A necessary condition for embedding a complex in S^{2k+2}

by

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Abstract. Let us suppose that a topological space X is of the homotopy type of a subcomplex of S^{2k+2}. Then if $a \in \mathrm{KU}(X)$, then $\mathrm{ch}_k(a) \in H^{2k}(X, Q)$ is an integral cohomology class.

0. Introduction. Let X be a topological space which is of the homotopy type of a subcomplex of S^{2k+2}. Then the KU-group $\mathrm{KU}(X)$ and the Chern character $\mathrm{ch}\colon \mathrm{KU}(X) \to H^*(X; Q)$ can be defined.

Our main result is the following:

THEOREM 0.1. *If a topological space X is of the homotopy type of a subcomplex of S^{2k+2} then for all $a \in \mathrm{KU}(X)$ the cohomology class $\mathrm{ch}_k(a) \in H^{2k}(X; Q)$ is integral, (namely it belongs to the image of the map $H^*(X; Z) \to H^*(X; Q)$).*

The proof of this theorem is based on an old idea which has been used before (see [6], [9], [10]) which is to exploit the existence of the Spanier–Whitehead dual of X in S^{2k+2}. The existence of this dual imposes certain restrictions on X. In this paper those restrictions are related with the integrability of certain rational characteristic classes on manifolds.

As an application of Theorem 0.1, we have the following result:

PROPOSITION 0.2. *Let X, Y be topological spaces which are of the homotopy type of finite complexes. We assume that there is an a $\mathrm{KU}(X)$ such that $\mathrm{ch}_k(a) \in H^{2k}(X; Q)$ is not integral. Furthermore we assume that $H^*(Y; Z)$ is torsion-free and $H^{2l}(Y; Z) \neq 0$. Then $X \times Y$ is not of the homotopy type of a subcomplex of $S^{2k+2l+2}$.*

Note. From now on we adopt the following notational convention. If we have an object which can be defined for both integral and rational coefficients (i.e. homology and cohomology groups, groups of the type $\mathrm{Hom}(A, *)$ where A is an abelian group e.t.c.) then we denote by s the map induced by the obvious inclusion $Z \to Q$ if such a map is naturally defined. For example in this terminology a cohomology class is integral if and only if it belongs to the image of s.

Proof of Proposition 0.2. Let $x \in H^{2l}(Y; Z)$ such that x is not divisible. Since $H^*(Y; Z)$ is torsion-free then there exists $b \in \mathrm{KU}(Y)$ such that $\mathrm{ch}_l(b) = s(x)$

and $\mathrm{ch}_i(b) = 0$ for $i = 0, 1, 2, \ldots, l-1$ (see [1]). Then since the Chern character is multiplicative we have

$$\mathrm{ch}_{k+l}(a\otimes b) = \mathrm{ch}_k(a)\cdot s(x)+\mathrm{ch}_{k-1}(a)\cdot \mathrm{ch}_{l+1}(b)+\ldots+\mathrm{ch}_0(a)\cdot \mathrm{ch}_{k+l}(b)\,.$$

But by the universal coefficient theorem $H^*(X\times Y; Z) = H^*(X; Z)\otimes H^*(Y; Z)$ and of course the same isomorphism holds in rational coefficients. So the non-integrability of $\mathrm{ch}_k(a)$ together with the non-divisibility of $\mathrm{ch}_l(b) = s(x)$ implies that $\mathrm{ch}_{k+l}(a\otimes b)$ is non-integral. And by Theorem 0.1, the result follows.

DEFINITION 0.3. As a notational convenience we use the symbolism $X<S^{2k+2}$ to mean that the space X is of the *homotopy type of a subcomplex of* S^{2k+2}. And by $X\not< S^{2k+2}$ we symbolize the negation.

In order to give some specific applications of the above results, let l be the canonical line bundle over $\mathrm{CP}(2)$, and let $t\in H^2(\mathrm{CP}(2); Z)$ be the generator of the cohomology. It is well known that $\mathrm{ch}(l) = e^t$, so $\mathrm{ch}_2(l) = \frac{1}{2}t^2$, which is non-integral. This implies that $\mathrm{CP}(2)\not< S^6$. This result was already known (see [6]). Beyond that in [6] it was proved that $\mathrm{CP}(2)<S^7$. But our theory, because of Proposition 0.2, gives beyond this example many others non-embeddability examples.

