The last set is $(n-1)$-dimensional as a countable union of compact, $(n-1)$-dimensional sets, so that Y is weakly n-dimensional.

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Equivariant maps of Z_p-actions into polyhedra

by

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Abstract. Let X be an n-dimensional compact metric space with a free Z_p-action. This paper shows that for any positive number ε there exists an equivariant ε-map from X into an n-dimensional polyhedron K with a free Z_p-action. Moreover, K can be equivariantly embedded in $(2n+1)$-dimensional euclidean space E with an orthogonal Z_p-action and there exists an equivariant ε-map arbitrarily close to a given equivariant map from X into E.

1. Introduction. Let X be an n-dimensional compact metric space with a map $a\colon X\to X$ of period p. The map a then defines a Z_p-action on X and (X, a) will denote the equivariant space (X, Z_p). Frequently, (X, a) is called a *Z_p-space*. An equivariant map $f\colon (X, a)\to(Y, b)$ between two Z_p-spaces is an equivariant ε-map if $\operatorname{diam} f^{-1}y<\varepsilon$ for every $y\in fX$.

In the following, if (Y, b) is a Z_p-space, then $y^* = \{y, by, \ldots, b^{p-1}y\}$ is called the *orbit of* Y, and $S^* = \bigcup_{j=0}^{p-1} b^jS$ is called the *orbit of* S, where y is an element in Y, and S is a subset of Y. A subset S of Y is called *sectional* if $S\cap y^* = \{y\}$ for each y in S, and any one-to-one function $\chi\colon (Y/Z_p)\to Y$ is called a *section*.

If the action on X is free, then an immediate consequence of (2.3) below is that for any positive number ε there exists an equivariant ε-map from X into an n-dimensional polyhedron K with a free Z_p-action. Moreover, by (3.1) below K can be equivariantly embedded in $(2n+1)$-dimensional euclidean space R^{2n+1} with an orthogonal Z_p-action. Finally, it is shown in (3.3) that there exists an equivariant ε-map arbitrarily close to a given equivariant map from X into R^{2n+1}.

A set C is called a *convex body in a euclidean space* if C is closed, convex and has a nonempty interior.

2. Replacement by polyhedra. (2.1), which is stated here and is used in proving (2.3) below, can be found in Jaworowski [7, p. 235].

COVERING LEMMA (2.1). *Let (X, a) be a compact metric Z_p-space and let A be an equivariant closed subspace of X such that Z_p acts freely outside of A. Suppose C is an equivariant open cover of $X-A$. Then there exists a countable, locally finite,* *equivariant, open cover B of $X-A$ which is a refinement of C and which satisfies the following*:

(i) $\lim\limits_{d(V,A)\to 0} (\operatorname{diam} \operatorname{St} V) = 0$ *for* $V \in B$;

(ii) *If* $V \in B$, *the* $\operatorname{Cl} V \subset X-A$;

(iii) *every neighborhood of A in X contains all but a finite number of elements of B*;

(iv) *for every* $V \in B$, *the sets* $\operatorname{St}_B V, a(\operatorname{St}_B V), \ldots, a^{p-1}(\operatorname{St}_B V)$ *are mutually disjoint*;

(v) *if* $\dim(X-A) \leqslant n$, *the* $\operatorname{Ord} B \leqslant p(n+1)-1$;

(vi) *if ε is a given positive number, then B can be chosen such that* mesh $B<\varepsilon$.

Observe that as a corollary to conditions (i), (ii), and (iii) above one obtains the following lemma.

(2.2) LEMMA. *If B is a covering constructed in Lemma* (2.1), *then, for every $x \in A$ and for every neighborhood U of x in X, there exists a neighborhood W of x in X such that if $V \in B$ and $V \cap W \neq \emptyset$ then $V \subset U$.*

POLYHEDRAL REPLACEMENT LEMMA (2.3). *Let (X, a) be a compact metric Z_p-space and let A be an equivariant, closed subspace such that Z_p acts freely outside A. For a given positive number ε, there exists a compact Hausdorff Z_p-space (Z, c) such that*:

(i) *Z contains A as an equivariant, closed subspace and* $c|_A = a|_A$;

(ii) *there exists a countable, locally finite, simplicial complex K and simplicial map* $b\colon K\to K$ *of period p with* $|K| = Z-A$ *and with* $c|_{|K|}\colon |K|\to|K|$ *a free simplicial map of period p*;

(iii) *an equivariant ε-map* $f\colon (X, a)\to(Z, c)$ *such that* $f|_A = 1_A$, $f(X-A)\subset|K|$, *and* $f^{-1}\{\operatorname{St}(V)|\ V\in K^0\}$ *forms a locally finite, equivariant, open cover of $X-A$ of mesh less than ε*; *and*

(iv) *if* $\dim(X-A)\leqslant n$, *then* $\dim K\leqslant n$.

Remark. Lemma (2.3) is a modification of Lemma 4.7 in [7, p. 237].

Proof. By a remark in [7], one can assume that a is isometric. Let B be an equivariant, locally finite, countable open cover of $X-A$ satisfying the conditions of the Covering Lemma (2.1). Let $K_1 = N(B)$ be the nerve of B and let Z_1 be the disjoint set sum of A and $|K_1|$. Then K_1 is a countable, locally finite simplicial complex. Given a member V of B, also denote by V the vertex of K_1 corresponding to V. Whenever it is necessary to make the distinction, $\operatorname{St}_{K_1}(V)$ will denote the open star of the vertex V in the simplicial complex K_1, while $\operatorname{St}_B(V)$ will denote the union of the members of B intersecting V.

For a subset S of X, let $\hat{S}$ denote the union of $A\cap S$ and of the open stars of the vertices of K_1 corresponding to the members of B which are contained in S; i.e., $\hat{S} = (A\cap S)\cup(\bigcup[\operatorname{St}_{K_1}(V)|\ V\subset S])$. The space Z_1 is topologized by means of the subbasis consisting of all the open subsets of $|K_1|$ and all the sets of the form $\hat{U}$, where U is an open subset of X.

Before proceeding with the rest of the proof, the following lemma is established:

(2.4) LEMMA. *For every x in A and every neighborhood U of x in X, there is a neighborhood O_U of x in Z_1 such that, if $y\in O_U\cap(Z-A)$ and $\langle s\rangle$ is an open simplex of $|K_1|$ containing y, then all the vertices of $\langle s\rangle$ (considered as members of the cover B) are contained in U.*

Proof. Given a neighborhood U, choose a neighborhood W of x according to Lemma (2.2). Let $O_U = \hat{W}$. Then, if $y\in O_U\cap(Z-A)$ and $\langle s\rangle$ is the carrier of y in $|K_1|$, some vertex V of $\langle s\rangle$ is contained in W; and all the other vertices are contained in U since they meet $V\subset W$.

