Closure properties of countable non-standard integers

by

J. B. Paris (Manchester) and G. Mills (Berkeley)

Abstract. Let $M$ be a model of Peano's Axioms (PA) and let $I^M_\omega$ be the largest initial segment of $M$ all of whose proper initial segments are countable. We investigate the closure properties of $I^M_\omega$ and show that in a certain sense multiplication and exponentiation are essentially the only functions for which the cardinality of the arguments limit the cardinality of the values.

Introduction. Let $M \models PA$. For $a \in M$ let

$$< a = \{ b \in M \mid M \models b < a \}, \quad ||a|| = \text{Card}(< a).$$

For $\lambda$ an infinite cardinal let

$$I^M_\lambda = \{ a \in M \mid ||a|| \leq \lambda \},$$

$$J^M(\lambda) = \text{Inf}(||a||), \quad a \in M \text{ and } ||a|| > \lambda.$$

For economy we will use "initial segment" to mean "infinite initial segment!". For $I$ an initial segment of $M$ we say $I$ is closed under multiplication (exponentiation) iff $a, b \in I$ implies $ab \in I$ ($a^b \in I$), or equivalently $a \in I$ implies $a^b \in I$ ($2^a \in I$). The equivalence for exponentiation follows from the inequality $a^b < 2^{ab}$. Using model and set theoretic ideas within non-standard models of arithmetic we shall show the following results:

1) If $||a|| = \omega$, $J^M(\omega) \leq 2^{\omega}$ then $I^M_\omega$ is closed under multiplication and this is the fastest function under which $I^M_\omega$ needs be closed.

2) If $||a|| = \omega$, $J^M(\omega) > 2^{\omega}$ then $I^M_\omega$ is closed under exponentiation and this is the fastest function under which $I^M_\omega$ needs be closed.

3) If $||a|| > \omega$ (i.e., $\omega_1$) then $I^M_\omega \not\models PA$.

1) follows from Lemma 1 and Theorem 2, 2) from Lemma 1 and Theorem 6 and 3) from Theorem 9. 1) and 2) can be viewed as saying that multiplication and exponentiation are the only sorts of function $F$ for which $||a||$ gives information about $|F(a)|$.

We conclude this paper by stating some extensions of these results to $I^M_\lambda$ for uncountable $\lambda$. 
Before giving any results we introduce some more notation. Let $A$ be a bounded subset of $M^c$. We say $A$ is in $M$, written $A \in M$, if $A$ is definable in $M$, equivalently if $A$ can be coded by an element of $M$. For $A \in M$ we let $|A|$ denote the unique element $a$ of $M$ such that

$M \vdash A$ has exactly $a$ elements

and write $A < b$ if $b > c$ for all $c \in A$.

**Lemma 1.** For $M \vdash PA$, $J^M_{\omega}$ is an initial segment of $M$ closed under multiplication. If $J^M_{2^n} > 2^n$, then $J^M_{\omega}$ is closed under exponentiation.

**Proof.** It is clear that $J^M_{\omega}$ is an initial segment. For $a \in M$ there is a map $G \in M$ such that $G$ maps $< a \times < a - 1$ onto $< a$. Thus if $a \in J^M_{\omega}$ then

$|\omega| = \text{Card}(|a| \times |a|) \leq \omega$

so $a^2 \in J^M_{\omega}$.

Since in $M$ there are exactly $2^n$ subsets of $< a$, $\||\omega|\| < 2^{3|a|}$. Thus if $J^M(\omega) > 2^n$,

$a \in J^M_{\omega} \Rightarrow |\omega| < 2^n \Rightarrow J^M_{\omega} \not= 2^n \Rightarrow 2^n \in J^M_{\omega}$. □

The next two theorems show that without further assumptions on $J^M$, Lemma 1 is the best possible.

**Theorem 2.** Let $M \vdash PA$, $M$ countable and $I$ an initial segment of $M$ closed under multiplication. Then there is an $N > M$ such that $J^N(\omega) = 2^n$ and $I = J^N_{\omega}$.

