\#\colon \pi_1(X,x_0)\to\pi_1(Y,y_0)$ and $f_*^s\colon \pi_s(X,x_0)\to\pi_s(Y,y_0)$ induced by f.

We assume that the reader is familiar with the theory of shape and knows the notion of procategory (for references see [2] and [8]).

In the last section a knowledge of cohomology groups with local coefficients (see [15] and [13] p. 281) will also be assumed.

The author acknowledges his gratitude to Dr. J. Dydak, who made several valuable remarks concerning the present note.

1. The fundamental dimension. Suppose that W is a finite CW complex and $f\colon X\to W$ is a map. We say (see [9]) that $\omega(f)\leqslant n$ iff there exists a homotopy $\varphi\colon X\times[0,1]\to W$ such that

$$\varphi(x,0)=f(x)\quad\text{for every } x\in X \tag{1.1}$$

and

$$\varphi(X\times\{1\})\subset W^{(n)}\,. \tag{1.2}$$

If (X,x_0), (W,w_0) are pointed CW complexes and $f\colon (X,x_0)\to(W,w_0)$, then the condition $\omega(f)\leqslant n$ implies that there exists a homotopy $\varphi\colon X\times[0,1]\to W$ which satisfies (1.1) and (1.2) and fixes x_0. We can infer this fact from the cellular approximation theorem (see [13] p. 404 and [13] p. 57, Exercise D4).

The following theorem characterizes compacta with the fundamental dimension $\leqslant n$ (see [9] p. 214 and compare [3] p. 80).

(1.3) THEOREM. *Let a compactum X be the inverse limit of an inverse sequence $\{X_k,p_k^{k+1}\}$ of finite* CW *complexes and let n be a natural number or* 0. *Then the following conditions are equivalent*:

(a) $\mathrm{Fd}(X)\leqslant n$,

(b) *for every k there exists a k' such that $\omega(p_k^{k'})\leqslant n$.*

If (X,x_0) is a pointed compactum and $\mathrm{Fd}(X)\leqslant n$, then Theorem (1.3) implies that we can assume that $(X,x_0)=\varprojlim\{(X_k,x_k),p_k^{k+1}\}$, where (X_k,x_k) is a pointed finite CW complex and $\omega(p_k^{k+1})\leqslant n$ for every $k=1,2,\dots$ It follows that p_k^{k+1} is homotopic (in the pointed sense) with a map $q_k^{k+1}\colon (X_{k+1},x_{k+1})\to(X_k,x_k)$ such that $q_k^{k+1}(X_{k+1})\subset X_k^{(n)}$. It is clear that $\mathrm{Sh}(X,x_0)=\mathrm{Sh}(Y,y_0)$ and $\dim Y\leqslant n$, where $(Y,y_0)=\varprojlim\{(X_k,x_k),q_k^{k+1}\}$.

Hence, we obtain

(1.4) PROPOSITION. *For every pointed compactum (X,x_0) there exists a pointed compactum (Y,y_0) such that* $\mathrm{Sh}(X,x_0)=\mathrm{Sh}(Y,y_0)$ *and* $\dim Y=\mathrm{Fd}(X)$.

Remark. Proposition (1.4) was first proved by S. Spież ([14]).

Let

$$d(X,Y)=\mathrm{Fd}(X)+\mathrm{Fd}(Y)-\mathrm{Fd}(X\times Y)$$

for every compactum X and every $Y\in\mathscr{F}$.

In [11] the author has proved the following

(1.5) THEOREM. *If* $3\leqslant\mathrm{Fd}(X)$, *then* $d(X,Y)=0$ *for every $\mathscr{T}$-continuum Y.*

One can easily prove that the following corollary holds:

(1.6) COROLLARY. *Suppose that X is a compactum and* $\mathrm{Fd}(X)<\infty$. *The number $d(X,Y)$ does not depend on $Y\in\mathscr{T}$ such that* $\mathrm{Fd}(X\times Y)\geqslant 3$. *If there exists a $Y_0\in\mathscr{F}$ such that $d(X,Y_0)=0$ and* $\mathrm{Fd}(X\times Y_0)\geqslant 3$, *then $d(X,Y)=0$ for every $Y\in\mathscr{F}$.*

Proof. Let Y be an arbitrary $\mathscr{F}$-continuum such that $\mathrm{Fd}(X\times Y)\geqslant 3$. From Theorem (1.5) we infer that

$$\begin{aligned}\mathrm{Fd}(X)+\mathrm{Fd}(Y)+3-d(X,S^3)&=\mathrm{Fd}(X\times S^3)+\mathrm{Fd}(Y)=\mathrm{Fd}((X\times S^3)\times Y)\\&=\mathrm{Fd}((X\times Y)\times S^3)=\mathrm{Fd}(X\times Y)+3\\&=\mathrm{Fd}(X)+\mathrm{Fd}(Y)+3-d(X,Y)\end{aligned}$$

and

$$d(X,Y)=d(X,S^3)\,.$$

This completes the proof of the first part of Corollary (1.6).