Continuation of the proof of (2.3). The fact that Z_1 is Hausdorff follows readily from Lemma (2.4). In [7] an explicit proof is given to show that Z_1 is compact. Since the cover B is equivariant, it follows that $a\colon X\to X$ of period p defines a simplicial map b_1 of period p on K_1. In fact, if $(V_{i_1}, \ldots, V_{i_r})$ denotes the simplex in K_1 with vertices $V_{i_1}, \ldots, V_{i_r}$, then $b_1(V_{i_1}, \ldots, V_{i_r}) = (aV_{i_1}, \ldots, aV_{i_r})$.

Define $c_1'\colon |K_1|\to|K_1|$ as follows. For each $V_i\in K_1^0$, denote by p_{V_i} the element in $|K_1|$ where

$$p_{V_i}(V_j) = \begin{cases} 0 & \text{for} \quad i\neq j\,,\\ 1 & \text{for} \quad i=j\,.\end{cases}$$

Note that the set of p_{V_i}'s in $|K_1|$ corresponds to the set of vertices of K_1. Define $c_1'\colon |K_1|\to|K_1|$ by, for $p\in|K_1|$, $c_1'p = \sum_{V_i\in K_1^0} p(V_i)p_{aV_i}$. It is an easy verification to show that c_1' is well-defined and a simplicial map of period p. The fact that c_1' is free follows from condition (iv) of the Covering Lemma.

Define $c_1\colon Z_1\to Z_1$ by a on A and by c_1' on $|K_1|$. The continuity of c_1 follows from the fact that B is equivariant and it is clear that $(c_1)^p = 1_{Z_1}$. Thus conditions (i) and (ii) hold for Z_1 and c_1.

Define $f_1'\colon X-A\to|K_1|$ by the canonical map of $X-A$ into the space $|K_1|$ of the nerve $N(B) = K_1$ (see Borsuk [2], p. 76): i.e., for each $x\in X-A$,

$$f_1'x = p_x\in|K_1| \quad \text{where} \quad p_x(V_i) = \frac{d(x, X-V_i)}{\sum_{V_j\in B} d(x, X-V_j)}$$

for every $V_i\in K_1^0$. Now B being an equivariant cover and a being isometric imply that for every $V_i\in K_i^0$

$$p_{ax}(V_i) = \frac{d(ax, X-V_i)}{\sum_{V_j\in B} d(ax, X-V_j)} = \frac{d(x, X-aV_i)}{\sum_{V_j\in B} d(x, X-V_j)} = p_x(aV_i)$$
$$= \Big(\sum_{V_k\in K_1^0} p_x(V_k)p_{aV_k}\Big)(V_i) = c_1p_x(V_i);$$

i.e., $f_1'ax = p_{ax} = c_1'p_x = c_1'f_1'x$ for every $x\in X-A$. Therefore, $f_1'a = c_1'f_1'$ and f_1' is equivariant.

Finally, let $f_1\colon (X, a)\to(Z_1, c_1)$ be defined as $f_1|_{X-A} = f_1'$ and $f_1|_A = 1_A$. f_1 is equivariant since f_1' is. The continuity of f_1 follows easily from the definition of the topology, just as in [4]. In fact, it suffices to show that f_1 is continuous at points of A. Let a_0 be in A and let $\hat{U}$ be an element of the subbasis of Z_1 containing $f_1(a_0)$. Then U is a neighborhood of a_0 in X. By Lemma (2.2) there exists a neighborhood W of a_0 in X which is contained in U such that, if V is in B and $V \cap W \neq \emptyset$, then $V \subset U$.

It is claimed that $f_1(W) \subset \hat{U}$. Let x be in $W \cap A$; then

$$f_1(x) = x \in W \cap A \subset U \cap A \subset \hat{U} = (A \cap U) \cup (\bigcup [\mathrm{St}_{K_1}(V)|\ V \in B \text{ and } V \subset U]).$$

Secondly, let x be in $W \cap (X-A)$. Suppose x is in V_1 and V_1 is a member of B. Then, in particular, $f_1 x = f_1' x$ is in an open simplex having V_1 as a vertex. Hence, $f_1 x$ is in $\mathrm{St}_{K_1}(V_1)$. Furthermore, $V_1 \cap W \neq \emptyset$, and so $V_1 \subset U$. All of this implies that $f_1 x$ is in $\hat{U}$. As a result, it is true that $f_1(W) \subset \hat{U}$, as was claimed.

From this point on in the proof, assume that the mesh of B is less than ε (see condition (vi) of the Covering Lemma). To prove that f_1 is an ε-map, it remains to show that f_1' is an ε-map on $X-A$. If $s = (V_{i_1}, \ldots, V_{i_t})$ is a simplex in K_1, then $\langle s\rangle = \langle (V_{i_1}, \ldots, V_{i_t})\rangle$ denotes the open simplex in $|K_1|$ corresponding to s. Let $p \in |K_1|$. Suppose $p \in \langle s\rangle = \langle (V_{i_1}, \ldots, V_{i_t})\rangle$, the unique open simplex in $|K_1|$ containing p. Let $x \in X$ be such that $f_1' x = p_x = p$. By the definition of f_1' and by the definition of what it means to be an element in $|K_1|$, it follows that $p_x(V_{i_j}) > 0$ and $x \in V_{i_j}$ for $j = 1, \ldots, t$. Therefore, in particular, $(f_1')^{-1} p \subset V_{i_1}$ where $\operatorname{diam} V_{i_1} < \varepsilon$. Hence, $\operatorname{diam}(f_1')^{-1} p < \varepsilon$, and therefore $f_1'\colon (X-A, a) \to (|K_1|, c_1')$ is an ε-map. This implies that f_1 is an ε-map since $f_1|_A = 1_A$.

Furthermore, suppose $V_i \in K_1^0$. $\mathrm{St}\, V_i$, the open star of the vertex V_i, denotes the union of all open simplexes in $|K_1|$ with vertex V_i. Let $O = \mathrm{St}\, V_i$. $x \in f_1^{-1} O$ implies that $f_1 x$ is in an open simplex having V_i as a vertex. This implies that $f_1 x$ is positive on V_i and so, by the definition of f_1, $x \in V_i$. Therefore, $f_1^{-1} O \subset V_i$ and $\operatorname{diam} f_1^{-1} O < \varepsilon$. The fact that $f_1^{-1}\{\mathrm{St}\, V_i|\ V_i \in K_1^0\}$ forms a locally finite, equivariant, open cover of $X-A$ follows from the observations that $\{\mathrm{St}\, V_i|\ V_i \in K_1^0\}$ is a locally finite, equivariant open cover of $|K_1|$, that f_1 is continuous, and that f_1 is equivariant. Hence, f_1 satisfies condition (iii).