**Proof.** By taking an end extension of $M$ if necessary we may assume $I \not= M$.

Let $I < \omega \in M$. Before proceeding with the construction of $N$ we need three propositions.

**Proposition 3.** Let $A \subseteq \langle \omega \rangle^+$, $A \in M$ and $|A| \geq d/b$ some $b \in I$. Then $\exists X \subseteq \langle \omega \rangle^+$, $X \in M$ such that

$\langle \omega \rangle^+ \cup 2 \subseteq \{\langle x_0, x_1, \tilde{y} \rangle \mid x_0 \in X, x_1 \in X' \times < x_0, \tilde{y} \rangle, \langle x_1, \tilde{y} \rangle \in A\}$.

**Proof.** For $n = 1$ the result is clear to suppose $n > 2$. For each $\langle \tilde{y} \rangle \in < \omega \rangle^+$ let

$e_{\tilde{y}} = \{x \mid \langle x, \tilde{y} \rangle \in A\}$.

Working in $M$ consider the sum

$k_{\tilde{y}} = \sum_{x \in M} |\langle x, \tilde{y} \rangle \times (\langle x \rangle \cap \tilde{y})|$

where $\tilde{y} = \langle x_0 - x, \tilde{y} \rangle$ and $\langle \tilde{y} \rangle \in < \omega \rangle^{n-1}$. Let $x_0, x_1$ be distinct elements of $e_{\tilde{y}}$.

Thus the above sum is $2^{2^n} \times$ (the number of ways of picking two distinct elements from $e_{\tilde{y}}$) $= 2^{2^n} \times (2^{2^n} - |\tilde{y}|)$.

Hence

$$\sum_{\langle y \rangle} k_{\tilde{y}} \geq 2^{2^n} \cdot \frac{a^{n+1}}{b^2}$$

since

$$\frac{a^{n+1}}{b^3} - |A| \geq \frac{a^{n+1}}{b^3} - d \geq \frac{a^{n+1}}{b^2}.$$

Reversing the order of summation gives

$$\sum_{\langle y \rangle} \sum_{\langle x \rangle} |\langle x, \tilde{y} \rangle \times (\langle x \rangle \cap \tilde{y})| \geq 2^{2^n} \cdot \frac{a^{n+1}}{b^3}.$$

Therefore since there are only $2^n \times \langle X \rangle \in M$ with $X \subseteq \langle \omega \rangle$, for some $X \subseteq \langle \omega \rangle$, $X \in M$

$$\sum_{\langle y \rangle} |\langle x, \tilde{y} \rangle \times (\langle x \rangle \cap \tilde{y})| \geq \frac{a^{n+1}}{b^3}.$$ 

This is the required $X$. □

Since $M$ is countable we can find a decreasing sequence $a_n, n \in \omega$ of elements of $M$ such that $a_n$ is exactly to the set of lower bounds of this sequence. Fix such a sequence.

**Proposition 4.** Let $G \in M$, $G: < \omega \rangle^{n+1} \rightarrow M$. Let $A \in M$, $A \subseteq < \omega \rangle^+$ and $|A| \geq d/b$ some $b \in I$. Then there is an $\exists X \in M$, $X \subseteq A$ such that $|X| \geq d/c$ some $c \in I$ and either

(i) $G$ is constant on $X$
(ii) for some $n \in \omega$, $a_n$ is a lower bound of $G'(X)$

**Proof.** Find $q \in M$ such that

$$||\langle \tilde{y} \rangle \in A \mid G(\tilde{y}) < q|| \leq ||\langle \tilde{y} \rangle \in A \mid G(\tilde{y}) \geq q||.$$ 

If $I < q$ set $X = \{\langle \tilde{y} \rangle \in A \mid G(\tilde{y}) \geq q\}$ so $|X| \geq 1/|A|$. If $q \in I$ set $Y = \{\langle \tilde{y} \rangle \in A \mid G(\tilde{y}) \leq q\}$ so $|Y| \geq 1/|A|$. By the pigeonhole principle in $M$ $|G^{-1}(\varepsilon) \cap Y| \geq |\varepsilon|/|\varepsilon| + 1$ for some $\varepsilon \leq q$. Take $X = G^{-1}(\varepsilon) \cap Y$. □

**Proposition 5.** Let $A \subseteq < \omega \rangle^+$, $A \in M$ and $|A| \geq d/b$ some $b \in I$. Let $a > c$. Then $\exists X_1, \ldots, X_n \subseteq < a$ such that $X_1, \ldots, X_n \in M$, $|X_1| = |X_2| = \ldots = |X_n| = c$ and $|\langle X_1 \times X_2 \times \ldots \times X_n \rangle \cup a^c| = a^n/b^c$.