Of course analogous results hold for all complex projective space $\mathrm{CP}(n)$, but for the higher dimensional ones probably there is a lot of room for improvement.

As another concrete application let X be the cell complex obtained by attaching a $(2n+4k)$-cell to S^{2n} by a map $f\colon S^{2n+4k-1}\to S^{2n}$ where f is in the class of the generator of the image of the J-homomorphism in the stable stem π_{4k-1}. Then it is known that there is a bundle $l\in \mathrm{KU}(X)$ such that the denominator of $\mathrm{ch}_{n+2k}(l)$ expressed as a fraction in lowest terms is M_k/b_k (see [4]). In the previous statement M_k is the denominator of the fraction $B_k/4k$, where B_k is the kth Bernoulli number expressed in lowest terms, and b_k is 1 or 2 as k is even or odd. But M_k/b_k is never ± 1 (see [8] p. 284) so $X\not< S^{2n+4k+2}$. And as previously by Proposition 0.2 we get new $\not<$ examples.

Essentially Theorem 0.1, is a "codimension" two result. It is not clear how it can be extended to greater "codimensions".

The paper has been arranged as follows. In the first section we prove some integrability results for characteristic classes of weakly almost complex manifolds. In the second section combining the results of Section 1 and Spanier–Whitehead duality we prove Theorem 0.1.

1. Integrability of characteristic classes. Let M be an m-dimensional closed connected C^∞ manifold with a complex structure on its stable tangent bindle. Following Hattori we will call such a manifold weakly almost complex manifold (see [5]) and in abbreviation w.a.c. manifold. For such a manifold we denote $\tau_M\colon M\to \mathrm{BU}$ the map classifying the stable tangent bundle. Note that BU is $\mathrm{BU}(N)$, the classifying space of N-dimensional complex bundles, where N is very big in comparison with m.

DEFINITION 1.1. We define $I_m^{2k} = \{x\in H^{2k}(\mathrm{BU}; Q)\colon \tau_M^*(x)\in H^{2k}(M; Q)$ is integral cohomology class for all w.a.c. m-dimensional manifolds $M\}$.

It is an important question to compute I_m^{2k} for the various values of k, m.

Let $K(Z, m-2k)$ be the well-known Eilenberg–McLane space and i its fundamental class. If $m = 2k$ we put $K(Z, m-2k) = $ point.

Next we consider the m-dimensional complex bordism groups

$$\mathrm{MU}_m(K(Z, m-2k))\,,$$

whose elements consists of equivalence classes of pairs (M, f), where M is an m-dimensional w.a.c manifold and $f\colon M\to K(Z, m-2k)$ is a map. For details on bordism groups e.t.c. a good reference is R. Stong's book [13].

Following Brown–Peterson (see [2]) we define a map

$$l\colon H^{2k}(\mathrm{BU}; Q)\to \mathrm{Hom}\big(\mathrm{MU}_m(K(Z, m-2k)),\, Q\big)$$

by the formula $l(x)(M, f) = \big(\tau_M^*(x)\cdot f^*(i)\big)(M)\in Q$. By (M) we denote the orientation class of the w.a.c. manifold M. It can be checked that l is well defined.

Our next result gives a reduction of the computation of I_m^{2k} to a homotopy question.

THEOREM 1.2. *The group I_m^{2k} consists of those elements $x\in H^{2k}(\mathrm{BU}; Q)$ such that $l(x)$ belongs to the image of s.*

In the statement of the Theorem above we make use of the notational convention described in the Introduction after Proposition 0.2.

The group I_{2k}^{2k} has been computed and the answer is commonly referred as the Hattori–Stong Theorem (see [5], [12]). Our next result gives a description of I_{2k+1}^{2k}.

THEOREM 1.3. $I_{2k+1}^{2k} = I_{2k}^{2k}$.

The rest of this section is occupied with the proof of Theorems 1.2 and 1.3. Essentially we follow the methods of Brown–Peterson (see [2]). I believe that Theorems 1.2 and 1.3 are known to other people too, but I have not seen them published, so I include them here.