Let us assume that Y, Y_0 are $\mathscr{F}$-continua such that $\mathrm{Fd}(X\times Y_0)\geqslant 3$ and $d(X,Y_0)=0\neq d(X,Y)$. From Theorem (1.5) we infer that

$$\mathrm{Fd}(X\times Y_0\times Y)=\mathrm{Fd}(X)+\mathrm{Fd}(Y_0)+\mathrm{Fd}(Y)\,.$$

On the other hand we have

$$\mathrm{Fd}(X\times Y_0\times Y)\leqslant\mathrm{Fd}(X\times Y)+\mathrm{Fd}(Y_0)<\mathrm{Fd}(X)+\mathrm{Fd}(Y_0)+\mathrm{Fd}(Y)\,.$$

The proof of our corollary is finished.

In section four we will use the notion of a generalized local system of coefficients (see [11]).

Let X be a continuum. By a *generalized local system of abelian groups on X* we understand a pair $(\{X_k,p_k^{k+1}\},\mathscr{L}_k)=\underline{\mathscr{L}}$ consisting of an inverse sequence of finite CW complexes $\{X_k,p_k^{k+1}\}$ associated with X (this means that $\mathrm{Sh}(X)=\mathrm{Sh}(\varprojlim\{X_k,p_k^{k+1}\})$) and a sequence $\mathscr{L}_k$, where $\mathscr{L}_k$ is a local system of abelian groups on X_k for every $k=1,2,\dots$ and $\mathscr{L}_{k+1}$ is induced on X_{k+1} by $\mathscr{L}_k$ and the map p_k^{k+1}.

For every generalized local system of abelian groups $\underline{\mathscr{L}}=(\{X_k,p_k^{k+1}\},\mathscr{L}_k)$ on a continuum X the direct limit $H^n(X;\underline{\mathscr{L}})$ of the direct sequence of abelian groups

$\{H^n(X_k; \mathscr{L}_k), (p_k^{k+1})^*\}$ ($H^n(X_k; \mathscr{L}_k)$ denotes the n-dimensional cohomology group of X_k with coefficients in $\mathscr{L}_k$ and $(p_k^{k+1})^*: H^n(X_k; \mathscr{L}_k) \to H^n(X_{k+1}; \mathscr{L}_{k+1})$ denotes the homomorphism which is induced by p_k^{k+1}) will be called an *n-dimensional cohomology group of X with coefficients in $\underline{\mathscr{L}}$.*

If $X \neq \emptyset$ is a continuum, then let us denote by $c[X]$ (see [11] the maximum of numbers n (finite or infinite) such that there is a generalized local system of coefficients $\underline{\mathscr{L}}$ on X such that $H^n(X; \underline{\mathscr{L}}) \neq 0$.

In [11] it is proved that

$$c[X] \leqslant \mathrm{Fd}(X) \leqslant \max(2, c[X]) \tag{1.7}$$

for every continuum X with $\mathrm{Fd}(X) < \infty$.

It follows from the analysis (see [11]) of the proof of (1.7) that the following theorem holds:

(1.8) THEOREM. *If X is a continuum with $3 \leqslant n = \mathrm{Fd}(X) < \infty$ and $\{X_k, p_k^{k+1}\}$ is an arbitrary sequence of n-dimensional finite CW complexes such that its inverse limit has the same shape as X, then for every $k = 1, 2, \ldots$ there exists a local system of abelian groups $\mathscr{L}_k$ on X_k such that $\underline{\mathscr{L}} = (\{X_k, p_k^{k+1}\}, \mathscr{L}_k)$ is a generalized local system on X and $H^n(X; \underline{\mathscr{L}}) \neq 0$.*

2. Two algebraic lemmas. If G is a multiplicative group and Z is the ring of integers, then the *integral group ring $Z(G)$ of G* is the set of all finite formal sums $\sum n_i g_i$, $n_i \in Z$ and $g_i \in G$, with addition and multiplication given by

$$\sum n_i g_i + \sum m_i g_i = \sum (n_i + m_i) g_i$$

and

$$\left(\sum n_i g_i\right)\left(\sum m_j g_j\right) = \sum (n_i m_j) g_i g_j .$$

We will employ the following lemma:

(2.1) LEMMA. *Let G be a non-trivial multiplicative group and*

$$0 \neq z = n_1 g_1 + n_2 g_2 + \ldots + n_k g_k \in Z(G) .$$

For every element $a \in G$ with the order $\geqslant k$ there exists a natural number $l \leqslant k$ such that $(1a^l - 1e)z \neq 0$, where e is the unit of G.