**Proof.** Work in $M$. Consider all possible $X_1, \ldots, X_n \subseteq < a$ such that $|X_1| = |X_2| = \ldots = |X_n| = c$. There are \(\binom{a}{c(a-c)}\) such sequences. Consider
Let \( C_n \) be the set within the modulus signs. Carry on like this to find \( C_n^2 \subseteq (\{a_0\})^{n+1} \) such that
\[
|C_n^2| \geq \frac{2^{n+1}}{c}
\]
for some \( c \in I \).

Now by Proposition 4 pick \( E_{n+1} \subseteq C_n^2 \) such that \( E_{n+1} \cap M, |E_{n+1}| \geq \frac{2^{n+1}}{c} \)
for some \( c \in I \) and all the \( 2^{n+1} \)-permutations of \( G_{x_{n1}}, G_{x_{n2}} \) are constant on \( E_{n+1} \) or bounded above \( I \). Finally using Proposition 5 pick \( X_{2_{n1}}^1, X_{2_{n2}}^1 \subseteq X^2_1 \) all of modulus \( a_{n+1} \) such that
\[
|E_{n+1} \cap (X_{2_{n1}}^1 \times X_{2_{n2}}^1 \times \ldots \times X_{2_{n2}}^1)| \geq \frac{2^{n+1}}{c}
\]
for some \( c \in I \).

Set
\[
A_n = E_{n+1} \cap \{x_{2_{n1}}^1 \times x_{2_{n2}}^1 \times \ldots \times x_{2_{n2}}^1\}.
\]

It clearly satisfies a), b), c), d), e). To see f) notice that
\[
\langle x_{11}, x_{12}, \ldots, x_{1k} \rangle \in C_1 \Rightarrow \langle x_{11}, x_{12}, \ldots, x_{1k} \rangle \in \mathcal{A}_3
\]
and so on.

For \( n \in \omega \) let \( G_1, G_2, \ldots, G_{2^n} \) enumerate \( 2^n \) in the usual lexicographic ordering. We shall construct \( N \) using a compactness argument. Add to the language of arithmetic new constants \( b \) for each \( b \in M \) and \( f \) for each \( f \). Let \( Z \) be the set of sentences \( \theta(e_{f1}, \ldots, e_{f2}, b_1, \ldots, b_k) \) in this language such that for all \( k \) eventually, if \( e_{f1b} = G_{f1} \), \( i = 1, \ldots, n \) and \( (x_{i1}, \ldots, x_{ik}) \in \mathcal{A}_3 \), then \( M \models \theta(x_{i1}, \ldots, x_{ik}, b_1, \ldots, b_k) \).

Clearly \( Z \) is consistent. Let \( N^+ \) be a model of \( Z \) and \( N \) the elementary substructure generated by \( \langle e_f \rangle \) and \( \{b \mid b \in M \} \). So up to isomorphism, \( M \prec N \).

It only remains to show that \( \Gamma = I, J(\omega) = 2^n \).

Certainly by e) for \( f, g \in 2^n, f \not= g \)
\[
(e_f \neq e_g) \in Z
\]
so \( N \) has cardinality \( 2^n \). Furthermore in \( M \), \( |A_3| \leq \omega^n \). From this it follows that \( 2^n = |A_3| \leq \omega^n \) and hence \( |a_0| = 2^n \). For suppose \( G_{n+1} \in f \in 2^n \) for \( i = 1, \ldots, 2^n \). Then by f),
\[
\langle x_{i1}, \ldots, x_{ik} \rangle \in A_3 \Rightarrow \langle x_{i1}, x_{i2}, \ldots, x_{ik} \rangle \in A_3 \text{ so } (\langle x_{i1}, \ldots, x_{ik} \rangle, \ldots, \langle x_{i1}, \ldots, x_{ik} \rangle) \in \mathcal{A}_3
\]
The result follows since there are \( 2^n \) possible \( \langle f_1, \ldots, f_k \rangle \).

