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Equivariant embeddings of Z_p-actions in euclidean space

by

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Abstract. This paper shows that a finite dimensional compact metric space on which Z_p acts freely outside the fixed point set equivariantly embeds in a euclidean space with an orthogonal Z_p-action. Moreover, a minimum dimension for the euclidean space is obtained.

1. Introduction. Mostow [6] first showed that every action of a compact Lie group with a finite number of non-conjugate isotropy subgroups on a finite dimensional, separable, metrizable space can be equivariantly embedded in a linear action of the group on some euclidean space. In the case that the group is Z_p, the embedding has a particularly simple form, which is all that is required for the purposes of this paper. First, embed X in R^{2n+1} via i, and, then, embed X equivariantly in $R^{(2n+1)p}$ via $ex = (ix, iax, \ldots, ia^{p-1}x)$, where $a \in Z_p$ and where $\sigma(x_1, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$ generates an orthogonal Z_p-action on $R^{(2n+1)p}$. However, Mostow's theorem said nothing as to the required dimensions of the euclidean space. Copeland and de Groot [2] went on to show that every action of a cyclic group of prime order on an n-dimensional, separable, metrizable space can be equivariantly embedded in a linear action on R^{3n+2} or R^{3n+3}. Finally, Kister and Mann [5] extended the result of Copeland and de Groot to actions of compact Abelian Lie groups with a finite number of distinct isotropy subgroups. They found a dimension for a euclidean space appropriate for the embedding which depends only upon the dimension of the original space, the structure of the Abelian transformation group, and the number of distinct isotropy subgroups.

In the present work improvements on the result of Copeland and de Groot are obtained in the case of a compact, finite dimensional metric space with an action of a cyclic group. In particular, let X be a compact n-dimensional metric space with a map $a\colon X \to X$ of period p whose fixed point set is F. The map a then defines a Z_p-action on X. In this paper, (X, a) will denote the equivariant space (X, Z_p). Suppose this action is free outside of F and suppose F is embeddable in k-dimensional euclidean space, R^k, via an embedding w. In the case of an involution (i.e., $a^2 = 1_X$), let $m = \max\{k, n\}$ and $\alpha\colon R^{n+1} \times R^m \to R^{n+1} \times R^m$, where $\alpha = (\alpha_1, 1_{R^m})$ and $\alpha_1\colon R^{n+1} \to R^{n+1}$ is defined by $\alpha_1(x) = -x$. The following theorem is then proved.

THEOREM (1.1). *(X, a) equivariantly embeds in $(\mathbf{R}^{n+1+m}, \alpha)$ via an embedding which extends w. Furthermore, the set of such embeddings which coincide with w on F forms a dense subset of the space of all continuous equivariant maps from (X, a) into $(\mathbf{R}^{n+1+m}, \alpha)$ coinciding with w on F.*

If the map a has period $p>2$, the cases of n odd and of n even must be distinguished. If n is odd (even), let $m = \max\{k, n\}$ ($m' = \max\{k, n-1\}$). Define β_2: $\mathbf{R}^2 \to \mathbf{R}^2$ as the rotation about the origin through the angle $2\pi/p$ and define β_1: $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ (β_1': $\mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$) as $\beta_1 = (\beta_2, \ldots, \beta_2)$, the $\frac{1}{2}(n+1)$-fold ($\frac{1}{2}(n+2)$-fold) product of β_2. Finally, let

$$\beta = (\beta_1, 1_{R^m})\colon \mathbf{R}^{n+1} \times \mathbf{R}^m \to \mathbf{R}^{n+1} \times \mathbf{R}^m ((\beta_1', 1_{R^{m'}})\colon \mathbf{R}^{n+2} \times \mathbf{R}^{m'} \to \mathbf{R}^{n+2} \times \mathbf{R}^{m'}).$$

The theorem stated below is proved.