LEMMA 1.4. *Let M be an m-dimensional, closed, connected, oriented manifold. Let x be an element of $H^q(M; Q)$. Then x is integral if and only if for every $y\in H^{m-q}(M; Z)$ we have $(x\cdot s(y))(M)\in Z\subseteq Q$.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} H^q(M; Q) & \xrightarrow{f_1} & \mathrm{Hom}(H^{m-q}(M; Q), Q) \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s} \\ H^q(M; Z) & \xrightarrow{f_2} & \mathrm{Hom}(H^{m-q}(M; Z), Z) \end{array}$$

where f_1 is defined by the formula $f_1(x)(y) = (x\cdot y)(M)\in Q$ for all $x\in H^q(M; Q)$, $y\in H^{m-q}(M; Q)$, and the map f_2 is defined similarly.

By Poincaré duality we have the isomorphisms $H^{m-q}(M; *)\simeq H^q(M; *)$ where $*$ is Z or Q. On the other hand by the universal coefficient theorem we have epimorphism $H^q(M; *)\to \mathrm{Hom}(H_q(M; *), *)$, which is isomorphism for the case $* = Q$. So f_1 is iso and f_2 is epi. Since homomorphism into Q kills torsion we have an isomorphism $s\colon \mathrm{Hom}(H^{m-q}(M; Z), Q)\to \mathrm{Hom}(H^{m-q}(M; Q), Q)$.

Because of the previous remarks the above diagram takes the following form:

$$\begin{tikzcd} H^q(M;Q) \arrow[r, "g_1"] & \mathrm{Hom}(H^{m-q}(M;Z),Q) \\ H^q(M;Z) \arrow[u, "s"] \arrow[r, "g_2"] & \mathrm{Hom}(H^{m-q}(M;Z),Z) \arrow[u, "s"'] \end{tikzcd}$$

where both g_1 and g_2 are induced by the cup product pairings:

$$H^q(M;Q)\times H^{m-q}(M;Z)\to Q \quad \text{and} \quad H^q(M;Z)\times H^{m-q}(M;Z)\to Z$$

respectively.

But since g_1 is iso and g_2 is onto our lemma follows free of charge.

Proof of Theorem 1.2. Let us assume that $l(x)$ belongs to the image of s. By the definition of l that means that $(\tau_M^*(x)\cdot f^*(i))(M)\in Z\subseteq Q$ for all m-dimensional w.a.c. manifolds M, and all maps $f\colon M\to K(Z,m-2k)$. By the fundamental property of Eilenberg-McLane spaces, $f^*(i)$ could be any cohomology class of $H^{m-2k}(M;Z)$, and according to the previous lemma, the previous remark implies that $\tau^*(x)$ is integral, and by the very definition $x\in I_m^{2k}$.

By MU we denote $\mathrm{MU}(N)$, the Thom space of the universal N-dimensional complex bundle, when N is very big in comparison with m. Let $U\in H^{2N}(\mathrm{MU};Z)$ be the Thom class of the universal bundle. Let $\beta\colon \mathrm{BU}\to\mathrm{BU}$ be the map classifying the inverse of the universal bundle. Clearly β is a homotopy equivalence and $\beta^2\simeq\mathrm{Id}$. For every w.a.c. manifold there is a naturally induced complex structure on its stable normal bundle. Let M be a w.a.c. manifold and $\nu_M\colon M\to\mathrm{BU}$ be the map classifying the stable normal bundle of M, then $\nu_M\simeq\beta\tau_M$, $\tau_M\simeq\beta\nu_M$.

It is well known that the group $\mathrm{MU}_m(K(Z,m-2k))$ is isomorphic with $\pi_{2N+m}(\mathrm{MU}\wedge K(Z,m-2k)_+)$. For abbreviation we put $X=\mathrm{MU}\wedge K(Z,m-2k)$. Through the isomorphism above, the map $l\colon H^{2k}(\mathrm{BU};Q)\to\mathrm{Hom}(\pi_{2N+m}(X),Q)$ can be described as follows: Let $a\colon S^{2N+m}\to X$ be a map and (a) the corresponding homotopy element, then $l(x)(a)=a^*(\beta^*(x)U\cdot i)(S^{2N+m})\in Q$ (see Stong's book).