Thus it only remains to show that if \( a \in N \) then either \( N \models a \in b \) for some \( b \in M \). Suppose then \( N \models a = b \), \( f \in 2^n \), \( f_1, \ldots, f_k \), distinct.
Let \( k \) be such that \( G_k: (a_0)^n \to M \),
\[ G_k(x_1, \ldots, x_n) = \mu x: \theta(x, x_1, \ldots, x_n, b_1, \ldots, b_n) \text{ if this exists}, \]
\[ = 0 \text{ otherwise.} \]

Let \( j > k \) be such that \( f_1 \upharpoonright f_2 \) are distinct, say \( f_1 \upharpoonright f_2 = G_{j_1}, \ldots, G_{j_n} \). Let \( G \) be the \( 2^n \)-permutation of \( G_k \) such that
\[ G(x_1, \ldots, x_n) = G_k(x_{j_1}, \ldots, x_{j_n}). \]
Then \( G \) is either constant on \( A_j \) or bounded below on \( A_j \) by some \( a_0 \). If the former occurs then for some \( b \in M \),
\[ b = \mu x: \theta(x, x_1, \ldots, x_n, b_1, \ldots, b_n) \text{ if this exists}, \]
\[ = 0 \text{ otherwise} \]
for all \( \langle x_1, \ldots, x_n \rangle \in A_j \). Clearly by \( f \) the corresponding result holds for \( f' \).

Hence
\[ (\exists x) \exists x' \theta(x, x_1, \ldots, x_n, b_1, \ldots, b_n) \iff \exists x \theta(x, x_1, \ldots, x_n) \in Z. \]

Thus \( N \models b = \mu x: \theta(x, x_1, \ldots, x_n, b_1, \ldots, b_n) \) so \( N \models a = b \). Similarly if the second option occurs we see \( N \models a \preceq b \). Thus Theorem 2 is proved. ■

Remark. Since we could take \( I \) to be of the form \( \{ b \in M \mid b < a, \alpha \in A \} \) for some non-standard \( a \in M \), multiplication is the fastest function under which \( I \) needs be closed.

A second consequence of this result is that there is a model \( N \) of PA and \( a \in N \) such that \( \| \theta \| = 2^\omega \) whilst \( \| \sigma \| = \omega \).

We now prove a result for the case \( J^\omega(a) = \omega \).

**THEOREM 6.** Let \( M \) be PA, \( M \) countable and \( I \) an initial segment of \( M \) closed under exponentiation. Let \( \kappa \) be an infinite cardinal. Then there is an \( N > M \) such that \( J^\kappa(a) = \kappa \) and \( I = I^\kappa \).

Proof. As with Theorem 2 we first need some propositions. The proof of the next proposition mimics the proof of the Erdős–Rado theorem in Set Theory.

**PROPOSITION 7.** Let \( A, E \in M, a \in \langle - A \rangle \) and \( F: \langle A \rangle \to \langle a \rangle \). Then \( \exists X \subseteq A, X \in M, I \subseteq \langle X \rangle \) such that \( F \) is constant on \( \langle X \rangle \).

Let \( \langle A \rangle^* = \{ \langle x_1, \ldots, x_n \rangle \mid x_1, \ldots, x_n \in A \} \). Suppose \( T(b) \), the elements of \( T \) of level \( b \), has been found and suppose \( \langle x, Y \rangle \in T(b) \to x \in A, A \equiv Y \in M \) and \( x < Y \) (i.e. \( x < y \) for all \( y \in Y \)).