THEOREM (1.2). *(X, a) equivariantly embeds in $(\mathbf{R}^{n+1+m}, \beta)$ $((\mathbf{R}^{n+2+m'}, \beta'))$ via an embedding which extends w. Furthermore, the set of such embeddings which coincide with w on F forms a dense subset of the space of all continuous equivariant maps from (X, a) into $(\mathbf{R}^{n+1+m}, \beta)$ $((\mathbf{R}^{n+2+m'}, \beta'))$ coinciding with w on F.*

The approach here is similar to the classical proof of Menger–Nobeling–Hurewicz [3] which shows that one can always embed a separable n-dimensional space in $\mathbf{R}^{2n+1}$. Essential to the proof is Jaworowski's idea [4] to replace $X-F$ by an n-dimensional, locally finite, equivariant polyhedron, K, without fixed points. One then equivariantly embeds K in $\mathbf{R}^{n+1+m}$ ($\mathbf{R}^{n+2+m'}$) in a way compatible with the embedding of F in $\mathbf{R}^k \subset \mathbf{R}^m (\mathbf{R}^k \subset \mathbf{R}^{m'})$.

The fact that for period greater than two, the dimension of X being odd or even plays an added role, is due essentially to the existence of a free orthogonal $\mathbf{Z}_2$-action on a sphere of any dimension, whereas free orthogonal $\mathbf{Z}_p$-actions for $p>2$ exist only on odd-dimensional spheres.

2. Equivariant spaces and equivariant maps. Frequently, (X, a) is called a $\mathbf{Z}_p$-space. An equivariant map f: $(X, a) \to (Y, b)$ between two $\mathbf{Z}_p$-spaces is an equivariant ε-map if $\operatorname{diam} f^{-1} y < \varepsilon$ for every $y \in fX$. If (X, a) is a compact metric $\mathbf{Z}_p$-space and (Y, b) is a separable metric $\mathbf{Z}_p$-space, then $(Y, b)^{(X,a)}$ is the subspace of the metric space Y^X (with metric defined by $d(f, g) = \sup_{x \in X} d(fx, gx)$) consisting of all equivariant maps from (X, a) to (Y, b). In fact, since $(Y, b)^{(X,a)}$ is closed in Y^X, then $(Y, b)^{(X,a)}$ is complete.

In the following, if (Y, b) is a $\mathbf{Z}_p$-space, then $y^* = \{y, by, \ldots, b^{p-1}y\}$ is called the *orbit of* y, and $S^* = \bigcup_{j=0}^{p-1} b^j S$ is called the *orbit of* S, where y is an element in Y, and S is a subset of Y. A subset S of Y is called *sectional* if $S \cap y^* = \{y\}$ for each y in S, and any one-to-one function χ: $(Y/\mathbf{Z}_p) \to Y$ is called a *section.*

(2.1), which is stated here and is used in proving (2.2) below, can be found in Jaworowski [4, p. 235].

COVERING LEMMA (2.1). *Let (X, a) be a compact metric $\mathbf{Z}_p$-space and let A be an equivariant closed subspace of X such that $\mathbf{Z}_p$ acts freely outside of A. Suppose C is an equivariant open cover of $X-A$. Then there exists a countable, locally finite, equivariant, open cover B of $X-A$ which is a refinement of C and which satisfies the following:*

(i) $\lim_{d(V,A)\to 0} (\operatorname{diam} \operatorname{St} V) = 0$ *for* $V \in B$;

(ii) *If* $V \in B$, *the* $\operatorname{Cl} V \subset X-A$;

(iii) *Every neighborhood of A in X contains all but a finite number of elements of B;*

(iv) *For every $V \in B$, the sets $\operatorname{St}_B V, a(\operatorname{St}_B V), \ldots, a^{p-1}(\operatorname{St}_B V)$ are mutually disjoint;*

(v) *If* $\dim(X-A) \leqslant n$, *the* $\operatorname{Ord} B \leqslant p(n+1)-1$;

(Concerning the notation $\operatorname{Ord} B$, *compare* [1].)