Proof. Without loss of generality we may assume that

$$n_i \neq 0 \quad \text{and} \quad g_i \neq g_j \quad \text{for all } i, j = 1, 2, \ldots, k \text{ such that } i \neq j. \tag{2.2}$$

Let us suppose that our lemma does not hold.

This means that there is an $a \in G$ satisfying the following condition:

$$a^s \neq e \quad \text{and} \quad n_1 a^s g_1 + n_2 a^s g_2 + \ldots + n_k a^s g_k = n_1 g_1 + n_2 g_2 + \ldots + n_k g_k \tag{2.3}$$

for $s = 1, 2, \ldots, k$.

From (2.2) and (2.3) we infer that for every $s = 1, 2, \ldots, k$ there exists a function $\varkappa_s: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ such that

$$n_i g_i = n_{\varkappa_s(i)} a^s g_{\varkappa_s(i)} \text{ for } i = 1, 2, \ldots, k \quad \text{and} \quad \varkappa_s(i) \neq \varkappa_s(j) \text{ for } i \neq j. \tag{2.4}$$

If $\varkappa_s(1) = 1$, then $n_1 g_1 = n_1 a^s g_1$, and $g_1 = a^s g_1$, and $a^s = e$. Therefore

$$\varkappa_s(1) \neq 1 \quad \text{for every } s = 1, 2, \ldots, k . \tag{2.5}$$

Let us observe also that

$$\varkappa_{s_1}(i) \neq \varkappa_{s_2}(i) \quad \text{for} \quad s_1, s_2, i = 1, 2, \ldots, k \text{ such that } s_1 \neq s_2 .$$

Indeed, from $s_1 > s_2$ and $\varkappa_{s_1}(i) = \varkappa_{s_2}(i)$ we conclude that $n_i g_i = n_{\varkappa_{s_1}(i)} a^{s_1} g_{\varkappa_{s_2}(i)} = n_{\varkappa_{s_2}(i)} a^{s_2} g_{\varkappa_{s_2}(i)}$ and $a^{s_1 - s_2} = e$.

Therefore $1 \in \{\varkappa_1(1), \varkappa_2(1), \ldots, \varkappa_k(1)\}$ contradicting (2.4) and (2.5). Thus the proof of Lemma (2.1) is completed.

If (W, w_0) is a pointed topological space, then every element σ of a multiplicative group $\pi_1(W, w_0)$ induces an authomorphism $h_\sigma: \pi_k(W, w_0) \to \pi_k(W, w_0)$ of an additive $(k \geqslant 2)$ group $\pi_k(W, w_0)$ (see [5], Theorem (14.1) of Chapter IV or [13] p. 379).

It is well known that for every pointed connected CW complex (W, w_0) and $k > 1$ the group $\pi_k(W, w_0)$ is a left $Z(\pi_1(W, w_0))$-module where

$$z\alpha = n_1 h_{\sigma_1}(\alpha) + n_2 h_{\sigma_2}(\alpha) + \ldots + n_1 h_{\sigma_1}(\alpha)$$

for $\alpha \in \pi_k(W, w_0)$ and $z = n_1 \sigma_1 + n_2 \sigma_2 + \ldots + n_1 \sigma_1 \in Z(\pi_1(W, w_0))$.

We recall ([7]) that if M is a left R-module and if there exists a $B \subset M$ such that for every $m \in M$ we have

$$m = r_1 b_1 + \ldots + r_k b_k \quad \text{where} \quad r_i \in R \text{ and } b_i \in B \text{ for } i = 1, 2, \ldots, k \tag{2.6}$$

and such that presentation (2.6) is unique, then M is said to be a *free R-module* and B is said to be a *basis* for M.

Let $(X, x_0) \underset{\text{top}}{+} (Y, y_0) = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ for all pointed compacta (X, x_0) and (Y, y_0) ([2] p. 136).

We shall also use the following

(2.7) LEMMA. *Let $k \geqslant 2$ and $(Y, y_0) = (X, x_0) \underset{\text{top}}{+} (S^k, s_0)$ where (X, x_0) is a pointed connected CW complex with $\pi_i(X, x_0) = 0$ for every $i = 2, 3, \ldots, k$. Then $\pi_k(Y, y_0)$ is a free left $Z(\pi_1(Y, y_0))$-module and the basis of $\pi_k(Y, y_0)$ consists of one element $\varepsilon \in \pi_k(Y, y_0)$.*

Proof. Let $p: (\tilde{X}, x) \to (X, x_0)$ be a universal covering projection for (X, x_0). It is easy to check that a map $q: (\tilde{Y}, y) = (X \times \{s_0\} \cup p^{-1}(x_0) \times S^k, (x, s_0)) \to (Y, y_0)$ given by the formula

$$q(x) = \begin{cases} (p(y), s_0) & \text{for} \quad x = (y, s_0) \in X \times \{s_0\}, \\ (y, s) & \text{for} \quad x = (y, s) \in p^{-1}(x_0) \times S^k \end{cases}$$

is a universal covering projection for (Y, y_0).