Set
\[ D(x, Y) = \{ u \mid \exists W, \langle u, W \rangle \subseteq \langle x, Y \rangle \in T \}, \]
so \( |D(x, Y)| = b + 1 \) for \( \langle x, Y \rangle \in T(b) \). For each \( G: D(x, Y) \to \langle a \rangle, G \in M \) let
\[ e_0 = \{ z \in Y \mid F(z, z) = G(z) \text{ for all } \langle z, Y \rangle \in D(x, Y) \} \]
Thus there are \( \{ e_0 \} \subseteq \langle a \rangle \) possible \( e_0 \)'s. The elements above \( \langle x, Y \rangle \in T(b) \) on level \( b+1 \) are the pairs
\[ \langle \min(e_0), e_0 - \langle \min(e_0) \rangle \rangle \text{ for } e_0 \neq \emptyset. \]

By induction in \( M, T(b) \leq \langle a \rangle^b \). We shall show \( T(b) \neq \emptyset \) for some \( b > I. \) Suppose not. Then \( T(b) = \emptyset \) for some \( b > I. \) This means that for every \( x \in A \) there is a \( c < b \) such that \( \langle x, Y \rangle \in T(c) \) for some \( Y \). But then
\[ |A| \leq \sum_{b>c}^\omega \langle a \rangle^b < \omega^\omega \leq \omega^{\omega^b} \in I - \text{ a contradiction.} \]

Now pick \( b > I, T(b) \neq \emptyset \) and let \( \langle x, Y \rangle \in T(b), U = D(x, Y) \). Then
\[ |U| = b + 1 > I \text{ and if } \langle x_1, \ldots, x_n \rangle \in |U|^a \text{ then} \]
\[ F(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, x_{n+1}) = H(x_1, \ldots, x_n) \]
By assumption pick \( X \subseteq U, X \in M, |X| > I \) such that \( H \) is constant on \( |X|^a \).

Then \( F \) is constant on \( |X|^a \). ■

The next proposition works in the same way that Proposition 4 did in the proof of Theorem 2.

**PROPOSITION 8.** Let \( G, A \in M, I \subseteq \langle A \rangle, \langle a \rangle \to \langle A \rangle \). Then \( \exists X \subseteq A, X \subseteq M, I \subseteq \langle X \rangle \) such that either
(i) \( \exists b > I, G^* \neq b \) or
(ii) \( G \) is constant on \( \langle X \rangle \).

Proof. For each \( a \in M \) let \( H_a: \langle A \rangle \to \langle a \rangle \) by \( H_a(x) = 0 \iff G(x) \sim a \). Working in \( M \) pick for each \( H_a \) a maximal homogenous set \( X \). If \( H_a^* \langle X \rangle = \emptyset \) for some \( a > I \) take \( X = X \), and notice that by Proposition 7, \( |X| > I \). Otherwise \( H_a^* \langle X \rangle = \{ x \} \) for some \( a > I \) and \( |X| > I \) so by using Proposition 7 we can find \( X \subseteq X \) to satisfy (i).

Proof of Theorem 6 continued. The method of proof is similar to the proof of Theorem 2.

Let \( \omega_0 \) be a decreasing sequence of elements of \( M \) whose set of lower bounds is precisely \( I. \) Let \( G_i, i \in \omega \) enumerate all maps \( G \in M, G: \langle a_0 \rangle^\omega \to \langle M \rangle \) for some \( n. \)

We define decreasing subsets \( A_n \) of \( \langle a_0 \rangle^\omega \) satisfying
\[ a \text{ a}_0 \subseteq M, \]
\[ b \text{ i} \neq n \text{ and } G_i^\omega \subseteq A \text{ then either } G_0 \text{ is constant on } \langle a \rangle^\omega \text{ or } G_i^\omega \subseteq a \text{ for some } k. \]

