Closure properties of countable non-standard integers

by

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Abstract. Let M be a model of Peano's Axioms (PA) and let I_ω^M be the largest initial segment of M all of whose proper initial segments are countable. We investigate the closure properties of I_ω^M and show that in a certain sense multiplication and exponentiation are essentially the only functions for which the cardinality of the arguments limit the cardinality of the values.

Introduction. Let $M \models \mathrm{PA}$. For $a \in M$ let

$$<a = \{b \in M \mid M \models b < a\}, \quad \|a\| = \mathrm{Card}(<a).$$

For λ an infinite cardinal let

$$I_\lambda^M = \{a \in M \mid \|a\| \leqslant \lambda\},$$
$$J^M(\lambda) = \mathrm{Inf}\{\|a\| \mid a \in M \text{ and } \|a\| > \lambda\}.$$

For economy we will use "initial segment" to mean "infinite initial segment". For I an initial segment of M we say I is closed under multiplication (exponentiation) iff $a, b \in I$ implies $ab \in I$ ($a^b \in I$), or equivalently $a \in I$ implies $a^2 \in I$ ($2^a \in I$). The equivalence for exponentiation follows from the inequality $a^a < 2^{2^a}$. Using model and set theoretic ideas within non-standard models of arithmetic we shall show the following results

1) If $|I_\omega^M| = \omega$, $J^M(\omega) \leqslant 2^\omega$ then I_ω^M is closed under multiplication and this is the fastest function under which I_ω^M needs be closed.

2) If $|I_\omega^M| = \omega$, $J^M(\omega) > 2^\omega$ then I_ω^M is closed under exponentiation and this is the fastest function under which I_ω^M needs be closed.

3) If $|I_\omega| > \omega$ (i e. $= \omega_1$) then $I_\omega^M \models \mathrm{PA}$.

1) follows from Lemma 1 and Theorem 2, 2) from Lemma 1 and Theorem 6 and 3) from Theorem 9. 1) and 2) can be viewed as saying that multiplication and exponentiation are the only sorts of function F for which $\|a\|$ gives information about $\|F(a)\|$.

We conclude this paper by stating some extensions of these results to I_λ^M for uncountable λ.

Before giving any results we introduce some more notation. Let A be a bounded subset of M^n. We say A *is in* M, written $A \in M$, if A is definable in M, equivalently if A can be coded by an element of M. For $A \in M$ we let $|A|$ denote the unique element a of M such that

$$M \models A \text{ has exactly } a \text{ elements}$$

and write $A<b$ if $b>c$ for all $c \in A$.

LEMMA 1. *For $M \models \mathrm{PA}$, I_ω^M is an initial segment of M closed under multiplication. If $J_{(\omega)}^M > 2^\omega$ then I_ω^M is closed under exponentiation.*

Proof. It is clear that I_ω^M is an initial segment. For $a \in M$ there is a map $G \in M$ such that G maps $<a \times <a$ 1-1 onto $<a^2$. Thus if $a \in I_\omega^M$ then

$$||a^2|| = \mathrm{Card}(||a|| \times ||a||) \leqslant \omega$$

so $a^2 \in I_\omega^M$.

Since in M there are exactly 2^a subsets of $<a$, $||2^a|| \leqslant 2^{||a||}$. Thus if $J^M(\omega) > 2^\omega$,

$$a \in I_\omega^M \to ||2^a|| \leqslant 2^\omega \to I_\omega^M \not< 2^a \to 2^a \in I_\omega^M. \blacksquare$$

The next two theorems show that without further assumptions on I_ω^M, Lemma 1 is the best possible.

THEOREM 2. *Let $M \models \mathrm{PA}$, M countable and I an initial segment of M closed under multiplication. Then there is an $N \succ M$ such that $J^N(\omega) = 2^\omega$ and $I = I_\omega^N$.*

Proof. By taking an end extension of M if necessary we may assume $I \neq M$. Let $I < a \in M$. Before proceeding with the construction of N we need three propositions.

PROPOSITION 3. *Let $A \subseteq (<a)^n$, $A \in M$ and $|A| \geqslant a^n/b$ some $b \in I$. Then $\exists X \subseteq <a$, $X \in M$ such that*

$$a^{n+1}/8b^2 \leqslant |\{\langle x_0, x_1, \vec{y}\rangle \mid x_0 \in X, x_1 \notin X \,\&\, \langle x_0, \vec{y}\rangle, \langle x_1, \vec{y}\rangle \in A\}|.$$

Proof. For $n = 1$ the result is clear to suppose $n \geqslant 2$. For each $\langle \vec{y}\rangle \in (<a)^{n-1}$ let

$$e_{\vec{y}} = \{x \mid \langle x, \vec{y}\rangle \in A\}.$$

Working in M consider the sum

$$k_{\vec{y}} = \sum_{\substack{X \subseteq <a \\ X \in M}} |(e_{\vec{y}} \cap X) \times (e_{\vec{y}} \cap \neg X) \times \{\vec{y}\}|$$

where $\neg X = <a - X$ and $\langle \vec{y}\rangle \in (<a)^{n-1}$. Let x_0, x_1 be distinct elements of $e_{\vec{y}}$. Then $\langle x_0, x_1, \vec{y}\rangle$ appears in 2^{a-2} of the sets $(e_{\vec{y}} \cap X) \times (e_y \cap \neg X) \times \{\vec{y}\}$.