Proof of Theorem 1.3. Because of the previous remarks we have:

$$\mathrm{MU}_{2k}(\mathrm{point})\simeq\pi_{2N+2k}(\mathrm{MU}),\qquad \mathrm{MU}_{2k+1}(K(Z,1))\simeq\pi_{2N+2k+1}(\mathrm{MU}\wedge S^1_+)\,.$$

But the groups $\pi_{2N+2k}(\mathrm{MU})$ and $\pi_{2N+2k+1}(\mathrm{MU}\wedge S^1_+)$ are isomorphic through the suspension isomorphism. Because of the description of l given above the following diagram commutes.

$$\begin{tikzcd} H^{2k}(\mathrm{BU};Q) \arrow[r, "l"] \arrow[dr, "l"'] & \mathrm{Hom}(\pi_{2N+2k}(\mathrm{MU}),Q) \arrow[d, "S"', "\simeq"] \\ & \mathrm{Hom}(\pi_{2N+2k+1}(\mathrm{MU}\wedge S^1_+),Q) \end{tikzcd}$$

(S is the map induced by the suspension map). Now the theorem follows easily.

Remark. By combining Theorem 1.2 and the approach of A. Hattori (see [5]) the group I_{2k+2}^{2k} can be computed, the basic special feature being the fact that the space $K(Z,2)=\mathrm{CP}(\infty)$ is torsion-free. But it seems that the method does not work for I_m^{2k} if $m>2k+2$.

2. The Todd character and Spanier–Whitehead duality. We begin by summarizing some well-known facts about complex bordism, KU-theory, Chern character, Todd character e.t.c. A good reference is Conner–Floyd's book [3].

Let Y be a finite complex. Then $\mathrm{MU}^*(Y)$ is the homology complex bordism group, $\mathrm{MU}^*(Y)$ is the cohomology complex bordism group, $\mathrm{KU}_*(Y)$ is the homology KU-theory group and $\mathrm{KU}^*(Y)$ is the cohomology KU-theory group. The grading in all these theories is over the integers. Furthermore $\mathrm{KU}^0(Y)$ is $\mathrm{KU}(Y)$.

It is well-known there is a natural transformation $\mu\colon \mathrm{MU}^*(Y)\to\mathrm{KU}^*(Y)$ and $\mu\colon \mathrm{MU}_*(Y)\to\mathrm{KU}_*(Y)$ such that $\mu\colon \mathrm{MU}^0(Y)\to\mathrm{KU}^0(Y)$ is onto. There is a natural transformation called *Chern character*

$$\mathrm{ch}_i\colon \mathrm{KU}^*(Y)\to H^{2i}(Y;Q),\qquad \mathrm{ch}_i\colon \mathrm{KU}^*(Y)\to H^{2i+1}(Y;Q)\,,$$
$$\mathrm{ch}_i\colon \mathrm{KU}_*(Y)\to H_{2i}(Y;Q),\qquad \mathrm{ch}_i\colon \mathrm{KU}_*(Y)\to H_{2i+1}(Y;Q)\,.$$

We consider the composition $\mathrm{th}_i=\mathrm{ch}_i\mu$, it is called the *Todd character.*

The Todd character $\mathrm{th}_i\colon \mathrm{MU}_*(Y)\to H_j(Y;Q)$, where j is $2i$ or $2i+1$, can be described in another way as follows.

Let $\mathrm{Td}=(\mathrm{Td}_0,\mathrm{Td}_1,\ldots,\mathrm{Td}_i,\ldots)$ be the total Todd class, where $\mathrm{Td}_i\in H^{2i}(\mathrm{BU};Q)$ (see [7]). Let $(M,f)\in\mathrm{MU}_*(Y)$, where M is a w.a.c manifold and $f\colon M\to Y$ is a map. The homology class $\mathrm{th}_i(M,f)\in H_j(Y;Q)$ can be described as an element of $\mathrm{Hom}(H^j(Y;Z),Q)$ in the following way: if $y\in H^j(Y;Z)$ then

$$\mathrm{th}_i(M,f)(y)=\big(f^*(y)\mathrm{Td}(M)\big)(M)\in Q$$

where $\mathrm{Td}(M)$ is $\tau^*(\mathrm{Td})\in H^*(M;Q)$ the total Todd class of M (see [3], p. 35).