Put \( A_0 \neq \langle a \rangle. \) Suppose \( A_0 \) constructed. Using Proposition 8 find \( X \subseteq A_n, X \subseteq M, I \subseteq \langle X \rangle \) such that \( b \text{ holds for } G_0 \). Now let \( A_{n+1} = X, A_{n+1} = M \text{ such that } I \subseteq \langle A_{n+1} \rangle \subseteq a_{n+1}. \)
Now add to the language of arithmetic $\kappa$ new constants $e_n, v < \kappa$ and $b$ for each $b \in M$. Let $Z$ be the set of sentences $\theta(e_1, \ldots, e_n, b_1, \ldots, b_n)$ such that $v_1 < v_2 < \cdots < v_n$ and for all $t$ eventually, all $\langle x_1, \ldots, x_n \rangle \in \{A_n\}^n$, 

$$M \models \theta(x_1, \ldots, x_n, b_1, \ldots, b_n).$$

Clearly $Z$ is consistent. Let $N^+$ be a model of $Z$ and $N$ the elementary substructure of $N^+$ generated by the $e_n, v < \kappa$ and $b, b \in M$. Then, up to isomorphism, $M \cong N$. From $b$ it follows that $\|a\|_n = x$ for $n \in \omega$ and hence using $c$ it follows that $J^\omega(\omega) = x, I^\omega_n = I$.

Remarks. If we define the super power function, $Sp$, by 

$$Sp(a, 0) = 1, \quad Sp(a, x+1) = d^{Sp(a, x)}$$

then the $I$ in Theorem 6 could be taken to be 

$$\{b \in M \mid M \models b < Sp(a, n), n \in \omega\}$$

for some non-standard $a \in M$. In this sense this exponentiation is the fastest function under which $I^\omega_n$ needs to be closed, assuming $J^\omega(\omega) > 2^\omega$.

The above construction also enables us to show the following:

For any complete extension $T$ of PA and any infinite cardinal $\kappa$, there is an elementary end extension of the minimal model $M_0$ of $T$ such that $\kappa < \kappa$ and every elementary submodel of $M_0$ is cofinal in $\kappa$.

This follows from the proof of Theorem 6 by letting $M$ be a minimal elementary end extension of $M_0$ and choosing $I$ so that $M_0 \subset I \subset M$ (strict inclusion). Then $M$ is cofinal in the model $N$ produced in the proof, and $N$ has cardinality $\kappa$. Suppose $N' < N$ is not cofinal in $N$. Then $N' < b$ for some $b \in M$. Setting

$$M' = \{x \in M \mid \exists y \in N' \ x < y\}$$

one can show by checking for closure under Skolem functions that $M' < N$. Since $M$ is minimal over $M_0$ and $M_0$ is the minimal model of $T$, it follows that $M' = M_0$. Thus $\forall x \in N', \forall y \in M - M_0 \ x < y$ and hence $\forall x \in N' \exists y \in I \ x < y$ (namely, $y$ can be any element of $I - M_0$). Since by construction

$$\{x \in N' \mid \exists y \in I \ x < y\} \subset M$$

we get that $N' \leq M$. Therefore $N' < N$, and since $N' \neq N$ it follows that $N' = M_{\omega_\alpha}$ as desired.

Finally, in the following remarks, the $I^\omega_n$ were both countable. As the next theorem shows this was essential. Notice $|I^\omega_n| \geq 2^\omega$, $|I^\omega_\omega| = \omega_1$.

**Theorem 9.** If $M \models PA$ and $|I^\omega_n| = \omega_1$, then $I^\omega_n \models PA$.

**Proof.** We first need an easy proposition.

**Proposition 10.** Let $I^\omega_n$ be as in Theorem 9, $\omega \in I^\omega_n$. Then $\theta(\bar{z}, \bar{b})$ is $I^\omega_n(\Sigma_n)$. Then there is a $\Delta_0$ formula $\psi(\bar{x}, \bar{b}, \bar{c})$, $\bar{c} \in I^\omega_n$ such that for all $\bar{x} < a$,

$$I^\omega_n \models \theta(\bar{x}, \bar{b}) \iff \psi(\bar{x}, \bar{b}, \bar{c}).$$