Thus the above sum is $2^{a-2} \times$(the number of ways of picking two distinct elements from $e_{\vec{y}}$) $= 2^{a-2} \cdot (|e_{\vec{y}}|^2 - |e_{\vec{y}}|)$. Since

$$\sum_{\langle y\rangle \in (<a)^{n-1}} |e_{\vec{y}}| = |A| \geqslant \frac{a^n}{b},$$

$$\sum_{\langle y\rangle \in (<a)^{n-1}} |e_{\vec{y}}|^2 \geqslant \frac{a^2}{b^2} \cdot a^{n-1} = \frac{a^{n+1}}{b^2}.$$

Hence

$$\sum_{\langle y\rangle \in (<a)^{n-1}} k_{\vec{y}} \geqslant 2^{a-2} \cdot \frac{a^{n+1}}{2b^2}$$

since

$$\frac{a^{n+1}}{b^2} - |A| \geqslant \frac{a^{n+1}}{b^2} - a^n \geqslant \frac{a^{n+1}}{2b^2}.$$

Reversing the order of summation gives

$$\sum_{\substack{X \subseteq <a \\ X \in M}} \Big(\sum_{\langle \vec{y}\rangle \in (<a)^{n-1}} |(e_{\vec{y}} \cap X) + (e_{\vec{y}} \cap \neg X) \times \{\vec{y}\}| \Big) \geqslant 2^{a-3} \cdot \frac{a^{n+1}}{b^2}.$$

Therefore since there are only 2^a $X \in M$ with $X \subseteq <a$, for some $X \subseteq <a$, $X \in M$

$$\sum_{\langle \vec{y}\rangle \in (<a)^{n-1}} |(e_{\vec{y}} \cap X) \times (e_{\vec{y}} \cap \neg X) \times \{\vec{y}\}| \geqslant \frac{a^{n+1}}{8b^2}.$$

This is the required X. $\blacksquare$

Since M is countable we can find a decreasing sequence a_n, $n \in \omega$ of elements of M such that I is exactly the set of lower bounds of this sequence. Fix such a sequence.

PROPOSITION 4. *Let $G \in M$, $G: (<a)^n \to M$. Let $A \in M$, $A \subseteq (<a)^n$ and $|A| \geqslant a^n/b$ for some $b \in I$. Then there is an $X \in M$, $X \subseteq A$ such that $|X| \geqslant a^n/c$ some $c \in I$ and either*

(i) G is constant on X or

(ii) for some $n \in \omega$, a_n is a lower bound of $G''X$.

Proof. Find $q \in M$ such that

$$|\{\langle \vec{x}\rangle \in A \mid G(\vec{x}) < q\}| \leqslant |\{\langle \vec{x}\rangle \in A \mid G(\vec{x}) \geqslant q\}| \leqslant |\{\langle \vec{x}\rangle \in A \mid G(\vec{x}) \leqslant q\}|.$$

If $I < q$ set $X = \{\langle \vec{x}\rangle \in A \mid G(\vec{x}) \geqslant q\}$ so $|X| \geqslant \frac{1}{2}|A|$. If $q \in I$ set $Y = \{\langle \vec{x}\rangle \in A \mid G(\vec{x}) \leqslant q\}$ so $|Y| \geqslant \frac{1}{2}|A|$. By the pigeon hole principle in M $|G^{-1}\{e\} \cap Y| \geqslant |Y|/(q+1)$ for some $e \leqslant q$. Take $X = G^{-1}\{e\} \cap Y$. $\blacksquare$

PROPOSITION 5. *Let $A \subseteq (<a)^n$, $A \in M$ and $|A| \geqslant a^n/b$ for some $b \in I$. Let $a > c > I$. Then $\exists X_1, \ldots, X_n \subseteq <a$ such that $X_1, \ldots, X_n \in M$, $|X_1| = |X_2| = \ldots = |X_n| = c$ and $|A \cap (X_1 \times X_2 \times \ldots \times X_n)| \geqslant c^n/b$.*

Proof. Work in M. Consider all possible $X_1, \ldots, X_n \subseteq <a$ such that $|X_1| = |X_2| = \ldots = |X_n| = c$. There are $\left(\dfrac{a!}{c!(a-c)!}\right)^n$ such sequences. Consider

$\langle x_1, \ldots, x_n \rangle \in A$. This appears in $X_1 \times X_2 \times \ldots \times X_n$ for $\left(\frac{(a-1)!}{(c-1)!(a-c)!}\right)^n$ possible sequences $X_1, \ldots, X_n$. Thus

$$\sum_{X_1,\ldots,X_n} |(X_1 \times X_2 \times \ldots \times X_n) \cap A| = |A| \cdot \left(\frac{(a-1)!}{(c-1)!(a-c)!}\right)^n .$$

Hence one of the sets $(X_1 \times X_2 \times \ldots \times X_n) \cap A$ must have at least

$$\left(\frac{(a-1)!}{(c-1)!(a-c)!}\right)^n \cdot \left(\frac{a!}{c!(a-c)!}\right)^{-n} \cdot |A| \geqslant \frac{c^n}{b} \text{ elements. } \blacksquare$$

Before returning to the proof of Theorem 2 we introduce some notation. Let $m \leqslant n$, $G \in M$ and $G\colon (<a)^m \to M$. We say $G'\colon (<a)^n \to M$ is an n-permutation of G if there are distinct $i_1, \ldots, i_m \leqslant n$ such that

$$G'(x_1, \ldots, x_n) = G(x_{i_1}, \ldots, x_{i_m}) \quad \text{for all } x_1, \ldots, x_n \in {<a} .$$

We are now ready to construct the N claimed in Theorem 2. Let G_i, $i \in \omega$ enumerate all maps G such that $G \in M$ and $G\colon (<a_0)^n \to M$ for some $n \in \omega$. We assume that if $G_m\colon (<a_0)^n \to M$ then $n \leqslant m$.