(vi) *If ε is a given positive number, then B can be chosen such that mesh$B < \varepsilon$.*

POLYHEDRAL REPLACEMENT LEMMA (2.2). *Let (X, a) be a compact metric $\mathbf{Z}_p$-space and let A be an equivariant, closed subspace such that $\mathbf{Z}_p$ acts freely outside A. For a given positive number ε, there exists a compact Hausdorff $\mathbf{Z}_p$-space (Z, c) such that:*

(i) *Z contains A as an equivariant, closed subspace and* $c|_A = a|_A$;

(ii) *there exists a countable, locally finite, simplicial complex K and simplicial map b: $K \to K$ of period p with $|K| = Z-A$ and with $c|_{|K|}$: $|K| \to |K|$ a free simplicial map of period p;*

(iii) *an equivariant ε-map f: $(X, a) \to (Z, c)$ such that $f|_A = 1_A$, $f(X-A) \subset |K|$, and $f^{-1}\{\operatorname{St}(V) | V \in K^0\}$ forms a locally finite, equivariant, open cover of $X-A$ of mesh less than ε; and*

(iv) *if $\dim(X-A) \leqslant n$, then* $\dim K \leqslant n$.

Remarks. Lemma (2.2) is a modification of Lemma 4.7 in [4, p. 237]. The space Z, the polyhedron K, and the equivariant map f are all constructed as in [4]. The fact that f can be shown to be an ε-map follows from using an equivariant open cover of mesh less than ε in the construction of K. Such a cover is guaranteed by Lemma (2.1). The result that $\dim K \leqslant n$ when $\dim(X-a) \leqslant n$ is obtained essentially through a finite number of modifications of the map f found in [4]. The modifications are made by utilizing results on stable and unstable values found in Hurewicz and Wallman ([3] Theorem VI, 1, p. 75; Proposition B, p. 78). The details of such modifications are very technical and can be found in [1].

3. Equivariant general position. The main result of this section is the Equivariant General Position Lemma. Before stating it, several definitions and a preliminary lemma are needed. Given a set S in $\mathbf{R}^N$, $L(S)$ denotes the affine span of S in $\mathbf{R}^N$. Let C be a convex body in $\mathbf{R}^N$ (i.e., C is closed, convex and has a nonempty interior in $\mathbf{R}^N$). If S is a subset of C, then $L_C(S) = L(S) \cap C$ and $L_C(S)$ is the affine span of S in C. Note that $L_{R^N}(S) = L(S)$ and $\dim(L_C(S \cup T)) \leqslant \dim(L_C(S)) + \dim(L_C(T)) + 1$.

DEFINITION (3.1). Let C be a convex set in $\boldsymbol{R}^N$ and let S and T be subsets of C. S is said to be in *T-position if* $L_C(S) \cap T = \emptyset$.

LEMMA (3.2). *Let C be a convex body in $\boldsymbol{R}^N$ and let T be a subset of C with $\dim(L_C(T)) = k$. Suppose S is a finite set of points in C such that any subset of S containing less than $N-k$ elements is in general position and in T-position. If U is an open subset of C, then*

(1) *$V = U - \bigcup\{L_C(S' \cup T)|\ S' \subset S;\ \#(S') \leqslant N-k-1\} \neq \emptyset$ and is open in C; and*

(2) *For any v in V and any $S' \subset S$ with $\#(S') \leqslant N-k-1$, $S' \cup \{v\}$ is in general position and in T-position.*

Proof. Let $S' \subset S$ and $\#(S') < N-k-1$. It follows that

$$\dim(L_C(S' \cup T)) \leqslant \dim(L_C(T)) + \#(S') \leqslant k+N-k-1 < N.$$

Define

$$P = \bigcup\{L_C(S' \cup T)|\ S' \subset S;\ \#(S') \leqslant N-k-1\}.$$

P is closed and nowhere dense in C; hence, $C-P$ is open and dense, which implies that $V = U \cap (C-P) \neq \emptyset$. Furthermore, by construction, any v in V satisfies (2) of the lemma.

DEFINITION (3.2). Let C be a convex set in $\boldsymbol{R}^N$ and let S and T be subsets of C. For a given positive integer q, S is said to be in *(q, T)-position* if every subset of S containing less than $(q+1)$-elements is in T-position.

DEFINITION (3.4). Suppose $(\boldsymbol{R}^N, \gamma)$ is a Z_p-space and $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\boldsymbol{R}^N, \gamma)$ where C is a convex set in $\boldsymbol{R}^N$. Let Q and T be equivariant subsets of C and let q be a positive integer.