At this point, it should be noted that f_1, Z_1, K_1 and c_1 satisfy all of the conditions of the lemma except (iv). The remainder of the proof will be devoted to proving that condition (iv) holds. The heart of this work will lie in modifying the map f_1.

Henceforth in this proof assume $\dim(X-A) \leqslant n$. By condition (v) of the Covering Lemma, it follows that $\operatorname{Ord} B \leqslant p(n+1)-1$, and this implies that $\dim(K_1) \leqslant p(n+1)-1$. Let $\dim(K_1) = d$. If $d \leqslant n$, then let $f_1 = f$, $Z_1 = Z$, $K_1 = K$, and $c_1 = c$. Thus, condition (iv) is satisfied. In this case, the proof of (2.3) is completed. If $d > n$, certain technical lemmas need to be established. In order to state and prove these lemmas which follow, the following notation will be used. Let $\partial(s)$ denote the simplicial complex consisting of all the faces of the simplex s; let $|\partial(s)|$ denote the space of this complex; and let $|(\partial(s))^j|$ denote the space of the jth skeleton of $\partial(s)$.

(2.5) LEMMA. *Let $N > n$ and $\langle s\rangle = \langle (V_1, \ldots, V_{N+1})\rangle$ be an open simplex in $|K_1^N|$ of dimension N. Suppose $h\colon X-A \to |K_1^N|$ is an equivariant map. Then there exists an equivariant map*

$$h_s\colon X-A \to |K_1^N| - (\bigcup_{k=0}^{p-1} c_1^k\langle s\rangle)$$

such that:

(1) *$h = h_s$ on $(X-A) - h^{-1}(\bigcup_{k=0}^{p-1} c_1^k\langle s\rangle)$;*

(2) *$h_s(h^{-1} c_1^k\langle s\rangle) \subset |(\partial(c_1^k\langle s\rangle))^{N-1}|$ for each $k = 0, 1, \ldots, p-1$;*

(3) *$h_s^{-1}(c_1^k \mathrm{St}_{K_1^{N-1}} V_i) \subset h^{-1}(c_1^k \mathrm{St}_{K_1^N} V_i)$ for each $i = 1, \ldots, N+1$ and for each $k = 0, \ldots, p-1$.*

Proof. Let $s = (V_{i_1}, \ldots, V_{i_{N+1}})$ be a simplex in K_1^N of dimension N and denote by $\langle s\rangle = \langle (V_{i_1}, \ldots, V_{i_{N+1}})\rangle$ the open simplex in $|K_1^N|$ corresponding to s. Similarly, $|s| = |(V_{i_1}, \ldots, V_{i_{N+1}})|$ denotes the closed simplex in $|K_1^N|$ corresponding to s. Suppose $\langle s\rangle \cap h(X-A) \neq \emptyset$. Let $q = q(s)$ be the barycenter of the simplex s; i.e., $q\colon K_1^0 \to [0, 1]$ where

$$q(V_i) = \begin{cases} \dfrac{1}{N+1}, & \text{if } V_i \text{ is a vertex of } s, \\ 0, & \text{if } V_i \text{ is not a vertex of } s. \end{cases}$$

Clearly, $q \in \langle s\rangle$. If $q \notin h(X-A)$, then define $h_q\colon (X-A) \to |K_1^N|$ by $h_q = h$. If $q \in h(X-A)$, then using the results on stable and unstable values in Hurewicz and Wallman ([6], Theorem VI 1, p. 75; Proposition B, p. 78), there exists

$$h_q\colon (X-A) \to |K_1^N|$$

such that

(1) $hx = h_q x$, if $hx \notin \langle s\rangle$;

(2) $h_q x \in \langle s\rangle$, if $hx \in \langle s\rangle$; and

(3) $q \notin h_q(X-A)$.

Define $t_q\colon X-A \to |K_1^N| - \langle s\rangle$ by $h_q = h$ on $(X-A) - h^{-1}\langle s\rangle$ and by $r_q \circ h_q$ on $h^{-1}|s|$, where r_q is the radial projection from the barycenter of q of $|s|$ onto the boundary of $|s|$. Then define

$$h_s\colon (X-A) \to |K_1^N| - (\bigcup_{k=0}^{p-1} c_1^k\langle s\rangle)$$

by t_q on $(X-A) - h^{-1}(\bigcup_{k=0}^{p-1} c_1^k\langle s\rangle)$ and by $c_1^j t_q a^{p-j}$ on $h^{-1} c_1^j |s|$ for $j = 1, \ldots, p-1$. By straightforward verification, h_s is well-defined, continuous, and equivariant.

It is clear from the definitions of t_q and r_q that (1) and (2) in the conclusion of Lemma (2.5) are satisfied. So it remains to verify (3). In fact, it suffices to prove (3) for the case of $k = 0$. Let x be in $h_s^{-1}(\mathrm{St}_{K_1^{N-1}} V_i)$. Then $h_s x = p$ is in $\mathrm{St}_{K_1^{N-1}} V_i$.

Case 1. $hx = h_s x = p$. This implies that hx is in $\mathrm{St}_{K_1^{N-1}} V_i \subset \mathrm{St}_{K_1^N} V_i$ and, hence, it follows that x is in $h^{-1}(\mathrm{St}_{K^N} V_i)$.

Case 2. $hx \neq h_s x = p$. Then, using the definition of h_q, there exists a point p' in $\langle s \rangle$ such that $hx = p'$. Hence, hx is in $\mathrm{St}_{K_1^N} V_i$ and, so, x is in $h^{-1}(\mathrm{St}_{K_1^N} V_i)$. Therefore, in either case, it is true that $h_s^{-1}(\mathrm{St}_{K_1^{N-1}} V_i) \subset h^{-1}(\mathrm{St}_{K_1^N} V_i)$; and, since h and h_s are both equivariant, (3) is true. This completes the proof of Lemma (2.5).

Remark. Both Lemma (2.5) above and Lemma (2.6) which follows are steps in the proof of Lemma (2.3). In words, Lemma (2.5) says that an equivariant map h from a space of dimension n into a simplicial complex of higher dimension can be appropriately modified so that the image of h misses the orbit of any open simplex of dimension greater than n. In addition, Lemma (2.6) will show that any equivariant map from a space of dimension n into a simplicial complex of higher dimension can be appropriately modified so that the image misses all open simplexes of a given dimension greater than n.