We now define $A_n \subseteq (<a_0)^{2^n}$ inductively so that

a) $A_n \in M$,

b) $|A_n| \geqslant a_n^{2^n}/b$ for some $b \in I$.

c) $\exists X_1^n, \ldots, X_{2^n}^n \subseteq {<a_0}$ such that $A_n \subseteq X_1^n \times X_2^n \times \ldots \times X_{2^n}^n$, $|X_1^n| = |X_2^n| = \ldots$ $\ldots = |X_{2^n}^n| = a_n$ and $X_1^n, \ldots, X_{2^n}^n \in M$,

d) if $m < n$ and G is a 2^n-permutation of G_m then either G is constant on A_n or $G''A_n > a_i$ for some $i \in \omega$,

e) if $\langle x_1, \ldots, x_{2^n} \rangle \in A_n$ then the $x_1, \ldots, x_{2^n}$ are all distinct,

f) if $\langle x_1, \ldots, x_{2^n} \rangle \in A_n$, $n > 0$ and $f\colon \{1, 2, \ldots, 2^{n-1}\} \to \{0, 1\}$ then

$$\langle x_{2-f(1)}, x_{4-f(2)}, \ldots, x_{2^n - f(2^{n-1})} \rangle \in A_{n-1} .$$

Set $A_0 = {<a_0} = X_1^0$. Now suppose A_n has been found successfully. In view of the existence of the X_i^n we can visualize A_n as a subset of $(<a_n)^{2^n}$. We shall apply Propositions 3, 4, 5 with this visualization in mind.

By Proposition 3 choose $B_1^n \subseteq X_1^n$ such that

$$\frac{a_n^{2^n+1}}{c} \leqslant |\{\langle x_1, x_1', \vec{y} \rangle |\ x_1 \in B_1^n, x_1' \in {'B_1^n} \ \&\ \langle x_1, \vec{y} \rangle, \langle x_1', \vec{y} \rangle \in A_n\}|$$

for some $c \in I$. Set C_1^n to be the set within the modulus signs. Now by Proposition 1 choose $B_2^n \subseteq X_2^n$ such that

$$\frac{a_n^{2^n+2}}{c} \leqslant |\{\langle x_1, x_1', x_2, x_2', \vec{y} \rangle |\ x_2 \in B_2^n, x_2' \in {'B_2^n}, \langle x_1, x_1', x_2, \vec{y} \rangle, \langle x_1, x_1', x_2', \vec{y} \rangle \in C_1^n\}|$$

for some $c \in I$.

Let C_2^n be the set within the modulus signs. Carry on like this to find $C_{2^n}^n \subseteq (<a_0)^{2^{n+1}}$ such that

$$|C_{2^n}^n| \geqslant \frac{a_n^{2^{n+1}}}{c} \quad \text{for some } c \in I .$$

Now by Proposition 4 pick $E_{n+1} \subseteq C_{2^n}^n$ such that $E_{n+1} \in M$, $|E_{n+1}| \geqslant \frac{a_n^{2^{n+1}}}{c}$ for some $c \in I$ and all the 2^{n+1}-permutations of $G_0, \ldots, G_n$ are constant on E_{n+1} or bounded above I. Finally using Proposition 5 pick $X_{2i}^{n+1}, X_{2i+1}^{n+1} \subseteq X_i^n$ all of modulus a_{n+1} such that

$$|E_{n+1} \cap (X_1^{n+1} \times X_2^{n+1} \times \ldots \times X_{2^{n+1}}^{n+1})| \geqslant \frac{a_{n+1}^{2^{n+1}}}{c}$$

for some $c \in I$. Set

$$A_{n+1} = E_{n+1} \cap (X_1^{n+1} \times X_2^{n+1} \times \ldots \times X_{2^{n+1}}^{n+1}) .$$

A_{n+1} clearly satisfies a), b), c), d), e). To see f) notice that

$$\langle x_1, x_1', \vec{y} \rangle \in C_1^n \to \langle x_1, \vec{y} \rangle, \langle x_1', \vec{y} \rangle \in A_n ,$$

$$\langle x_1, x_1', x_2, x_2', \vec{y} \rangle \in C_2^n \to \langle x_1, x_1', x_2, \vec{y} \rangle, \langle x_1, x_1', x_2', \vec{y} \rangle \in C^n$$

$$\to \langle x_1, x_2, \vec{y} \rangle, \langle x_1, x_2', \vec{y} \rangle, \langle x_1', x_2, \vec{y} \rangle,$$

$$\langle x_1', x_2', \vec{y} \rangle \in A_n$$

and so on.