(i) Q is said to be in *equivariant general position in C* if every sectional subset S of Q is in general position in C.

(ii) Q is said to be in *equivariant (q, T)-position* if every sectional subset of Q is in (q, T)-position.

(iii) Q is said to be in equivariant (q, T)-general position if Q is in equivariant general position and in equivariant (q, T)-position.

EQUIVARIANT GENERAL POSITION LEMMA (3.5). *Suppose γ: $\boldsymbol{R}^N \to \boldsymbol{R}^N$ is an isometric map of period p and $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\boldsymbol{R}^N, \gamma)$ where C is a convex body in $\boldsymbol{R}^N$. Let $Q = \{q_1, q_2, \ldots\}$ be a countable set; let φ: $Q \to C$ be a function; and let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers. If T is an equivariant closed subset of C and $\dim(L_C(T)) = k < N$, then there exists a function ψ: $Q \to C$ satisfying the following*:

(1) *$(\psi Q)^*$ is in equivariant $(N-k, T)$-general position in C; and*

(2) *$d(\varphi q_i, \psi q_i) < \varepsilon_i$.*

Proof. Let $B(\varphi q_i, \varepsilon_i) = \{y \in C|\ d(\varphi q_i, y) < \varepsilon_i\}$. Pick ψq_1 to be any point in $B(\varphi q_1, \varepsilon_1) \cap (C-T)$. Then $E_1 = \{\psi q_1, \gamma\psi q_1, \ldots, \gamma^{p-1}\psi q_1\} = \{\psi q_1\}^*$ satisfies conditions (1) and (2).

Assume $E_j = \{\psi q_i, \gamma\psi q_i, \ldots, \gamma^{p-1}\psi q_i|\ \ i = 1, \ldots, j\} = \{\psi q_i|\ \ i = 1, \ldots, j\}^*$ is defined and satisfies conditions (1) and (2). Let

$$P_1 = \bigcup\{L_C(S)|\ S \subset E_j;\ \#(S) \leqslant N\colon\ S \text{ is sectional}\}.$$

It follows that $P_1 = P_1^*$ is a closed, nowhere dense set in C. Therefore, $C-P_1^*$ is an equivariant, open, dense subset of C. If c is any element in $C-P_1^*$, then $E_j \cup c^*$ is in equivariant general position.

Let

$$P = \bigcup\{L_C(S \cup T)|\ S \subset E_j;\ \#(S) \leqslant N-k-1;\ S \text{ is sectional}\}.$$

As was the case above for P_1, it follows that $P = P^*$ is a closed, nowhere dense set in C. Therefore, $C-P^*$ is an equivariant, open, dense subset of C and $(C-P^*) \cap T = \emptyset$. Furthermore, $O = (C-P_1^*) \cap (C-P^*)$ is an open, dense subset of C which does not intersect T.

Let $D = B(\varphi q_{j+1}, \varepsilon_{j+1}) \cap (C-T)$. D is open in C; thus D^* is open in C. And, since γ is isometric, $D^* \cap T = \emptyset$. Hence, $U = O \cap D^*$ is open and equivariant. Let y be in D such that $y^* \subset U$. Define $\psi q_{j+1} = y$ and $E_{j+1} = E_j \cup y^*$. E_{j+1}, so constructed, satisfies (1) and (2).

Thus, using induction, a function ψ that satisfies the conclusion of the lemma can be constructed.

Lemma (3.5) is used in the next section to equivariantly embed equivariant polyhedra.

4. Polyhedral replacement embedding lemma.

LEMMA (4.1). *Let K be a countable, locally finite, n-dimensional, simplicial complex where b: $|K| \to |K|$ is a free, simplicial map of period p and $K^0 = (\{v_i\}_{i=1}^{\infty})^*$. Suppose γ: $\boldsymbol{R}^{N+k} \to \boldsymbol{R}^{N+k}$, $N > n$, is an isometric, linear map of period p and $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\boldsymbol{R}^{N+k}, \gamma)$, where C is a convex body in $\boldsymbol{R}^{N+k}$. Let $Q = \{q_1, q_2, \ldots\}$ be a countable set of points in C and let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers. If T is the fixed point set of γ_C (γ_C is free outside T) and $\dim(L_C(T)) = k \geqslant n$, then there exists an equivariant embedding h: $(|K|, b) \to (C-T, \gamma_C)$ such that $d(hv_i, q_i) \leqslant \varepsilon_i$.*