Recall that f_1 on $X-A$ is the canonical map of $X-A$ into the space $|K_1|$ of the nerve $N(B) = K_1$, where B is an equivariant open cover of $X-A$.

(2.6) LEMMA. *Let $1 \leqslant i < d-n+1$. Suppose g_i: $X-A \to |K_1^{d-i+1}|$ is an equivariant ε-map satisfying the following properties*:

(1) $g_i^{-1}\{\mathrm{St}_{K_1^{d-i+1}}(V) |\ V \in K_1^0\}$ *is a locally finite, equivariant, open cover of $X-A$ of mesh less than* ε;

(2) $g_i(f_1^{-1}\langle s \rangle) \subset |(\partial(s))^{d-i+1}|$ *for each open simplex $\langle s \rangle$ in $|K_1|$; and*

(3) $g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V) \subset f_1^{-1}(\mathrm{St}_{K_1} V)$.

Then there exists an equivariant ε-map

$$g_{i+1}:\ X-A \to |K_1^{d-i}|$$

such that

(1) $g_{i+1}^{-1}\{\mathrm{St}_{K_1^{d-i}}(V) |\ V \in K_1^0\}$ *is a locally finite, equivariant, open cover of $X-A$ of mesh less than* ε;

(2) $g_{i+1}(f_1^{-1}\langle s \rangle) \subset |(\partial(s))^{d-i}|$ *for each open simplex $\langle s \rangle$ in $|K_1|$; and*

(3) $g_{i+1}^{-1}(\mathrm{St}_{K_1^{d-i}} V) \subset f_1^{-1}(\mathrm{St}_{K_1} V)$.

Proof. Let $O = \{S_0, S_1, \ldots, S_{p-1}\}$ be an orbit decomposition of the open simplexes in $|K_1^{d-i+1}|$ of dimension $d-i+1 > n$. Suppose

$$S_i = \{c_1^i \langle s_j \rangle\}_{j=1}^{\infty};$$

$$B_j = \bigcup_{k=0}^{p-1} g_i^{-1} c_1^k |s_j|, \quad j = 1, 2, \ldots;$$

and

$$B_0 = (X-A) - \bigcup_{j=1}^{\infty} \Big(\bigcup_{k=0}^{p-1} g_i^{-1} c_1^k \langle s_j \rangle \Big).$$

Let $C = \{B_j\}_{j=0}^{\infty}$. It is clear that C is a closed cover of $X-A$. Secondly, it is claimed that C is a neighborhood-finite cover (i.e., each x in $X-A$ has a neighborhood U such that $B_j \cap U \neq \emptyset$ for at most finitely many indices j).

Let x be in $X-A$ and let U be any neighborhood of x. Suppose U intersects non-trivially an infinite number of distinct B_j's, say $B_{j_1}, B_{j_2}, \ldots$ Then $g_i(U)$ intersects non-trivially $c_1^{k_{j_1}}|s_{j_1}|$, $c_1^{k_{j_2}}|s_{j_2}|, \ldots$, and all of these closed simplexes are distinct. Furthermore, since K_1^{d-i+1} is locally finite, any given vertex appears in at most a finite number of the $c_1^{k_{j_1}}|s_{j_1}|$, $c_1^{k_{j_2}}|s_{j_2}|, \ldots$ This implies that $g_i(U)$ intersects non-trivially the stars of an infinite number of distinct vertices, say, $V_{l_1}, V_{l_2}, \ldots$ Hence, U intersects non-trivially $g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V_{l_1})$, $g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V_{l_2}), \ldots$ Therefore, any neighborhood of x intersects non-trivially an infinite number of elements of $g_i^{-1}\{\mathrm{St}_{K_1^{d-i+1}}(V) |\ V \in K_1^0\}$, and this contradicts the local finite property of this cover. So, C is a neighborhood-finite cover.

Define g_{i+1}: $X-A \to |K_1^{d-i}|$ by

(i) $g_{i+1} = g_i$ on B_0,

(ii) $g_{i+1} = h_{s_j}$ on B_j (where h_{s_j} satisfies Lemma (2.5)).

By definition, g_{i+1} is equivariant, and, by Theorem (9.4) found in [5], g_{i+1} is well-defined and continuous (this theorem concerns the piecewise definition of a map on a closed, neighborhood-finite cover).

Let $V \in K_1^0$ and consider $O' = g_{i+1}^{-1}(O) = g_{i+1}^{-1}(\mathrm{St}_{K_1^{d-i}} V)$. If $\mathrm{St}_{K_1^{d-i}} V = \mathrm{St}_{K_1^{d-i+1}} V$, then $g_i^{-1}(\mathrm{St}_{K_1^{d-i}} V) \subset B_0$ and $g_i^{-1}(O) = g_{i+1}^{-1}(O) = O'$. Consequently, $O' \subset B_0$ and

$$O' = g_i^{-1}(O) = g_i^{-1}(\mathrm{St}_{K_1^{d-i}} V) = g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V)$$

has diameter less then ε by (1) of the hypotheses. Secondly, if $\mathrm{St}_{K_1^{d-i}} V \neq \mathrm{St}_{K_1^{d-i+1}} V$, let $\mathrm{St}_{K_1^{d-i+1}} V = (\mathrm{St}_{K_1^{d-i}} V) \cup \langle s_1 \rangle \cup \ldots \cup \langle s_t \rangle$, where the $\langle s_l \rangle$'s are all the open simplexes of dimension $d-i+1$ having V as a vertex. Then, by the definition of g_{i+1},

$$g_{i+1}^{-1} \mathrm{St}_{K_1^{d-i}} V \subset \bigcup_{l=1}^{t} h_{s_l}^{-1}(\mathrm{St}_{K_1^{d-i}} V) \subset \bigcup_{l=1}^{t} g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V) = g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}} V),$$

where the second inclusion is true by (3) of Lemma (2.5). Consequently, using assumption (1) in the statement of Lemma (2.6), it is true in this case that O' has diameter less than ε. Hence, $g_{i+1}^{-1}\{\mathrm{St}_{K_1^{d-i}}(V) |\ V \in K_1^0\}$ has mesh less than ε.