Using the fact that $q_*^k: \pi_k(Y, y) \to \pi_k(Y, y_0)$ is an isomorphism one can easily verify that for every $g \in \pi_k(Y, y_0)$ there exists an $n_1 \sigma_1 + \ldots + n_1 \sigma_1 \in Z(\pi_1(Y, y_0))$ such that

$$g = (n_1 \sigma_1 + \ldots + n_1 \sigma_1)\varepsilon$$

where ε is an element of $\pi_k(Y, y_0)$ which is induced by a map $\alpha\colon (S^k, s_0)\to(Y, y_0)$ defined by the formula

$$\alpha(s) = (x_0, s) \quad \text{for every } s\in S^k .$$

It is clear that $z_1\varepsilon \neq z_2\varepsilon$ for all $z_1, z_2 \in Z(\pi_1(Y, y_0))$ such that $z_1 \neq z_2$.

3. Main results. Let us prove the following

(3.1) THEOREM. *If* $\mathrm{Fd}(X) = 1$ *and* $Y\in\mathcal{F}$, *then* $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(Y)+1$.

Before beginning the proof of Theorem (3.1) we have to show the following lemma:

(3.2) LEMMA. *Let* (K, k_0), (W, w_0) *be a finite pointed connected* CW *complexes,* $s_0\in S^2$, *and let* $f\colon (W, w_0)\to(K\times S^2, (k_0, s_0))$ *be a map. The following conditions are equivalent*:

(a) $\omega(f)\leqslant 2$,

(b) *there exists a homotopy* $\varphi\colon W\times[0, 1]\to K\times S^2$ *such that*

$$\varphi(x, t) = f(x) \quad \textit{for every } (x, t)\in W\times\{0\}\cup\{w_0\}\times[0, 1]$$

and

$$\varphi(W\times\{1\})\subset K^{(2)}\times\{s_0\}\cup\{k_0\}\times S^2 .$$

Proof. It is evident that (b)⇒(a).

Now let us assume that condition (a) is satisfied. S^2, K and $K\times S^2$ are finite CW complexes and $(S^2)^{(0)} = (S^2)^{(1)} = \{s_0\}$ and

$$(K\times S^2)^{(2)} = K^{(2)}\times\{s_0\}\cup K^{(2)}\times(s_0)\cup K^{(0)}\times S^2 .$$

From the definition of ω we infer that there is a homotopy $\varphi'\colon W\times[0, 1]\to K\times S^2$ such that

$$\varphi'(x, t) = f(x) \quad \text{for} \quad x\in W\times\{0\}\cup\{w_0\}\times[0, 1]$$

and

$$\varphi'(W\times\{1\})\subset(K\times S^2)^{(2)} .$$

It is easy to check that a homotopy $\varphi\colon W\times[0, 1]\to K\times S^2$ defined by the formula

$$\varphi(x, t) = \begin{cases}\varphi'(x, 2t) & \text{for } x\in W \text{ and } 0\leqslant t\leqslant \tfrac12 ,\\ \big(\psi(p_1\varphi'(x, 1), 2t-1), p_2\varphi'(x, 1)\big) & \text{for } x\in W \text{ and } \tfrac12\leqslant t\leqslant 1 ,\end{cases}$$

where $p_1\colon K\times S^2\to K$ and $p_2\colon K\times S^2\to S^2$ are projections and $\psi\colon K\times[0, 1]\to K$ is a homotopy which compress $K^{(0)}$ to k_0 and fixes k_0, satisfies the required conditions.

Proof of Theorem (3.1). It is sufficient to show that $\mathrm{Fd}(X\times S^2) = 3$ (see Corollary (1.6)).

For simplicity, we will assume that $(X, x_0) = \underline{\lim}\{(X_k, x_k), p_k^{k+1}\}$ where (X_k, x_k) is a finite pointed connected CW complex with $\dim X_k\leqslant 1$.

This implies that $(X\times S^2, (x_0, s_0)) = \underline{\lim}\{(X_k\times S^2, (x_k, s_0)), p_k^{k+1}\times\mathrm{id}_{S^2}\}$.