For $n \in \omega$ let $G_1^n, \ldots, G_{2^n}^n$ enumerate ${}^n2$ in the usual lexicographic ordering. We shall construct N using a compactness argument. Add to the language of arithmetic new constants $\underline{b}$ for $b \in M$ and e_f for $f \in {}^\omega 2$. Let Z be the set of sentences $\theta(e_{f_1}, \ldots, e_{f_n}, \underline{b}_1, \ldots, \underline{b}_n)$ in this language such that for all k eventually, if $e_{f_i \restriction k} = G_{j_i}^k$, $i = 1, \ldots, n$ and $\langle x_1, \ldots, x_{2^k} \rangle \in A_k$ then $M \models \theta(x_{j_1}, \ldots, x_{j_n}, b_1, \ldots, b_n)$.

Clearly Z is consistent. Let N^+ be a model of Z and N the elementary substructure generated by $\{e_f |\ f \in {}^\omega 2\} \cup \{\underline{b} |\ b \in M\}$. So up to isomorphism. $M \prec N$. It only remains to show that $I_\omega^N = I$, $J^N(\omega) = 2^\omega$.

Certainly by e) for $f, g \in {}^\omega 2$, $f \neq g$

$$(e_f \neq e_g) \in Z$$

so N has cardinality 2^ω. Furthermore in M, $|A_k| \leqslant a_k^{2^k}$. From this it follows that $2^\omega = |A_k|_N \leqslant ||a_k^{2^k}||_N$ and hence $||a_k||_N = 2^\omega$. For suppose $G_i^k \subset f_i \in {}^\omega 2$ for $i = 1, \ldots, 2^k$. Let $j \geqslant k$ and $f_i \restriction j = G_{p_i}^j$ for $i = 1, \ldots, 2^k$. Then by f),

$$\langle x_1, \ldots, x_{2^j} \rangle \in A_j \to \langle x_{p_1}, \ldots, x_{p_{2^k}} \rangle \in A_k \quad \text{so} \quad (\langle e_{f_1}, \ldots, e_{f_{2^k}} \rangle \in A_k) .$$

The result follows since there are 2^ω possible $\langle f_1, \ldots, f_{2^k} \rangle$.

Thus it only remains to show that if $a \in N$ then either $N \models \underline{a}_k \leqslant a$ some k or $N \models a = \underline{b}$ some $b \in M$. Suppose then $N \models a = \mu x\colon \theta(x, e_{f_1}, \ldots, e_{f_n}, \underline{b}_1, \ldots, \underline{b}_m)$, $f_1, \ldots, f_n$ distinct.

Let k be such that G_k: $(<a_0)^n \to M$,

$$G_k(x_1, \dots, x_n) = \mu x\colon \theta(x, x_1, \dots, x_n, b_1, \dots, b_m) \text{ if this exists,}$$
$$= 0 \text{ otherwise.}$$

Let $j>k$ be such that $f_1 \upharpoonright j, \dots, f_n \upharpoonright j$ are all distinct, say $f_i \upharpoonright j = G^j_{p_i}$, $i = 1, \dots, n$. Let G be the 2^j-permutation of G_k such that

$$G(x_1, \dots, x_{2^j}) = G_k(x_{p_1}, \dots, x_{p_n}) .$$

Then G is either constant on A_j or bounded below on A_j by some a_k. If the former occurs then for some $b \in M$,

$$b = \mu x\colon \theta(x, x_{p_1}, \dots, x_{p_n}, b_1, \dots, b_m) \text{ if this exists,}$$
$$= 0 \text{ otherwise}$$

for all $\langle x_1, \dots, x_{2^j}\rangle \in A_j$. Clearly by f) the corresponding result holds for $j'>j$. Hence

$$((\exists x)[\theta(x, e_{f_1}, \dots, e_{f_n}, \underline{b}_1, \dots, \underline{b}_m) \to \underline{b} \text{ is least such } x]) \in Z .$$

Thus $N \models \underline{b} = \mu x\colon \theta(x, e_{f_1}, \dots, e_{f_n}, \underline{b}_1, \dots, \underline{b}_m)$, so $N \models a = \underline{b}$. Similarly if the second option occurs we see $N \models a \geqslant \underline{a}_k$. Thus Theorem 2 is proved. ■

Remark. Since we could take I to be of the form $\{b \in M \mid M \models b < a^n, n \in \omega\}$ for some non-standard $a \in M$, multiplication is the fastest function under which I^N_ω needs be closed.

A second consequence of this result is that there is a model N of PA and $a \in N$ such that $\|a\|_N = \|2^a\|_N = \omega$ whilst $\|a^a\|_N = 2^\omega$.

We now prove a result for the case $\mathrm{J}^N(\omega) > 2^\omega$.

THEOREM 6. *Let $M \models$ PA, M countable and I an initial segment of M closed under exponentiation. Let $\varkappa$ be an infinite cardinal. Then there is an $N \succ M$ such that $J^N(\omega) = \varkappa$ and $I = I^N_\omega$.*

Proof. As with Theorem 2 we first need some propositions. The proof of the next proposition mimics the proof of the Erdös–Rado theorem in Set Theory.