Let $\langle s \rangle$ be an open simplex in $|K_1|$. If the dimension of $\langle s \rangle$ is less than $d-i+1$, then $g_i(f_1^{-1}\langle s \rangle) \subset |\partial(s)^{d-i+1}| = |\partial(s)^{d-i}|$ and, consequently, $f_1^{-1}\langle s \rangle \subset B_0$ which implies that $g_{i+1}(f_1^{-1}\langle s \rangle) = g_i(f_1^{-1}\langle s \rangle) \subset |\partial(s)^{d-i}|$. If, on the other hand, the dimension of $\langle s \rangle$ equals $d-i+1$, then

$$f_1^{-1}\langle s \rangle \subset g_i^{-1} g_i(f_1^{-1}\langle s \rangle) \subset g_i^{-1}|(\partial(s))^{d-i+1}| = g_i^{-1}|(\partial(s))^{d-i}| \cup g_i^{-1}\langle s \rangle$$

which is contained in one of the B_j's, say B_{j_0}. Then

$$\begin{aligned} g_{i+1}(f_1^{-1}\langle s\rangle) &= h_s(f_1^{-1}\langle s\rangle) \subset h_s(g_i^{-1}|(\partial(s))^{d-i}| \cup g_i^{-1}\langle s\rangle) \\ &\subset h_s(g_i^{-1}|(\partial(s))^{d-i}|) \cup h_s(g_i^{-1}\langle s\rangle) \\ &\subset |(\partial(s))^{d-i}| \cup |(\partial\langle s\rangle)^{d-i}| = |(\partial(s))^{d-i}|, \end{aligned}$$

where the last inclusion follows by (2) in the conclusion of Lemma (2.5). Hence, the proof of (2) of the conclusion of (2.6) is finished.

If $\mathrm{St}_{K_1^{d-i}}V = \mathrm{St}_{K_1^{d-i+1}}V$, then $g_i^{-1}\mathrm{St}_{K_1^{d-i}}V \subset B_0$ and, so,

$$g_{i+1}^{-1}\mathrm{St}_{K_1^{d-i}}V = g_i^{-1}\mathrm{St}_{K_1^{d-i+1}}V \subset f_1^{-1}\ \mathrm{St}_{K_1}V,$$

where the last inclusion follows from (3) of the hypothesis of the lemma. Secondly, if $\mathrm{St}_{K_1^{d-i}}V \neq \mathrm{St}_{K_1^{d-i+1}}V$, then $\mathrm{St}_{K_1^{d-i+1}}V = (\mathrm{St}_{K_1^{d-i}}V) \cup \langle s_1\rangle \cup \ldots \cup \langle s_t\rangle$, where the $\langle s_l\rangle$'s are all the open simplexes of dimension $d-i+1$ having V as a vertex. Then, by the definition of g_{i+1}^{-1},

$$(\mathrm{St}_{K_1^{d-i}}V) \subset \bigcup_{l=1}^{t} h_{s_l}^{-1}(\mathrm{St}_{K_1^{d-i}}V) \subset \bigcup_{l=1}^{t} g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}}V) = g_i^{-1}(\mathrm{St}_{K_1^{d-i+1}}V) \subset f_1^{-1}(\mathrm{St}_{K_1}V),$$

where the last inclusion follows from (3) of the hypothesis of the lemma. This completes the proof of Lemma (2.6).

Using the preceding lemma the proof of (2.3) will be completed. Note that $f_1' = g_1: X-A \to |K_1^d|$ satisfies the conditions of Lemma (2.6) for $i = 1$. Define $f' = g_{d-n+1}: X-A \to |K_1^n|$ where the existence of g_{d-n+1} is guaranteed by Lemma (2.6) and satisfies the conclusions of Lemma (2.6), for $i = d-n$.

Let $K = K_1^n$, $b = b_1|_K$, $Z = A \cup |K|$ with the subspace topology, $c = c_1|_Z$. The fact that Z is a closed subspace of a compact Hausdorff space implies that Z itself is a compact Hausdorff space. Furthermore, let f be defined by the identity on A and by f' on $X-A$. It remains to see that f so defined is continuous.

As for the case of f_1, it suffices to show that f is continuous at points of A. Let a_0 be in A and let $\hat{O}$ be an element of the subbasis of Z containing $f(a_0)$. Then $\hat{O} = \hat{U} \cap Z$, where $\hat{U}$ is an element of the subbasis of Z_1. In fact, for any subbasis element $\hat{O}$ in Z,

$$\begin{aligned} \hat{O} = \hat{U} \cap Z &= [(A \cap U) \cup (\bigcup [\mathrm{St}_{K_1}(V)|\ V \in B \text{ and } V \subset U])] \cap (A \cup |K_1^n|) \\ &= (A \cap U) \cup [(\bigcup [\mathrm{St}_{K_1}(V)|\ V \in B \text{ and } V \subset U]) \cap |K_1^n|] \\ &= (A \cap U) \cup (\bigcup [\mathrm{St}_{K_1^n}(V)|\ V \in B \text{ and } V \subset U]) \\ &= (A \cap U) \cup (\bigcup [\mathrm{St}_K(V)|\ V \in B \text{ and } V \subset U]). \end{aligned}$$

Then a_0 is in $\hat{U}$ and, consequently, U is a neighborhood of a_0 in X. By Lemma (2.2) there exists a neighborhood W of a_0 in X which is contained in U such that if V is in B and $V \cap W \neq \emptyset$, then $V \subset U$.

It is claimed that $f(W) \subset \hat{O} = \hat{U} \cap Z$. Let x be in $W \cap A$; then

$$f(x) = x \in W \cap A \subset U \cap A \subset \hat{O}.$$

Secondly, let x be in $W \cap (X-A)$. Suppose x is in the elements $V_1, \ldots, V_r$ of B and in no other. Then $f_1'x$ is in the open simplex $\langle s\rangle = \langle (V_1, \ldots, V_r)\rangle$. By Lemma (2.6) $fx = f'x = g_{d-n+1}x$ is in $|(\partial(s))^n| \subset \bigcup_{i=1}^{r} \mathrm{St}_K V_i$. For each $i = 1, \ldots, r$, $V_i \cap W \neq \emptyset$, and so $V_i \subset U$. This implies, in particular, that $\bigcup_{i=1}^{r} \mathrm{St}_K V_i$ is contained in $\hat{O}$. Hence, fx is in $\hat{O}$ and, therefore, it is true that $f(W) \subset \hat{O}$, as was claimed.

The fact that f' is an equivariant ε-map, and hence also f, follows from Lemma (2.6). Similarly, it follows from Lemma (2.6) that

$$(f')^{-1}\{\mathrm{St}_K(V)|\ V \in K^0\} = f^{-1}\{\mathrm{St}_K(V)|\ V \in K^0\}$$

is a locally finite, equivariant, open cover of $X-A$ of mesh $<\varepsilon$. This completes the proof of Lemma (2.3).

Remarks.

(1) The equivariant ε-map $f_1|_{X-A}: X-A \to |K_1|$ is called the *canonical equivariant ε-map* of the equivariant space $(X-A, a|_{X-A})$ into the equivariant polyhedron $(|K_1|, c_1|_{|K_1|})$, where $|K_1|$ and c_1 are generated by the nerve of a locally finite, equivariant cover of $(X-A, a|_{X-A})$.