Let us suppose that $\mathrm{Fd}(X\times S^2) = 2$.

From Theorem (1.3) and Lemma (3.2) we infer that for every k there exist a $k'>k$ and a homotopy $\varphi\colon (X_{k'}\times S^2)\times[0, 1]\to X_k\times S^2$ such that

$$\varphi(y, t) = (p_k^{k'}(x), s) \quad \text{for} \quad (y, t) = ((x, s), t)\in(X_{k'}\times S^2)\times\{0\}\cup\{(x_k, s_0)\}\times[0, 1]$$

and

$$\varphi(Y_1\times\{1\})\subset X_k\times\{s_0\}\cup\{x_k\}\times S^2 = (X_k, x_k)\underset{\text{top}}{+}(S^2, s_0) = (Y_2, y_2)$$

and such that $(p_k^{k'})_\#\colon \pi_1(X_{k'}, x_{k'})\to\pi_1(X_k, x_k)$ is a non-trivial homomorphism, where $(Y_1, y_1) = (X_{k'}\times S^2, (x_k, s_0))$.

Let $\varepsilon_1\in\pi_2(Y_1, y_1)$ be the generator of $\pi_2(Y_1, y_1)$ and let ε_2 be an element of $\pi_2(Y_2, y_2)$ such that $\{\varepsilon_2\}$ is a basis for $Z(\pi_1(Y_2, y_2))$-module $\pi_2(Y_2, y_2)$ (see Lemma (2.7)).

Setting

$$q(x, s) = \varphi((x, s), 1) \quad \text{for} \quad (x, s)\in Y_1 = X_{k'}\times S^2 ,$$

we obtain a map $q\colon (Y_1, y_1)\to(Y_2, y_2)$.

It is clear that the homomorphism $q_\#\colon \pi_1(Y_1, y_1)\to\pi_1(Y_2, y_2)$ is non-trivial and that the homomorphism $q^2_*\colon \pi_2(Y_1, y_1)\to\pi_2(Y_2, y_2)$ is a monomorphism.

We have

$$q^2_*(\varepsilon_1) = (n_1\sigma_1+\ldots+n_k\sigma_k)\varepsilon_2 ,$$

where $0 \neq n_1\sigma_1+\ldots+n_k\sigma_k\subset Z(\pi_1(Y_2, y_2))$.

Let us denote by τ an element of $\pi_1(Y_1, y_1)$ such that $a = q_\#(\tau)$, is a non-trivial element of $\pi_1(Y_2, y_2)$.

Since $\pi_1(Y_2, y_2)$ is isomorphic with a free group $\pi_1(X_k, x_k)$, we conclude that $q_\#(\tau^s) = a^s$ is a non-trivial element of $\pi_1(Y_2, y_2)$ for every $s = 1, 2, \ldots$

Lemma (2.1) implies that there exists a natural number $s_0\leqslant k$ such that

$$(1e-1a^{s_0})(n_1\sigma_1+\ldots n_k\sigma_k) \neq 0$$

(e is the unit of $\pi_1(Y_2, y_2)$).

It follows that

$$n_1\sigma_1+n_2\sigma_2+\ldots+n_k\sigma_k \neq n_1q_\#(\tau^{s_0})\sigma_1+\ldots+n_kq_\#(\tau^{s_0})\sigma_k$$

and

$$q^2_*(\varepsilon_1) = (n_1\sigma_1+n_2\sigma_2+\ldots+n_k\sigma_k)\varepsilon_2 \neq \big(n_1q_\#(\tau^{s_0})\sigma_1+\ldots+n_kq_\#(\tau^{s_0})\sigma_k\big)\varepsilon_2 .$$

On the other hand,

$$q^2_*(\varepsilon_1) = q^2_*(\tau^{s_0}\varepsilon_1) = \big(1q_\#(\tau^{s_0})\big)\cdot q^2_*(\varepsilon_1) = 1a^{s_0}\cdot q^2_*(\varepsilon_1)$$
$$= n_1q_\#(\tau^{s_0})\sigma_1+\ldots+n_kq_\#(\tau^{s_0})\sigma_k .$$

Thus the proof is finished.

In this section, for every space X and for every map $f\colon X\to Y$ we denote by $H_n(X)$ the n-dimensional singular homology group of X with integer coefficients and by $(\widehat{f})_n\colon H_n(X)\to H_n(Y)$ the homomorphism which is induced by f.

Let (X, x_0), (Y, y_0) be connected pointed CW complexes and let $p\colon (\tilde{X}, a_0)\to(X, x_0)$, $q\colon (\tilde{Y}, b_0)\to(Y, y_0)$ be universal covering projections. Then for every map $f\colon (X, x_0)\to(Y, y_0)$ we will denote by $\tilde{f}\colon (\tilde{X}, a_0)\to(\tilde{Y}, b_0)$ the (unique) lifting of $fp\colon (\tilde{X}, a_0)\to(Y, y_0)$.