Proof. Let ψ: $Q \to C$ be a function that satisfies Lemma (3.5). Let $(\psi Q)^* = \{\psi q_i = r_i\}^*$. Define h: $K^0 \to C$ by $hb^j v_i = \gamma^j r_i$, $j = 0, 1, \ldots, p-1$. Extend h linearly to all the simplexes of K. It is clear that h is continuous since it is defined by continuous operations. Moreover, $d(hv_i, q_i) = d(r_i, q_i) = d(\psi q_i, q_i) < \varepsilon_i$. The details which show that h is one-to-one on $|K|$, and that $(h(|K|), \gamma_C)$ is an equivariant polyhedron in $(C-T, \gamma_C)$ can be found in [1].

The following notation pertains to (4.2) below. Let (X, a) be a compact metric Z_p-space of dimension $\leqslant n$, where Z_p acts freely outside of a closed equivariant subspace A. Suppose γ: $\boldsymbol{R}^{N+k} \to \boldsymbol{R}^{N+k}$, $N > n$, is an isometric linear map of period p and suppose $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\boldsymbol{R}^{N+k}, \gamma)$, where C is a convex body in R^{N+k}. Furthermore, let γ_C be free outside T, the fixed point set of γ_C, and let $\dim(L_C(T)) = k \geqslant n$.

If $w\colon A \to T$ is a fixed embedding and $g\colon (X, a) \to (C, \gamma_C)$ is an equivariant map such that $g|_A = w$, then, corresponding to a given positive number η, the uniform continuity of g implies that there exists a positive number δ such that, if $d(x, x') < \delta$, then $d(gx, gx') < \frac{1}{6}\eta$. In addition, corresponding to δ, let (K, b), $(Z = |K| \cup A, c)$, and $f\colon (X, a) \to (Z, c)$ be as in (2.2).

Finally, denote by $K^0 = (\{v_i\}_{i=1}^{\infty})^*$ and by $B^* = \{V_i\}^*$ the locally finite, equivariant, open cover of $X - A$, where K is generated by the nerve of B.

POLYHEDRAL REPRESENTATION EMBEDDING LEMMA (4.2). *There exists* $h\colon (Z, c) \to (C, \gamma_C)$ *such that*:

(i) *h is an equivariant embedding*;

(ii) $h|_A = w$;

(iii) *$h|_{Z-A}$ is a simplicial homeomorphism*; *and*

(iv) $d\big(\gamma^j h(v_i), g(f^{-1}(\mathrm{St}\, b^j v_i))\big) < \frac{1}{2}\eta$, *for each* $j = 0, \ldots, p-1$.

Proof. Define $D = \{V_i \in B|\ d(V_i, A) < \frac{1}{2}\delta\}$ and let $D' = B - D$.

For each $V_i \in D'$, choose $x_i \in V_i$. Then choose $p_i \in (C-T) \cap B(g(x_i), \varepsilon_i)$, where $\varepsilon_i = \frac{1}{4}\eta$ and where $B(g(x_i), \varepsilon_i)$ is the open ball of radius ε_i in C around $g(x_i)$. Similarly, $p_i^j = \gamma^j p_i \in (C-T) \cap B(\gamma^j g(x_i), \varepsilon_i)$ for each $j = 0, \ldots, p-1$.

For each $V_i \in D$ there exists $a_i \in A$ such that $d(V_i, A) = d(\overline{V}_i, A) = d(\overline{V}_i, a_i)$, and there exists $x_i \in V_i$ such that $d(x_i, a_i) < \delta$. Choose $p_i \in (C-T) \cap B(w(a_i), \varepsilon_i)$, where $\varepsilon_i = \min\{\frac{1}{2}d(\overline{V}_i, a_i), \frac{1}{6}\eta\}$. Similarly, $p_i^j = \gamma^j p_i \in (C-T) \cap B(\gamma^j w(a_i), \varepsilon_i)$ for each $j = 0, \ldots, p-1$.