**Proof.** By induction on $n$. Result clear if $n = 0$ so assume $n > 0$. Let $\theta(\bar{x}, \bar{b}) = (\forall \bar{c})\eta(\bar{x}, \bar{b}, \bar{c})$ where $\eta$ is $\Sigma_n$. Since there are only countably many $\bar{x} < a$, and $I^\omega_n$ has cofinality $\omega_1$ we can find $\bar{c} \in I^\omega_{\omega_\alpha}$ such that for all $\bar{x} < a$,

$$I^\omega_n \models (\exists \bar{c})\eta(\bar{x}, \bar{b}, \bar{c}) \iff (\exists \bar{c} < \bar{b}) \eta(\bar{x}, \bar{b}, \bar{c}).$$

By inductive hypothesis there is a $\Delta_0$ formula $\chi(\bar{x}, \bar{b}, \bar{c}, \bar{d}) \in I^\omega_n$, such that for all $\bar{x} < a, \bar{b} < \bar{c}$,

$$I^\omega_n \models (\exists \bar{d}) \chi(\bar{x}, \bar{b}, \bar{c}, \bar{d}) \iff (\exists \bar{d} < \bar{c}) \chi(\bar{x}, \bar{b}, \bar{c}, \bar{d}).$$

Thus for $\bar{x} < a$,

$$I^\omega_n \models \theta(\bar{x}, \bar{b}) \iff (\forall \bar{c}) \chi(\bar{x}, \bar{b}, \bar{c}, \bar{d})$$

and $I^\omega_n \models \theta(\bar{x}, \bar{b}) \iff (\forall \bar{c} < \bar{a}) \chi(\bar{x}, \bar{b}, \bar{c}, \bar{d})$.

this last formula being the required $\psi$. 

**Proof of Theorem 9 continued.** It is enough to show that the axiom of induction holds in $I^\omega_n$. Suppose $I^\omega_n \models \theta(0) \land (\forall x \theta(x) \to \theta(x + 1))$. Let $\omega \in I^\omega_n$ and by Proposition 10, find a $\Delta_0$ formula $\psi(x)$ (maybe containing elements of $I^\omega_n$) such that for all $x < a$,

$$I^\omega_n \models \psi(x) \iff (\exists \bar{c}) \chi(\bar{x}, \bar{b}, \bar{c}, \bar{d}).$$

Then $I^\omega_n \models \psi(0) \land (\forall x < a) (\psi(x) \to \psi(x + 1))$. Since $I^\omega_n$ is an initial segment of $M$ and this last formula is $\Delta_0$, $I^\omega_n \models \psi(0) \land (\forall x < a) (\psi(x) \to \psi(x + 1))$.

Since $M \models PA$, (indeed bounded induction is enough),

$$M \models \psi(\bar{c}).$$

Thus $I^\omega_n \models \psi(\bar{c})$ and so $I^\omega_n \models \theta(\bar{c})$. Therefore $I^\omega_n \models (\forall x \theta(x))$ and the theorem is proved.

**Extension to the uncountable case.** If we replace $\omega$ by an uncountable cardinal $\lambda$ then Lemma 1 and Theorem 9 (with $\lambda^+$ in place of $\omega_1$) go through as before.

**Theorem 10.** If $M \models PA$ and $|\omega| = \omega^\omega$, then $I^\omega_n \models PA$.

**Proof.** We first need an easy proposition.

**Proposition 11.** Let $I^\omega_n$ be as in Theorem 9, $\omega \in I^\omega_n$. Let $\theta(\bar{z}, \bar{b})$ be $I^\omega_n(\Sigma_n)$. Then there is a $\Delta_0$ formula $\psi(\bar{x}, \bar{b}, \bar{c})$, $\bar{c} \in I^\omega_n$ such that for all $\bar{x} < a$,

$$I^\omega_n \models \theta(\bar{x}, \bar{b}) \iff \psi(\bar{x}, \bar{b}, \bar{c}).$$

**Proof.** We need the following generalization of Proposition 10:
Proposition 12. Let $I_1, I_2$ be initial segments of $M$ closed under exponentiation. Let $A, B, F, G$ define $I_1 < A$, $I_2 < B$, and $F : (A)^n \times (B)^m \to < d$ for some $m, n \in \omega$. Then there exist $X, Y \in M$ such that $X \subseteq A, Y \subseteq B, I_1 < X, I_2 < Y$ and $F(G, B)$ is independent of $\bar{G}$ for $\langle \bar{G}, \bar{B} \rangle \in (X)^n \times (Y)^m$.