Let $\mathfrak{W}_0$ be a category whose objects are connected pointed CW complexes and whose morphisms are homotopy classes (in the pointed sense) of maps. It is well known that the tilde $\sim$ induces a functor from $\mathfrak{W}_0$ to $\mathfrak{W}_0$. This functor assigns to every object (W, w_0) of $\mathfrak{W}_0$ its universal covering space $(\tilde{W}, w)$ and to every morphism $[f]$ of $\mathfrak{W}_0$ represented by a map $f\colon (W, w_0)\to(V, v_0)$ the homotopy class $[\tilde{f}]$ of $\tilde{f}\colon (\tilde{W}, w)\to(\tilde{V}, v)$.

As an immediate consequence of this fact we obtain the following

(3.3) Lemma. *If* $\underline{X} = \{X_k, p_k^{k+1}\}$ *and* $\underline{Y} = \{Y_k, q_k^{k+1}\}$ *are sequences of polyhedra and* $\operatorname{Sh}(\varprojlim\{X_k, p_k^{k+1}\}) = \operatorname{Sh}(\varprojlim\{Y_k, q_k^{k+1}\})$, *then*

$$\underline{H} = \{H_n(\tilde{X}_k), \widehat{(\widetilde{p_k^{k+1}})_n}\} \quad \text{and} \quad \{H_n(Y_k), \widehat{(\widetilde{q_k^{k+1}})_r}\} = \underline{G}$$

are isomorphic progroups for every $n = 0, 1, 2, \ldots$

Now let us prove the following

(3.4) Theorem. *If* (X, x_0) *is not approximatively* 2*-connected pointed continuum and* $\operatorname{Fd}(X) = 2$, *then* $\operatorname{Fd}(X\times Y) = \operatorname{Fd}(X)+\operatorname{Fd}(Y)$ *for every* $Y\in\mathcal{F}$.

Proof. Without loss of generality we may assume that (X, x_0) is the inverse limit of an inverse sequence of 2-dimensional polyhedra. $\{(X_k, x_k), p_k^{k+1}\}$.

Let $\gamma_k\colon (\tilde{X}_k, a_k)\to(X_k, x_k)$ be a universal covering projection for every $k = 1, 2, \ldots$

We know that $(X\times S^2, (x_0, s_0)) = \varprojlim\{(X_k\times S^2, (x_k, s_0)), p_k^{k+1}\times \mathrm{id}_{S^2}\}$ and that $\gamma_k\times \mathrm{id}_{S^2}\colon (\tilde{X}\times S^2, (a_k, s_0))\to(X_k\times S^2, (x_k, s_0))$ is a universal covering projection for every $k = 1, 2, \ldots$

It is clear that for every $k = 1, 2, \ldots$ we have

$$(\gamma_k)^2_*(p_k^{k+1})^2_* = (p_k^{k+1})^2_*(\gamma_{k+1})^2_*$$

and

$$\Theta_2^k(p_k^{k+1})^2_* = \widehat{(\widetilde{p_k^{k+1}})_2}\circ\Theta_2^k$$

where $\Theta_2^i\colon \pi_2(\tilde{X}_i, a_i)\to H_2(\tilde{X}_i)$ is the Hurewicz homomorphism.

This means that the pair $\alpha = (\mathrm{id}_N, \Theta_2^k)$ is a morphism from a progroup $\{\pi_2(X_k, x_k), (p_k^{k+1})_*\}$ to a progroup $\underline{H}' = \{H_2(\tilde{X}_k), \widehat{(\widetilde{p_k^{k+1}})_2}\}$, where N denotes the set of natural numbers.

Since $\Theta_2^k\colon \pi_2(X_k, a_k)\to H_2(\tilde{X}_k)$ is an isomorphism, we conclude that α is an isomorphism of progroups and $\underline{H}'$ is not trivial.

The Künneth theorem for singular homology ([13] p. 235) implies that the progroups $\{H_4(\tilde{X}_k\times S^2), \widehat{(\widetilde{p_k^{k+1}}\times \mathrm{id}^{S^2})_4}\} = \underline{H}''$ and $\underline{H}'$ are isomorphic.

Therefore

(3.5) $\underline{H}''$ is not a trivial progroup.

The hypothesis that $\operatorname{Fd}(X\times S^2)\leqslant 3$ implies that $X\times S^2$ has the same shape as the inverse limit of an inverse sequence $\{Y_k, q_k^{k+1}\}$ of 3-dimensional polyhedra. From Lemma (3.4) we infer that the progroups $\underline{H}''$ and $\underline{H}''' = \{H_4(\tilde{Y}_k), \widehat{(\widetilde{q_k^{k+1}})_4}\}$ are isomorphic.