By the Equivariant General Position Lemma (3.5), there exists a countable, equivariant set $\{q_i|\ i = 1, 2, \ldots\}^*$ in C with the property that $d(\gamma^j p_i, \gamma^j q_i) < \varepsilon_i$ for each $j = 0, \ldots, p-1$. Furthermore, the following inequalities hold:

(1) For $V_i \in D'$, $d(g(x_i), q_i) \leqslant d(g(x_i), p_i) + d(p_i, q_i) < \varepsilon_i + \varepsilon_i = \frac{1}{4}\eta + \frac{1}{4}\eta = \frac{1}{2}\eta$.

(2) For $V_i \in D$, $d(g(x_i), q_i) \leqslant d(g(x_i), w(a_i)) + d(w(a_i), p_i) + d(p_i, q_i)$
$$< \tfrac{1}{6}\eta + \tfrac{1}{6}\eta + \tfrac{1}{6}\eta = \tfrac{1}{2}\eta.$$

Finally, define $h\colon (Z, c) \to (C, \gamma_C)$ as follows: $h|_A = w$.

For each $j = 0, \ldots, p-1$, let $h(b^j v_i) = \gamma^j h(q_i)$ for the vertices $v_i \in K^0$. Then extend h linearly from K^0 to all of $|K| = Z - A$.

By definition, h is equivariant, $h|_A = w$, and $h|_{Z-A}$ is simplicial. (1) and (2) above imply that condition (iv) of the lemma is satisfied. On A, h is clearly one-to-one. Lemma (4.1) tells us that on $|K|$ h is an embedding into the complement of the fixed point set of γ. The details which show that h is continuous can be found in [1].

5. Function space lemmas. Results relating to some function spaces will be discussed here. For this discussion, (X, a), (C, γ_C), A, T and w are the same as in Section 4. Then, one defines

$$M = (C, \gamma_C)^{(X, a)};$$
$$M' = \{\varphi \in M|\varphi|_A = w\};$$
$$M'(\varepsilon) = \{\varphi \in M'|\ \varphi \text{ is an } \varepsilon\text{-map}\};$$

and

$$E = \{\varphi \in M'|\varphi \text{ is an embedding}\}.$$

(5.1) *M' is complete.*

Since M is complete and M' is closed in M, statement (5.1) follows.

Note that the map $f\colon (X, a) \to (Z, c)$ used in the introduction to (4.2) above was an equivariant δ-map. Consequently, given an $\varepsilon > 0$, δ could have been chosen less than ε from the beginning.

(5.2) *$M'(\varepsilon) \neq \emptyset$ and, hence, $M' \neq \emptyset$.*

Statement (5.2) is obtained by letting $k = h \circ f$, where h satisfies (4.2). Since f is an equivariant δ-map, and hence an ε-map from (X, a) to (Z, c), then $k \in M'(\varepsilon)$.

Suppose S_1 is the set of ε-maps in C^X and suppose S_2 is the set of embeddings in C^X. In [3, pp. 57–59] it is shown that S_1 is open and dense in C^X and that $S_2 = \bigcap_{\varepsilon} S_1$. By adjusting the proof in [3] using (2.2), one proves the following.

(5.3) *For every positive number ε, $M'(\varepsilon)$ is dense in M'.*

Furthermore, $M'(\varepsilon) = M' \cap S_1$ and this implies that

$$\bigcap_{\varepsilon} M'(\varepsilon) = \bigcap_{\varepsilon} (M' \cap S_1) = M' \cap \big(\bigcap_{\varepsilon} S_1\big) = M' \cap S_2 = E.$$

The next two statements follow from the above observation.

(5.4) *For every positive number ε, $M'(\varepsilon)$ is open in M'.*

(5.5) *$h \in \bigcap M'(\varepsilon)$ if and only if $h\colon (X, a) \to (C, \gamma_C)$ with $h|_A = w$ is an equivariant embedding. In particular, $\bigcap_{\varepsilon} M'(\varepsilon) = E$.*

Finally, using (5.5), (5.4), (5.3), and (5.1), the following result is established.