(2) The equivariant ε-map $f|_{X-A}: X-A \to |K|$ is called the (*canonically*) *modified canonical equivariant ε-map* of the equivariant space $(X-A, a|_{X-A})$ into the equivariant polyhedron $(|K|, c|_{|K|})$ where $|K|$ and c are generated from the nerve of a locally finite equivariant open cover of $(X-A, a|_{X-A})$.

(3) As mentioned in [7] it is easily seen that Z is metrizable. In fact, this is also an immediate consequence of the Representation Embedding Lemma of Section 3 below.

(4) Some of the ideas in this and the previous section date back to the classical constructions of Kuratowski [9] and Dugundji [4].

3. Embedding lemmas.

Polyhedral Replacement Embedding Lemma (3.1). *Let K be a countable, locally finite, n-dimensional, simplicial complex where $b: |K| \to |K|$ is a free, simplicial map of period p and $K^0 = (\{v_i\}_{i=1}^{\infty})^*$. Suppose $\gamma: \mathbf{R}^{N+k} \to \mathbf{R}^{N+k}$, $N > n$, is an isometric, linear map of period p and $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\mathbf{R}^{N+k}, \gamma)$, where C is a convex body in $\mathbf{R}^{N+k}$. Let $Q = \{q_1, q_2, \ldots\}$ be a countable set of points in C and let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers. If T is the fixed point set of γ_C (γ_C is free outside T) and $\dim(L_C(T)) = k \geqslant n$, then there exists an equivariant embedding $h: (|K|, b) \to (C-T, \gamma_C)$ such that $d(hv_i, q_i) < \varepsilon_i$.*

Proof. Let $\psi: Q \to C$ be a function that satisfies Lemma (3.5) found in [1]. Let $(\psi Q)^* = \{\psi q_i = r_i\}^*$. Define $h: K^0 \to C$ by $hb^j v_i = \gamma^j r_i$, $j = 0, 1, \ldots, p-1$.

Extend h linearly to all the simplexes of K. It is clear that h is continuous since it is defined by continuous operations. Moreover,

$$d(hv_i, \varphi q_i) = d(r_i, \varphi q_i) = d(\psi q_i, \varphi q_i) < \varepsilon_i .$$

It remains to show that h is one-to-one on $|K|$, and that $(h(|K|), \gamma_C)$ is an equivariant polyhedron in $(C-T, \gamma_C)$.

Let $s = (v_0, \ldots, v_t)$ be a simplex in K. Note that $\{v_0, \ldots, v_t\}$ is sectional and $\dim(s) = t \leqslant n$. Then $\{hv_0, \ldots, hv_t\} \subset (\psi Q)^*$ is a linearly independent, sectional subset of C since $(\psi Q)^*$ is in equivariant general position (see [1]). Hence, $\{hv_0, \ldots, hv_t\}$ spans a simplex of dimension t in C. Since h is defined by a linear extension on $hv_0, \ldots, hv_t$, h is one-to-one on each simplex of K. Now, suppose that x and y are two points of $|K|$ not lying in a single simplex of K. If s and t are simplexes of K containing x and y, respectively, then the union U of their vertices contains at most $2n+2$ vertices and U is sectional (since b is free). Since $2n+2 = (n+1)+n+1 \leqslant N+k+1$ and U is sectional, then $h(U)$ is in general position in C and is sectional. It follows that $h(U)$ spans a simplex u in C of dimension equal to $\dim s + \dim t + 1$. Then, $h(x)$ and $h(y)$ lie on the faces $h(s)$ and $h(t)$ of u and neither lies on the face $h(s) \cap h(t)$. Therefore, $h(x) \neq h(y)$ and h is one-to-one on all of $|K|$.

Clearly, $h(|K|)$ is a polyhedron in C with vertices in $(\psi Q)^* = \{r_i\}^*$ since $y_0, \ldots, y_l$ in $\{r_i\}^*$ are vertices of a simplex in $h(|K|)$ if and only if $h^{-1}y_0, \ldots, h^{-1}y_l$ are vertices of a simplex in K. Furthermore, suppose y is in $(v'_0, \ldots, v'_l)$, the open simplex in $|K|$ spanned by $v'_0, \ldots, v'_l$ in K^0; i.e., $y = a_0 v'_0 + \ldots + a_l v'_l$, where $a_0 + \ldots + a_l = 1$ and $a_j \geqslant 0$ for $j = 0, \ldots, l$. Then

$$\begin{aligned} hbx &= h(a_0 bv'_0 + \ldots + a_l bv'_l) \\ &= a_0(hbv'_0) + \ldots + a_l(hbv'_l) = a_0(\gamma_C hv'_0) + \ldots + a_l(\gamma_C hv'_l) \\ &= \gamma_C(a_0 hv'_0 + \ldots + a_l hv'_l) = \gamma_C h(a_0 v'_0 + \ldots + a_l v'_l) = \gamma_C hx . \end{aligned}$$

Thus, h is equivariant and $(h(|K|), \gamma_C)$ is an equivariant polyhedron in C.

To complete the proof, it suffices to show that $h(|K|) \cap T = \emptyset$. If there exists x in $h(|K|)$ such that $\gamma_C x = x$ and if s is the unique open simplex in $h(|K|)$ containing x, then $\gamma_C(s) \cap T \neq \emptyset$ which implies that $h(|K|)^0 = (\psi Q)^*$ is not in equivariant T-position (see [1]). This is a contradiction to the manner in which $(\psi Q)^*$ was chosen. Therefore, $h(|K|) \subset C-T$, and the lemma is proved.

The following notation pertains to (3.2) below. Let (X, a) be a compact metric Z_p-space of dimension $\leqslant n$, where Z_p acts freely outside of a closed equivariant subspace A. Suppose $\gamma: \boldsymbol{R}^{N+k} \to \boldsymbol{R}^{N+k}$, $N > n$, is an isometric linear map of period p and suppose $(C, \gamma_C = \gamma|_C)$ is an equivariant subspace of $(\boldsymbol{R}^{N+k}, \gamma)$, where C is a convex body in $\boldsymbol{R}^{N+k}$. Furthermore, let γ_C be free outside T, the fixed point set of γ_C, and let $\dim(L_C(T)) = k \geqslant n$.

If $w: A \to T$ is a fixed embedding and $g: (X, a) \to (C, \gamma_C)$ is an equivariant map such that $g|_A = w$, then, corresponding to a given positive number η, the uniform

continuity of g implies that there exists a positive number δ such that, if $d(x, x') < \delta$, then $d(gx, gx') < \frac{1}{6}\eta$. In addition, corresponding to δ, let (K, b), $(Z = |K| \cup A, c)$, and $f: (X, a) \to (Z, c)$ be as in (2.3).