PROPOSITION 7. *Let A, $F \in M$, $a \in I < |A|$ and F: $[A]^n \to <a$. Then $\exists X \subseteq A$, $X \in M$, $I < |X|$ such that F is constant on $[X]^n$.*

$$([A]^n = \{\langle x_1, \dots, x_n\rangle \mid x_1, \dots, x_n \in A \text{ and } x_1 < x_2 < \dots < x_n\}) .$$

Proof. If $n = 1$ the result is clear so suppose $n > 1$ and the result proved for $n-1$. Working in M define a tree T as follows. T has initial element

$$\langle \min(A), A - \{\min(A)\}\rangle .$$

Suppose $T(b)$, the elements of T of level b, has been found and suppose

$$\langle x, Y\rangle \in T(b) \to x \in A,\ A \supseteq Y \in M \text{ and } x < Y \text{ (i.e. } x < y \text{ for all } y \in Y) .$$

Set

$$D(x, Y) = \{u \mid \exists W, \langle u, W\rangle \leqslant \langle x, Y\rangle \text{ in } T\} ,$$

so $|D(x, Y)| = b+1$ for $\langle x, Y\rangle \in T(b)$. For each G: $[D(x, Y)]^{n-1} \to <a$, $G \in M$ let

$$e_G = \{z \in Y \mid F(\vec{x}, z) = G(\vec{x}) \text{ for all } \langle \vec{x}\rangle \in [D(x, Y)]^{n-1}\} .$$

Thus there are $a^{|[D(x,Y)]^{n-1}|} (\leqslant a^{(b+1)^{n-1}})$ possible e_G's. The elements above $\langle x, Y\rangle \in T(b)$ on level $b+1$ are the pairs

$$\langle \min(e_G), e_G - \{\min(e_G)\}\rangle \quad \text{for} \quad e_G \neq \varnothing .$$

By induction in M, $|T(b)| \leqslant a^{(b+1)^n}$. We shall show $T(b) \neq \varnothing$ for some $b > I$. Suppose not. Then $T(b) = \varnothing$ for some $b \in I$. This means that for every $x \in A$ there is a $c < b$ such that $\langle x, Y\rangle \in T(c)$ for some Y. But then

$$|A| \leqslant \Big|\bigcup_{c<b} T(c)\Big| \leqslant \sum_{i=0}^{b-1} a^{(i+1)^n} < b \cdot a^{(b+1)^n} \in I \text{ — a contradiction.}$$

Now pick $b > I$, $T(b) \neq \varnothing$ and let $\langle x, Y\rangle \in T(b)$, $U = D(x, Y)$. Then $|U| = b+1 > I$ and if $\langle x_1, \dots, x_{n+1}\rangle \in |U|^{n+1}$ then

$$F(x_1, \dots, x_n) = F(x_1, \dots, x_{n-1}, x_{n+1}) = H(x_1, \dots, x_{n-1}) \text{ say.}$$

By assumption pick $X \subseteq U$, $X \in M$, $|X| > I$ such that H is constant on $[X]^{n-1}$. Then F is constant on $[X]^n$. ■

The next proposition works in the same way that Proposition 4 did in the proof of Theorem 2.

PROPOSITION 8. *Let $G, A \in M$, $I < |A|$, $G\colon [A]^n \to M$. Then $\exists X \subseteq A$, $X \in M$, $I < |X|$ such that either*

(i) *$\exists b > I$, $G''[X]^n > b$ or*

(ii) *G is constant on $[X]^n$.*

Proof. For each $a \in M$ let $H_a\colon [A]^n \to 2$ by $H_a(x) = 0 \leftrightarrow G(x) \geqslant a$. Working in M pick for each H_a a maximal homogeneous set X_a. If $H''_a[X_a]^n = \{0\}$ for some $a > I$ take $X = X_a$ and notice that by Proposition 7, $|X| > I$. Otherwise $H''_a[X_a]^n = \{1\}$ for some $a \in I$, and $|X_a| > I$ so by using Proposition 7 we can find $X \subseteq X_a$ to satisfy (i). ■

Proof of Theorem 6 continued. The method of proof is similar to the proof of Theorem 2.

Let a_n be a decreasing sequence of elements of M whose set of lower bounds is precisely I. Let G_i, $i \in \omega$ enumerate all maps $G \in M$, $G\colon [<a_0]^n \to M$ for some n.

We define decreasing subsets A_n of $<a_0$ satisfying

a) $A_n \in M$,

b) $I < |A_n| \leqslant a_n$,

c) if $i < n$ and $G_i\colon [<a_0]^m \to M$

then either G_i is constant on $[A_n]^m$ or $G''_i[A_n]^m \geqslant a_k$ for some k.

Put $A_0 = <a_0$. Suppose A_n constructed. Using Proposition 8 find $X \subseteq A_n$, $X \in M$, $I < |X|$ such that c) holds for G_n. Now let $A_{n+1} \subseteq X$, $A_{n+1} \in M$ such that $I < |A_{n+1}| \leqslant a_{n+1}$.