Lemma 2.1. *Td_k is an element of I_{2k+1}^{2k}.*

Proof. It is well known that Td_k belongs to I_{2k}^{2k} (see [5]) so because of Theorem 1.3 our lemma follows.

The next proposition describes the relation between Chern character, Todd character and Spanier–Whitehead duality (see [14]).

Proposition 2.2. *Let X be of the homotopy type of a subcomplex of S^{2k+2}, and Y be its Spanier–Whitehead dual. Then the following diagram commutes.*

$$\begin{tikzcd} \mathrm{MU}_{2k+1}(Y) \arrow[r, leftrightarrow, "D"] \arrow[d, "\mu"'] \arrow[dd, bend right=60, "\mathrm{th}_k"'] & \mathrm{MU}^0(X) \arrow[d, "\mu"] \arrow[dd, bend left=60, "\mathrm{th}_k"] \\ \mathrm{KU}_{2k+1}(Y) \arrow[r, leftrightarrow, "D"] \arrow[d, "\mathrm{ch}_k"'] & \mathrm{KU}^0(X) \arrow[d, "\mathrm{ch}_k"] \\ H_1(Y;Q) \arrow[r, leftrightarrow, "D"] & H^{2k}(X;Q) \end{tikzcd}$$

where D is the Spanier-Whitehead duality isomorphism.

Proof of Theorem 0.1. We consider the diagram of the previous proposition. Since $\mathrm{Td}_k \in I^{2k}_{2k+1}$, by the very description of th_k that we gave, the image of th_k in this diagram consists of integral elements. Since μ: $MU^0(Y) \to KU^0(Y)$ is onto it is clear that the image of ch_k, in the diagram, consists of integral elements, which is exactly what we want to prove.

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Retraceable homogeneous sets *

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Abstract. We show that every recursive partition $\langle P_1, P_2\rangle$ of $[N]^2$ admits a retraceable infinite homogeneous set X whose Turing jump is $\leqslant 0''$; this refines a result of Jockusch [1] by adding the condition of retraceability (in fact, retraceability by a finite-one retracing function).

1. Introduction. This article is intended as a contribution to the "fine analysis" of Ramsey's Theorem in terms of recursion-theoretic notions. Previous analyses of this sort ([3], [4]) led to Jockusch's paper [1] which from one point of view can be regarded as the last word on the subject. Further words can be said, however, if we consider other "descriptive" notions in place of or in addition to *classification within the Kleene hierarchy*. One such notion is that of retraceability: recall that an infinite sequence $a_0, a_1, a_2, \ldots$ of natural numbers is said to be *retraceable* if $(\forall n)[a_n < a_{n+1}]$ & there is a partial recursive function p such that $(\forall n)[p(a_{n+1})$ is defined and $= a_n]$. We shall replace *Π^0_2 classification*, in [1, Theorem 4.2], by the combination of retraceability and a fairly strong condition on jumps of Turing degrees. It remains an open question (as far as we know) whether retraceability can be *added to Π^0_2* representability, in Theorem 4.2 of [1]; a brief discussion of this question is included at the end of the paper. Our terminology and notation, where not explicitly defined or entirely standard, is in line with that of [1].

2. Recursive partitions and a theorem of Jockusch. If N = the natural numbers and $X \subseteq N$ is infinite, then $[X]^2$ denotes the set $\{\{x, y\}|\ x \in X \,\&\, y \in X \,\&\, x \neq y\}$. The classical theorem of Ramsey asserts that if $[N]^2$ is divided into complementary subsets P_1 and P_2 then there is an infinite set $X \subseteq N$ such that either $[X]^2 \subseteq P_1$ or $[X]^2 \subseteq P_2$; such an X is called a *homogeneous set* (or a *set of indiscernibles*) for the partition $\langle P_1, P_2\rangle$. One can ask about the degree of constructivity possible for X in case P_1 and P_2 are *recursive* sets of pairs. This question, first dealt with by Specker [3], has been answered in a definitive way relative to the Kleene hierarchy in Jockusch [1] (which paper, in addition, contains instructive commentary on what is probably the most elegant possible proof (due independently to various mathematicians) of the

* A version of the central result of this paper, Theorem 3, was independently proved by Gordon Phillips, a student of Jockusch. Our inquiries have led us to conclude that Phillips does not presently intend to publish his proof.