Finally, denote by $K^0 = (\{v_i\}_{i=1}^{\infty})^*$ and by $B^* = \{V_i\}^*$ the locally finite, equivariant, open cover of $X-A$, where K is generated by the nerve of B.

POLYHEDRAL REPRESENTATION EMBEDDING LEMMA (3.2). *There exists* $h: (Z, c) \to (C, \gamma_C)$ *such that*:

(i) h *is an equivariant embedding*;

(ii) $h|_A = w$;

(iii) $h|_{Z-A}$ *is a simplicial homeomorphism*; *and*

(iv) $d(\gamma^j h(v_i), g(f^{-1}(\mathrm{St}\, b^j v_i))) < \frac{1}{2}\eta$, *for each* $j = 0, \ldots, p-1$.

Proof. Define $D = \{V_i \in B \mid d(V_i, A) < \frac{1}{2}\delta\}$ and let $D' = B - D$.

For each $V_i \in D'$, choose $x_i \in V_i$. Then choose

$$p_i \in (C-T) \cap B(g(x_i), \varepsilon_i),$$

where $\varepsilon_i = \frac{1}{4}\eta$ and where $B(g(x_i), \varepsilon_i)$ is the open ball of radius ε_i in C around $g(x_i)$. Similarly,

$$p_i^j = \gamma^j p_i \in (C-T) \cap B(\gamma^j g(x_i), \varepsilon_i) \quad \text{for each } j = 0, \ldots, p-1 .$$

For each $V_i \in D$ there exists $a_i \in A$ such that $d(V_i, A) = d(\overline{V}_i, A) = d(\overline{V}_i, a_i)$, and there exists $x_i \in V_i$ such that $d(x_i, a_i) < \delta$. Choose

$$p_i \in (C-T) \cap B(w(a_i), \varepsilon_i),$$

where $\varepsilon_i = \min\{\frac{1}{2}d(\overline{V}_i, a_i), \frac{1}{6}\eta\}$. Similarly,

$$p_i^j = \gamma^j p_i \in (C-T) \cap B(\gamma^j w(a_i), \varepsilon_i) \quad \text{for each } j = 0, \ldots, p-1 .$$

By the Equivariant General Position Lemma (3.5) [1], there exists a countable, equivariant set $\{q_i \mid i = 1, 2, \ldots\}^*$ in C with the property that $d(\gamma^j p_i, \gamma^j q_i) < \varepsilon_i$ for each $j = 0, \ldots, p-1$. Furthermore, the following inequalities hold:

(1) For $V_i \in D'$, $d(g(x_i), q_i) \leqslant d(g(x_i), p_i) + d(p_i, q_i) < \varepsilon_i + \varepsilon_i = \frac{1}{4}\eta + \frac{1}{4}\eta = \frac{1}{2}\eta$.

(2) For $V_i \in D$, $d(g(x_i), q_i) \leqslant d(g(x_i), w(a_i)) + d(w(a_i), p_i) + d(p_i, q_i) < \frac{1}{6}\eta + \frac{1}{6}\eta + \frac{1}{6}\eta = \frac{1}{2}\eta$.

Finally, define $h: (Z, c) \to (C, \gamma_C)$ as follows:

(3) $h|_A = w$.

(4) For each $j = 0, \ldots, p-1$, let $h(b^j v_i) = \gamma^j h(q_i)$ for the vertices $v_i \in K^0$. Then extend h linearly from K^0 to all of $|K| = Z-A$.

By definition h is equivariant, $h|_A = w$, and $h|_{Z-A}$ is simplicial. (1) and (2) above imply that condition (iv) of the lemma is satisfied. On A, h is clearly one-to-one. Lemma (3.1) tells us that on $|K|$ h is an embedding into the complement of the fixed point set of γ. To prove the lemma one need only show that h is continuous.

Since $|K|$ is open in Z and $h|_{|K|}$ is linear, it follows that h is continuous at each point in $Z-A$. So it remains to show that h is continuous at each point a in A. Let $a \in A$ and let $B_1 = B(w(a), r) \subset C$, $r > 0$. One wants a neighborhood N of a in Z such that $h(N) \subset B_1$.

Given r there exists a positive number M_1 such that, if $d(x, y) < M_1$, then $d(gx, gy) < \frac{1}{13}r$. By condition (i) of the Covering Lemma there exists a positive number M_2 such that if $d(V, A) < M_2$, then $\operatorname{diam}\operatorname{St} V < \min\{\frac{1}{4}\delta, \frac{1}{3}M_1, \frac{1}{13}r\}$. Note that this implies that if $d(V, A) < M_2$, than it is also true that $\operatorname{diam} V_i < \min\{\frac{1}{4}\delta, \frac{1}{3}M_1, \frac{1}{13}r\}$ for every $V_i \in B$ such that $V_i \subset \operatorname{St} V$. Let $B_2 = B(a, M_3) \subset X$, where

$$M_3 = \min\{\tfrac{1}{3}M_1, M_2, \tfrac{1}{13}r, \tfrac{1}{4}\delta\},$$

and define

$$N = \hat{B}_2 = (A \cap B_2) \cup \left(\bigcup [\operatorname{St}_K V| \ V \subset B_2]\right).$$

Note that $V \subset B_2$ implies that $d(V, A) < M_3$. By definition N is an open neighborhood of a in Z.

If $p \in A \cap B_2$, then $d(p, a) < M_1$ and it is true that $d(hp, ha) = d(wp, wa) < r$. Hence $hp \in B_1$.

If $p \in \bigcup [\operatorname{St}_K V| \ V \subset B_2]$, then $p \in \operatorname{St}_K V$ for some $V \subset B_2$. Let v_i be a vertex of some simplex containing p. By the construction of the simplicial complex K it follows that:

(5) $V_i \cap V \neq \emptyset$. This implies that

(6) $d(V_i, A) \leqslant \operatorname{diam}(V) + d(V, A) < \frac{1}{13}r + \frac{1}{13}r = \frac{2}{13}r$. Similarly, it follows that

(7) $d(V_i, A) \leqslant \operatorname{diam}(V) + d(V, A) < \frac{1}{4}\delta + \frac{1}{4}\delta = \frac{1}{2}\delta$, which implies that $V_i \in D$. Hence, it follows that

(8) $d(q_i, p_i) < d(V_i, a_i) = d(V_i, A)$ for every such vertex v_i.