Now add to the language of arithmetic $\varkappa$ new constants e_ν, $\nu<\varkappa$ and $\underline{b}$ for each $b\in M$. Let Z be the set of sentences $\theta(e_{\nu_1},\dots,e_{\nu_n},\underline{b}_1,\dots,\underline{b}_m)$ such that $\nu_1<\nu_2<\dots<\nu_n$ and for all t eventually, all $\langle x_1,\dots,x_n\rangle\in[A_t]^n$,

$$M\models\theta(x_1,\dots,x_n,b_1,\dots,b_m).$$

Clearly Z is consistent. Let N^+ be a model of Z and N the elementary substructure of N^+ generated by the e_ν, $\nu<\varkappa$ and $\underline{b}$, $b\in M$. Then, up to isomorphism, $M\prec N$. From b) it follows that $\|a_n\|_N=\varkappa$ for $n\in\omega$ and hence using c) it follows that $J^N(\omega)=\varkappa$, $I_\omega^N=I$. ■

Remarks. If we define the super power function, Sp, by

$$\mathrm{Sp}(a,0)=1,\quad \mathrm{Sp}(a,x+1)=a^{\mathrm{Sp}(a,x)}$$

then the I in Theorem 6 could be taken to be

$$\{b\in M\mid M\models b<\mathrm{Sp}(a,n),n\in\omega\}$$

for some non-standard $a\in M$. In this sense then exponentiation is the fastest function under which I_ω^N needs be closed, assuming $J^N(\omega)>2^\omega$.

The above construction also enables us to show the following:

For any complete extension T of PA and any infinite cardinal $\varkappa$, there is an elementary end extension N of the minimal model M_0 of T such that $\operatorname{card}N=\varkappa$ and every elementary submodel of N except M_0 is cofinal in N.

This follows from the proof of Theorem 6 by letting M be a minimal elementary end extension of M_0 and choosing I so that $M_0\subset I\subset M$ (strict inclusion). Then M is cofinal in the model N produced in the proof, and N has cardinality $\varkappa$. Suppose $N'\prec N$ is not cofinal in N. Then $N'<b$ for some $b\in M$. Setting

$$M'=\{x\in M\mid \exists y\in N'\ x<y\}$$

one can show by checking for closure under Skolem functions that $M'\prec M$. Since M is minimal over M_0 and M_0 is the minimal model of T, it follows that $M'=M_0$. Thus $\forall x\in N'$, $\forall y\in M-M_0\ x<y$ and hence $\forall x\in N'\exists y\in I\ x<y$ (namely, y can be any element of $I-M_0$). Since by construction

$$\{x\in N\mid \exists y\in I\ x<y\}\subseteq M$$

we get that $N'\subseteq M$. Therefore $N'\prec M$, and since $N'\neq M$ it follows that $N'=M_0$, as desired.

In the two previous theorems the I_ω^N were both countable. As the next theorem shows this was essential. Notice $|I_\omega^M|>\omega\to|I_\omega^M|=\omega_1$.

THEOREM 9. *If $M\models$ PA and $|I_\omega^M|=\omega_1$ then $I_\omega^M\models$ PA.*

Proof. We first need an easy proposition.

PROPOSITION 10. *Let I_ω^M be as in Theorem 9, $a,\vec{b}\in I_\omega^M$. Let $\theta(\vec{x},\vec{b})$ be $\Pi_n(\Sigma_n)$. Then there is a Δ_0 formula $\psi(\vec{x},\vec{b},\vec{e})$, $\vec{e}\in I_\omega^M$ such that for all $\vec{x}<a$,*

$$I_\omega^M\models\theta(\vec{x},\vec{b})\leftrightarrow\psi(\vec{x},\vec{b},\vec{e}).$$

Proof. By induction on n. Result clear if $n=0$ so assume $n\geq 1$. Let $\theta(\vec{x},\vec{b})=(\forall\vec{z})\eta(\vec{x},\vec{z},\vec{b})$ where η is Σ_n. Since there are only countably many $\vec{x}<a$ and since I_ω^M has cofinality ω_1 we can find $c\in I_\omega^M$ such that for all $\vec{x}<a$,

$$I_\omega^M\models(\exists\vec{z})\neg\eta(\vec{x},\vec{z},\vec{b})\leftrightarrow(\exists\vec{z}<c)\neg\eta(\vec{x},\vec{z},\vec{b}).$$

By inductive hypothesis there is a Δ_0 formula $\chi(\vec{x},\vec{z},\vec{b},\vec{d})$ $\vec{d}\in I_\omega^M$, such that for all $\vec{x}<a$, $\vec{z}<c$,

$$I_\omega^M\models\neg\eta(\vec{x},\vec{z},\vec{b})\leftrightarrow\chi(\vec{x},\vec{z},\vec{b},\vec{d}).$$

Thus for $\vec{x}<a$,

$$\begin{aligned}I_\omega^M\models\theta(\vec{x},\vec{b})&\leftrightarrow(\forall\vec{z})\eta(\vec{x},\vec{z},\vec{b})\\&\leftrightarrow(\forall\vec{z}<c)\eta(\vec{x},\vec{z},\vec{b})\\&\leftrightarrow(\forall\vec{z}<c)\neg\chi(\vec{x},\vec{z},\vec{b},\vec{d})\end{aligned}$$

this last formula being the required ψ. ■

Proof of Theorem 9 continued. It is enough to show that the axiom of induction holds in I_ω^M. So suppose $I_\omega^M\models\theta(0)\wedge(\forall x)(\theta(x)\to\theta(x+1))$. Let $a\in I_\omega^M$, and by Proposition 10, find a Δ_0 formula $\psi(x)$ (maybe containing elements of I_ω^M) such that for all $x\leq a$,

$$I_\omega^M\models\psi(x)\leftrightarrow\theta(x).$$

Then $I^M\models\psi(0)\wedge(\forall x<a)(\psi(x)\to\psi(x+1))$. Since I_ω^M is an initial segment of M and this last formula is Δ_0,

$$M\models\psi(0)\wedge(\forall x<a)(\psi(x)\to\psi(x+1)).$$

Since $M\models$ PA, (indeed bounded induction is enough),

$$M\models\psi(a).$$

Thus $I_\omega^M\models\psi(a)$ and so $I_\omega^M\models\theta(a)$. Therefore $I_\omega^M\models(\forall x)\theta(x)$ and the theorem is proved. ■