Since $\dim\tilde{Y}_k\leqslant 3$ and $H_4(\tilde{Y}_k) = 0$, we conclude that $\underline{H}''$ and $\underline{H}'''$ are trivial progroups, in contradiction to (3.5). Thus the proof of (3.4) is finished.

Theorems (1.5), (3.1) and (3.4) give the following

(3.6) Theorem. *If* X *is a compactum and* $\operatorname{Fd}(X)\neq 2$ *or* $\operatorname{Fd}(X) = 2$ *and* X *is not approximatively* 2*-connected, then* $\operatorname{Fd}(X\times Y) = \operatorname{Fd}(X)+\operatorname{Fd}(Y)$ *for every* $Y\in\mathcal{F}$.

Remark. The last theorem partially answers the question of K. Borsuk whether $\operatorname{Fd}(X\times S^n) = \operatorname{Fd}(X)+n$ for $X\neq\emptyset$ (see [1] and compare [11]).

It is known ([9] pp. 219 and 220) that if M is a PL n-manifold or a topological n-manifold with $n\geqslant 6$ and a compactum X is a proper subset of M, then $\operatorname{Fd}(X)\leqslant n-1$.

Let M be a topological n-manifold and let $X\subsetneqq M$ be a compactum such that $\operatorname{Fd}(X) = n$, where $n = 4.5$. Then $X\times S^6\subsetneqq M\times S^6$ and $\operatorname{Fd}(X\times S^6)\leqslant n+5$. On the other hand, Theorem (3.6) implies that $\operatorname{Fd}(X\times S^6) = n+6$.

We get the corollary

(3.7) Corollary. *If* M *is a topological* n*-manifold and a compactum* X *is a proper subset of* M, *then* $\operatorname{Fd}(X)<n$.

Remark. Corollary (3.7) answers Problem (3.11) of [9].

4. Final remarks and problems. In this section we shall prove the following theorem:

(4.1) Theorem. *Let* X *be a continuum with* $\operatorname{Fd}(X)<\infty$. *Then the following conditions are equivalent*:

(a) $\operatorname{Fd}(X\times Y) = \operatorname{Fd}(X)+\operatorname{Fd}(Y)$ *for every* $Y\in\mathcal{F}$.

(b) $c[X] = \operatorname{Fd}(X)$.

Proof. (a)$\Rightarrow$(b). We can assume that $X = \varprojlim\{X_k, p_k^{k+1}\}$, where X_k is a polyhedron and $\dim X_k = \operatorname{Fd}(X) = \dim X = n$.

It follows from Theorem (1.8) that there exists a generalized local system of coefficients $\underline{\mathcal{L}} = (\{X_k\times S^3, p_k^{k+1}\times\mathrm{id}_{S^3}\}, \mathcal{L}_k')$ on $X\times S^3$ such that

(4.2) $$H^{n+3}(X\times S^3; \underline{\mathcal{L}})\neq 0\,.$$

Let $s_0\in S^3$ and let $\mathcal{K}_k$ be a local system on $X_k\times\{s_0\} = Y_k$ which is a restriction of $\mathcal{L}_k$ to Y_k.

Setting

$$q_k^{k+1}(x, s_0) = (p_k^{k+1}(x), s_0) \quad \text{for every } x \in X_k$$

we get a map $q_k^{k+1}\colon Y_{k+1} \to Y_k$.

Let $\alpha_k\colon X_k \times S^3 \to Y_k$ be a map defined by the formula

$$\alpha_k(x, s) = (x, s_0) \quad \text{for every } (x, s) \in X_k \times S^3$$

and let $\mathscr{L}'_k$ be a local system induced on $X_k \times S^3$ by α_k and $\mathscr{K}_k$.

It is clear that $\underline{\mathscr{K}} = (\{Y_k, q_k^{k+1}\}, \mathscr{K}_k)$ and $\underline{\mathscr{L}'} = (\{X_k \times S^3, p_k^{k+1} \times \mathrm{id}_{S^3}\}, \mathscr{L}'_k)$ are generalized local systems of abelian groups on X and $X \times S^3$.

Let us assume that $H^n(X; \underline{\mathscr{K}}) = 0$.