Let $S = \{v_{i_1}, \ldots, v_{i_k}\}$ be the set of vertices of the (unique) open simplex in $|K|$ containing p. Then one has that $d(hp, ha) < r$, since, for each $v_i \in S$,

$$d(hv_i, ha) = d(q_i, wa) \leqslant d(q_i, p_i) + d(p_i, p_V) + d(p_V, wa_V) + d(wa_V, wa) = J.$$

By (6) above it follows that

(9) $d(q_i, p_i) < \frac{2}{13}r$.

$V \subset B_2$ implies that $d(V, A) < \frac{1}{4}\delta$ and hence

(10) $d(p_V, wa_V) < d(V, A) < \frac{1}{13}r$.

Furthermore, $V \subset B_2$ implies that $d(V, A) \leqslant d(V, a) < M_1$. Let $y_1, y_2 \in \overline{V}$ be such that $d(y_1, a_V) = d(\overline{V}, a_V) = d(V, a_V) = d(V, A)$ and $d(y_2, a) = d(\overline{V}, a) = d(V, a)$. Since $\operatorname{diam} V = \operatorname{diam} \overline{V} < M_1$, $d(y_1, a_V) < M_1$, and $d(y_2, a) < M_1$, it follows that

(11) $d(wa_V, wa) \leqslant d(wa_V, wy_1) + d(wy_1, wy_2) + d(wy_2, wa) < \frac{1}{13}r + \frac{1}{13}r + \frac{1}{13}r = \frac{3}{13}r$.

By (5) above one has $V_i \cap V \neq \emptyset$. Let $z \in V_i \cap V$. Then it follows that

(12) $d(p_i, p_V) \leqslant d(p_i, wa_i) + d(wa_i, gx_i) + d(gx_i, gz) + d(gz, gx_V) + d(gx_V, wa_V) + d(wa_V, p_V) = I$ where

(13) $d(p_i, wa_i) < d(V_i, A) < \frac{2}{13}r$,

(14) $d(a_i, x_i) \leqslant d(a_i, V_i) + \operatorname{diam}(V_i) \leqslant \operatorname{diam} V + d(V, A) + \operatorname{diam} V_i < \frac{1}{3}M_1 + \frac{1}{3}M_1 + \frac{1}{3}M_1 = M_1$, which implies $d(ga_i, gx_i) < \frac{1}{13}r$,

(15) $x_i, z \in V_i$ implies $d(x_i, z) < M_1$, which gives $d(gx_i, gz) < \frac{1}{13}r$,

(16) $z, x_V \in V$ implies $d(x_V, z) < M_1$, which also gives $d(gx_V, gz) < \frac{1}{13}r$,

(17) $d(x_V, a_V) \leqslant \operatorname{diam} V + d(a_V, V) < \frac{1}{3}M_1 + \frac{1}{3}M_1 < M_1$, which implies $d(gx_V, ga_V) < \frac{1}{13}r$,

(18) $d(wa_V, p_V) < d(V, a_V) = d(V, A) < \frac{1}{13}r$.

(13)–(17) imply

(19) $I < \frac{7}{13}r$.

(9), (10), (11), and (12) then imply that

(20) $d(hv_1, ha) = d(q_i, wa) \leqslant J < \frac{2}{13}r + \frac{1}{13}r + \frac{7}{13}r + \frac{3}{13}r = \frac{13}{13}r = r$.

In concluding from (20) that $d(hp, ha) < r$, one uses the fact that hp is in the simplex spanned by the vertices $q_{i_1}, \ldots, q_{i_k}$ and that each vertex is less than r distance from $h(a)$. Therefore, $h(N) \subset B_1$ and the proof is completed.

Note that the map $f\colon (X, a) \to (Z, c)$ used in Lemma (3.2) above was an equivariant δ-map. Consequently, given an $\varepsilon > 0$, δ could have been chosen less than ε from the beginning. This implies the following result.

LEMMA (3.3). *Let X be an n-dimensional compact metric space with a free Z_p-action. Suppose $\gamma\colon \boldsymbol{R}^{N+k} \to \boldsymbol{R}^{N+k}$, $N > n$, is an isometric, linear map of period p with fixed point set T (γ is free outside T) and $\dim(T) = k \geqslant n$. Let ε be a positive number and g an equivariant map from X into $\boldsymbol{R}^{N+k}$. Then there exists an ε-map arbitrarily close to g.*

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Equivariant embeddings of Z_p-actions in euclidean space

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Abstract. This paper shows that a finite dimensional compact metric space on which Z_p acts freely outside the fixed point set equivariantly embeds in a euclidean space with an orthogonal Z_p-action. Moreover, a minimum dimension for the euclidean space is obtained.

1. Introduction. Mostow [6] first showed that every action of a compact Lie group with a finite number of non-conjugate isotropy subgroups on a finite dimensional, separable, metrizable space can be equivariantly embedded in a linear action of the group on some euclidean space. In the case that the group is Z_p, the embedding has a particularly simple form, which is all that is required for the purposes of this paper. First, embed X in R^{2n+1} via i, and, then, embed X equivariantly in $R^{(2n+1)p}$ via $ex = (ix, iax, \ldots, ia^{p-1}x)$, where $a \in Z_p$ and where $\sigma(x_1, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$ generates an orthogonal Z_p-action on $R^{(2n+1)p}$. However, Mostow's theorem said nothing as to the required dimensions of the euclidean space. Copeland and de Groot [2] went on to show that every action of a cyclic group of prime order on an n-dimensional, separable, metrizable space can be equivariantly embedded in a linear action on R^{3n+2} or R^{3n+3}. Finally, Kister and Mann [5] extended the result of Copeland and de Groot to actions of compact Abelian Lie groups with a finite number of distinct isotropy subgroups. They found a dimension for a euclidean space appropriate for the embedding which depends only upon the dimension of the original space, the structure of the Abelian transformation group, and the number of distinct isotropy subgroups.

In the present work improvements on the result of Copeland and de Groot are obtained in the case of a compact, finite dimensional metric space with an action of a cyclic group. In particular, let X be a compact n-dimensional metric space with a map $a\colon X \to X$ of period p whose fixed point set is F. The map a then defines a Z_p-action on X. In this paper, (X, a) will denote the equivariant space (X, Z_p). Suppose this action is free outside of F and suppose F is embeddable in k-dimensional euclidean space, R^k, via an embedding w. In the case of an involution (i.e., $a^2 = 1_X$), let $m = \max\{k, n\}$ and $\alpha\colon R^{n+1} \times R^m \to R^{n+1} \times R^m$, where $\alpha = (\alpha_1, 1_{R^m})$ and $\alpha_1\colon R^{n+1} \to R^{n+1}$ is defined by $\alpha_1(x) = -x$. The following theorem is then proved.