(5.6) *E is a dense G_δ set in M' and, hence, $E \neq \emptyset$.*

6. Proofs of embedding Theorems (1.1) **and** (1.2). In the case of (1.1), let $F = A$, $C = R^{n+1} \times R^m$, $T = \{0\} \times R^m$, and $\gamma = \alpha$. Then (1.1) is a corollary of (5.6). An immediate important consequence of (1.1) is the following analogue to the classical result on embeddings found in [3, p. 56].

COROLLARY (6.1). *If (X, a) is a compact n-dimensional metric Z_2-space with only one fixed point (or none), then (X, a) equivariantly embeds in (R^{2n+1}, α).*

In the case of (1.2), let $F = A$. If n is odd, let $C = R^{n+1} \times R^m$, $T = \{0\} \times R^m$, and $\gamma = \beta$; otherwise, let $C = R^{n+2} \times R^{m'}$, $T = \{0\} \times R^{m'}$, and $\gamma = \beta'$. Then (1.2) is also a corollary of (5.6). Similar to (6.1) for involutions is the following corollary for periodic maps of period p.

COROLLARY (6.2). *If* (X, a) *is a compact* n*-dimensional metric* Z_p*-space with one fixed point (or none) and with the action free outside the fixed point set, then* (X, a) *equivariantly embeds in* (R^{2n+1}, β) *if* n *is odd and in* (R^{2n+1}, β') *if* n *is even.*

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Some remarks concerning the fundamental dimension of the cartesian product of two compacta

by

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Abstract. It is proved that $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ if Y is an n-dimensional continuum such that $H^n(Y; G) \neq 0$ for every $G \neq 0$ and if X is a compactum and $\mathrm{Fd}(X) \neq 2$ or $\mathrm{Fd}(X) = 2$ and X is not approximatively 2-connected.

We will consider the problem of computing the fundamental dimension of $X\times Y$, where X and Y are compacta. From this point of view the notion of an $\mathcal{F}$*-continuum* will be very convenient. A continuum X with $0 \neq \mathrm{Fd}(X) = n<\infty$ belongs to a class $\mathcal{F}$ (in other words X is an $\mathcal{F}$*-continuum*) iff for every abelian group $G \neq 0$ the n-dimensional Čech cohomology group $H^n(X; G)$ of X with coefficients in G is non-trivial.

Using the universal coefficient theorem and the Künneth formula for homology and cohomology ([13] p. 244 and p. 336), one can check that the class $\mathcal{F}$ contains all connected n-dimensional ANR-sets with the non-trivial n-dimensional Čech homology group $H_n(X)$ over the group Z of integer numbers (in particular, all closed orientable manifolds) and that $X\times Y\in\mathcal{F}$ for all Γ, X, $Y\in\mathcal{F}$.

It is known ([10] p. 74) that there exists a sequence $G_1, G_2, \ldots$ of non-trivial countable abelian groups such that if X is an n-dimensional compactum with $H^n(X; G_k) = 0$ for every $k = 1, 2, \ldots$, then $H^n(X; G) = 0$ for every group G.

This fact together with the theorem which states ([6] p. 137) that for every countable group G and its character group G^* the group $H_n(X; G^*)$ is the character group of $H^n(X; G)$ and with the Pontriagin duality ([12] p. 259) imply that if X is an n-dimensional continuum and $H_n(X; G) \neq 0$ for every $G \neq 0$, then $X\in\mathcal{F}$ ($H_n(X; H)$ denotes the Čech homology group of X over H).

It is clear that the class $\mathcal{F}$ is closed with respect to the one-point union.

In [11] it is proved that $\mathrm{Fd}(X\times Y) = \mathrm{Fd}(X)+\mathrm{Fd}(Y)$ for every compactum X with $\mathrm{Fd}(X)\geqslant 3$ and every $Y\in\mathcal{F}$.

The purpose of this note is to generalize the last theorem and to show that the assumption that $\mathrm{Fd}(X)\geqslant 3$ may be replaced by the assumption that $\mathrm{Fd}(X) \neq 2$ or $\mathrm{Fd}(X) = 2$ and X is not approximatively 2-connected.