Simce S^3 is simply connected, for every path $\tau\colon [0, 1] \to X_k \times S^3$ from $(x, s_0) \in Y_k$ to $(y, s_0) \in Y_k$ the isomorphism $\mathscr{L}_k(\tau)\colon (\mathscr{L}_k)_{(x,s_0)} \to (\mathscr{L}_k)_{(y,s_0)}$ is the same as the isomorphism $\mathscr{L}_k(\alpha_k \tau)\colon (\mathscr{L}_k)_{(x,s_0)} \to (\mathscr{L}_k)_{(y,s_0)}$, where $(\mathscr{L}_k)_{(x,s_0)}$ and $(\mathscr{L}_k)_{(y,s_0)}$ are groups of $\mathscr{L}_k$ which correspond to the points (x, s_0) and (y, s_0) and $\mathscr{L}_k(\tau)$ and $\mathscr{L}_k(\alpha_k \tau)$ are isomorphisms which are induced by τ and $\alpha_k \tau$.

This means that $\mathscr{L}_k$ and $\mathscr{L}'_k$ are canonicaly equivalent and that the groups $H^{n+3}(X \times S^3; \underline{\mathscr{L}})$ and $H^{n+3}(X \times S^3; \underline{\mathscr{L}'})$ are isomorphic.

Condition (4.2) implies that $H^{n+3}(X \times S^3; \underline{\mathscr{L}'}) \neq 0$.

On the other hand, in the proof of Theorem (1.5) (see [11]) a more general case is considered and it is proved that

$$H^{n+3}(X \times S^3; \underline{\mathscr{L}'}) \approx H^n(X; \underline{\mathscr{K}}) \otimes H^3(S^3; Z) = 0 .$$

Therefore $H^n(X; \underline{\mathscr{K}}) \neq 0$.

The proof of Theorem (1.5) (see [11]) contains also the proof of the implication (b)$\Rightarrow$(a).

The proof of Theorem (4.1) is finished.

The following corollary is an immediate consequence of Theorems (3.6) and (4.1).

(4.3) Corollary. *Let X be a continuum with* $\mathrm{Fd}(X) < \infty$. *Then* $c[X] \leqslant \mathrm{Fd}(X) \leqslant \max(2, c[X])$. *If* $\mathrm{Fd}(X) \neq 2$ *or* $\mathrm{Fd}(X) = 2$ *and X is not approximatively 2-connected, then* $\mathrm{Fd}(X) = c[X]$.

Let us formulate some problems.

(4.4) Problem. *Is it true that there exists an approximatively 2-connected continuum X such that $c[X] < \mathrm{Fd}(X)$?*

(4.5) Problem. *Is it true that* $\mathrm{Fd}(X) = c[X]$ *for every continuum X with* $\mathrm{Fd}(X) < \infty$?

(4.6) Problem. *Is it true that* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ *if X is a continuum and $Y \in \mathscr{F}$.*

A positive answer to Problem (4.4) would give negative answers to Problems (4.5) and (4.6).

It is also clear that if X is an approximatively 2-connected continuum and $c[X] < \mathrm{Fd}(X) = 2$, then $\mathrm{Fd}(X \times S^1) = 2$ and $\mathrm{Fd}(X \times Y) < \mathrm{Fd}(X) + \mathrm{Fd}(Y)$ for every $Y \in \mathscr{F}$.

References

[1] K. Borsuk, *On several problems of the theory of shape*, Proc. Topology Conference at Charlotte 1974, Studies in Topology, pp. 67–79.
[2] — *Theory of Shape*, Warszawa 1975.
[3] J. Dydak, *A generalization of cohomotopy groups*, Fund. Math. 90 (1975), pp. 77–98.
[4] P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge 1960.
[5] S. T. Hu, *Homotopy Theory*, New York 1959.
[6] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941.
[7] J. Lambek, *Lectures on Rings and Modules*, London 1966.
[8] S. Mardešić, *On the Whitehead theorem in shape theory I*, Fund. Math. 91 (1976), pp. 51–64.
[9] S. Nowak, *Some properties of fundamental dimension*, Fund. Math. 85 (1974), pp. 211–227.
[10] — *On the fundamental dimension of approximatively 1-connected compacta*, Fund. Math. 89 (1975), pp. 61–79
[11] — *On the fundamental dimension of the Cartesian product of two compacta*, Bull. Acad. Polon. Sci. 24 (1976), pp. 1021–1028.
[12] П. С. Понтрягин, *Непрерывные Группы*, Москва 1973.
[13] E. H. Spanier, *Algebraic Topology*, New York 1966.
[14] S. Spież, *On characterization of shapes of several compacta*, Bull. Acad. Polon. Sci. 24 (1976), pp. 225–263.
[15] N. E. Steenrod, *The Topology of Fibre Bundles*, Princeton 1951.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW
INSTYTUT MATEMATYKI, UNIWERSYTET WARSZAWSKI

Accepté par la Rédaction le 27. 9. 1976