Extension to the uncountable case. If we replace ω by an uncountable cardinal λ then Lemma 1 and Theorem 9 (with λ^+ in place of ω_1) go through as before.

Theorem 6 generalizes immediately but trivially to the uncountable case by noting that $\omega<\lambda<\varkappa\ \&\ J^N(\omega)=\varkappa\to J^N(\lambda)=\varkappa\ \&\ I_\lambda^N=I_\omega^N$. However we can also obtain the following generalization with nontrivial I_λ^N.

THEOREM 11. *Let $M\models$ PA with M countable, and let I be an initial segment of M closed under exponentiation and containing a nonstandard element. Let $\varkappa,\lambda$ be infinite cardinals with $\lambda<\varkappa$. Then there exists $N\succ M$ such that $J^N(\lambda)=\varkappa$,*

$$I_\lambda^N=\{x\in N\mid \exists y\in I\,x<y\},$$

and $\exists a\in N\ \|a\|=\lambda$

Proof. We need the following generalization of Proposition 8:

PROPOSITION 12. *Let I_1, I_2 be initial segments of M closed under exponentiation. Let A, B, F, $d \in M$ satisfy $I_1 < |A| \in I_2$, $d \in I_2 < |B|$, and F: $[A]^m \times [B]^n \to <d$ for some m, $n \in \omega$. Then there exist X, $Y \in M$ such that $X \subseteq A$, $Y \subseteq B$, $I_1 < |X|$, $I_2 < |Y|$ and $F(\vec{a}, \vec{b})$ is independent of $\vec{b}$ for $\langle \vec{a}, \vec{b} \rangle \in [X]^m \times [Y]^n$.*

PROPOSITION 13. *If we assume $d \in I_1$ in Proposition 12, then we can choose X, Y so that F is constant on $[X]^m \times [Y]^n$.*

Proof. For each $\vec{b} \in [B]^n$ consider the induced function $F_{\vec{b}}$: $[A]^m \to <d$ given by $F_{\vec{b}}(\vec{a}) = F(\vec{a}, \vec{b})$. This is a function inside M. Since there are at most $e = d^{|[A]^m|}$ such functions in M, we may consider the map $\vec{b} \to F_{\vec{b}}$ to be coded by $G \in M$, G: $[B]^n \to <e$. Here $e \in I_2$ since I_2 is closed under exponentiation. Therefore by Proposition 7 there is $Y \in M$, $I_2 < |Y|$, $Y \subseteq B$ such that G is constant on $[Y]^n$. Thus $F(\vec{a}, \vec{b})$ is independent of $\vec{b}$ for $\langle \vec{a}, \vec{b} \rangle \in [A]^m \times [Y]^n$. This proves Proposition 12. If $d \in I_1$, then picking any $\vec{b} \in [Y]^n$ and applying Proposition 7 to $F_{\vec{b}}$, we get $X \in M$, $X \subseteq A$, $I_1 < |X|$ such that $F_{\vec{b}}$ is constant on $[X]^m$. Since $F(\vec{a}, \vec{b})$ is independent of $\vec{b}$ on $[A]^m \times [Y]^n$, it is constant on $[X]^m \times [Y]^n$. This proves Proposition 13. ■

Proof of Theorem 11 (continued). As in the proof of Theorem 6, let b_n, $n \in \omega$, be a decreasing sequence of elements of M whose set of lower bounds is precisely I. Let a_0 be a nonstandard element of I. Let G_i, $i \in \omega$, enumerate all maps $G \in M$ such that G: $[<a_0]^m \times [<b_0]^n \to M$ for some m, $n \in \omega$. We define a sequence of pairs $\langle A_i, B_i \rangle$, $i \in \omega$, satisfying

a) $A_i, B_i \in M$, $A_i \subseteq A_{i-1} \subseteq <a_0$, $B_i \subseteq B_{i-1} \subseteq <b_0$,

b) $\omega < |A_i|$, $I < |B_i| \leqslant b_i$,

c) if $j < i$, G_j: $[<a_0]^m \times [<b_0]^n \to M$, and $C = [A_i]^m \times [B_i]^n$ then either

i) $G_j'' C \geqslant d$ for some $d \in M - I$ or

ii) $d > G_j'' C$ for some $d \in I$ and $G_j(\vec{a}, \vec{b})$ is independent of $\vec{b}$ for $\langle \vec{a}, \vec{b} \rangle \in C$.

Put $A_0 = <a_0$, $B_0 = <b_0$. Suppose $\langle A_i, B_i \rangle$ is given. By taking a subset if necessary we may assume $|B_i| \leqslant b_{i+1}$. Let

$$I_1 = \{x \in M|\ \forall n \in \omega \operatorname{Sp}(x, n) < |A_i|\} \quad \text{and} \quad I_2 = \{x \in M| \forall n \in \omega \operatorname{Sp}(x, n) < |B_i|\}\,.$$

Then I_1, I_2 are closed under exponentiation and $\omega \subset I_1 \subset I \subset I_2$, $I_1 < |A_i|$, $I_2 < |B_i|$. Let $S(<, d, e)$ be the statement

$$\exists X \subseteq A_i \exists Y \subseteq B_i (\operatorname{Sp}(|X|, e) \geqslant |A_i| \wedge \operatorname{Sp}(|Y|, e) \geqslant |B_i| \wedge G_i''[X]^m \times [Y]^n < d)\,.$$

Let $S(\geqslant, d, e)$ be the same statement but with $\geqslant d$ replacing $<d$. Note: e can be chosen standard, $e \in \omega$, iff X, Y can be chosen so that $I_1 < |X|$, $I_2 < |Y|$. Also $S(<, d, e) \wedge e < e' \to S(<, d, e')$ and similarly for $S(\geqslant, d, e)$. Finally, for every $d \in M$, by applying Proposition 13 to I_1, I_2, A_i, B_i, H_d, 2 where H_d: $[A_i]^m \times [B_i]^n \to 2$ is given by

$$H_d(\vec{a}, \vec{b}) = \begin{cases} 1 & \text{if} \quad G(\vec{a}, \vec{b}) \geqslant d\,, \\ 0 & \text{if} \quad G_i(\vec{a}, \vec{b}) < d \end{cases}$$

there is $e \in \omega$ such that

$$M \vDash S(<, d, e) \vee S(\geqslant, d, e)\,.$$

Now, let $r = \max G_i''[A_i]^m \times [B_i]^n$. Then $M \vDash S(<, r+1, 0) \wedge \neg S(\geqslant, r+1, 0)$. Let d_0 be the least d such that $M \vDash \exists e (S(<, d, e) \wedge \forall f \leqslant e \neg S(\geqslant, d, f))$. If $d_0 > I$, let $A_{i+1} = X$, $B_{i+1} = Y$ where X, Y witness $S(\geqslant, d_0 - 1, e)$ for some $e \in \omega$. Then a), b) and c, i) hold.

If $d_0 \in I$, let X, Y witness $S(<, d_0, e)$ for some $e \in \omega$ and continue as follows: By Proposition 12 choose X_0, $Y_0 \in M$ such that $X_0 \subseteq X$, $Y_0 \subseteq Y$, $\omega < |X_0|$, $I < |Y_0|$ and $G_i(\vec{a}, \vec{b})$ is independent of $\vec{b}$ for $\langle \vec{a}, \vec{b} \rangle \in [X_0]^m \times [Y_0]^n$. Then letting $A_{i+1} = X_0$, $B_{i+1} = Y_0$ we have a), b) and c, ii) holding. This completes the description of the sequence $\langle A_i, B_i \rangle$, $i \in \omega$.

Now add three new sets of constants to the language of arithmetic:

$$C = \{\underset{\sim}{c}|\ c \in M\}\,,$$
$$D = \{d_\alpha|\ \alpha < \lambda\}\,,$$
$$E = \{e_\beta|\ \beta < \varkappa\}\,.$$

Let Z be the set of sentences $\theta(d_{\alpha_1}, \ldots, d_{\alpha_m}, e_{\beta_1}, \ldots, e_{\beta_n}, \underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_r)$ such that $\alpha_1 < \alpha_2 < \ldots < \alpha_m < \lambda$, $\beta_1 < \ldots < \beta_n < \varkappa$ and for all sufficiently large $i \in \omega$, all $\langle a_1, \ldots, a_m \rangle \in [A_i]^m$, $\langle b_1, \ldots, b_n \rangle \in [B_i]^n$,

$$M \vDash \theta(a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_r)\,.$$

As in the proof of Theorem 6, let $N^+ \vDash Z$ and let $N \prec N^+$ be generated by $C \cup D \cup E$, so that $M \prec N$. Then $||a||_N = \lambda$ for all $a \in I$, $a \geqslant a_0$ by a) and c). Also $||b_k||_N = \varkappa$ for all $k \in \omega$ by a) and b) and the fact that $\operatorname{card} N = \varkappa$. Finally for all $a \in N$ there exists $b \in M$ such that $a \leqslant b \in I$ or $I < b \leqslant a$, by c). Thus I_λ^N and $J^N(\lambda)$ are as required. ■

By taking $I = \{b \in M|\ \exists n \in \omega\, M \vDash b < \operatorname{Sp}(a, n)\}$ for some nonstandard $a \in M$, we have that exponentiation is the fastest function under which I_λ^M needs be closed, no matter how large $J^M(\lambda)$ is, provided $I_\lambda^M \leqslant \lambda$.

Using Theorem 2 and Chang's Theorem (assuming G.C.H.) we can show that for each regular uncountable λ there is a model $M \vDash$ PA in which $J^M(\lambda) = \lambda^+$ and I_λ^M is not closed under exponentiation. We do not know whether the G.C.H. can be removed or whether this I_λ^M can be of the form

$$b \in \{M|\ M \vDash b < a^n, n \in \omega\}$$

for some non-standard $a \in M$.

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