Proposition 13. If we assume $d \in I_1$ in Proposition 12, then we can choose $X$ so that $F$ is constant on $(X)^n \times (Y)^m$.

Proof. For each $\bar{B} \in [B]^m$ consider the induced function $F_\bar{B} : [A]^n \to < d$ given by $F_\bar{B}(G) = F(G, \bar{B})$. This is a function inside $M$. Since there are at most $d^{d^{d^{d^n}}}$ such functions in $M$, we may consider the family $\mathcal{F} = \{ F_\bar{B} \}$, and for each $\bar{B} \in [B]^m$, we may choose a constant function $\phi_{\bar{B}} : [A]^n \to < d$, such that $\phi_{\bar{B}}(G) = F_\bar{B}(G)$ for all $G \in M$. Then $F(G, \bar{B})$ is independent of $\phi_{\bar{B}}$ on $(A)^n \times (B)^m$, and therefore on $[A]^n \times [B]^m$. This proves Proposition 12.

Proposition 13. If we assume $d \in I_1$, then picking any $\bar{B} \in [B]^m$ and applying Proposition 7 to $F_\bar{B}$, we get $X \in M, X \subseteq A, I_1 < X$ such that $F_\bar{B}$ is constant on $(X)^n \times (Y)^m$. Since $F(G, \bar{B})$ is independent of $\phi_{\bar{B}}$ on $(A)^n \times (B)^m$, it is constant on $(X)^n \times (Y)^m$. This proves Proposition 13.

Proof of Theorem 11 (continued). As in the proof of Theorem 6, let $b_\alpha, \alpha \in \omega$, be a decreasing sequence of elements of $M$ whose set of lower bounds is precisely $I$. Let $\alpha_0$ be a nonstandard element of $I$. Let $G_\alpha, \alpha \in \omega$, enumerate all maps $G \in M$ such that $G : (A_\alpha)^n \to (B_\alpha)^m$ for some $m, n \in \omega$. We define a sequence of pairs $\langle A_\alpha, B_\alpha \rangle, \alpha \in \omega$, satisfying:

a) $\alpha_0 \in A_\alpha, \alpha_1 \in A_{\alpha+1}, \alpha_1 \in A_{\alpha+2}, \alpha_2 \in B_{\alpha+1}, \alpha_2 \in B_{\alpha+2}, \alpha_3 \in B_{\alpha+3}, \alpha_3 \in B_{\alpha+4}, \ldots$;

b) $\alpha_2 \in A_{\alpha+2}, \alpha_2 \in B_{\alpha+2}, \alpha_2 \in B_{\alpha+2}$;

c) if $\alpha < \beta$, $G_\alpha : (A_\alpha)^n \to (B_\alpha)^m$ then either $G_\beta : (A_\beta)^n \to (B_\beta)^m$.

Proof of Theorem 11 (continued). As in the proof of Theorem 6, let $N^* \subseteq \mathbb{N}$ and $N < N^*$ be generated by $C \cup \mathbb{D} \cup E$, so that $M = N^*$. Then $|\mathbb{N}| = \lambda$, for all $a \in \mathbb{N}$, $\mathbb{D}$, and $c \in \mathbb{C}$. Also $|\mathbb{C}| = \lambda$ for all $c \in \mathbb{C}$, and the same for $\mathbb{D}$. Finally for all $a \in \mathbb{N}$ there exists $b \in M$ such that $a \in b$ or $b < c.a. b \in a$. From this $\lambda$ and $\lambda^+$ are both as required.

By taking $I = \{ b \in M \mid b \in \mathbb{N} \cap \mathbb{D} \}$ for some nonstandard $\alpha \in M$, we have that exponentiation is the fastest function under which $\lambda^+$ needs to be closed, no matter how large $\lambda^+$ is, provided $\lambda^+ \leq \lambda$.

Using Theorem 2 and Chang's Theorem (assuming G.C.H.) we can show that for each regular uncountable $\lambda$ there is a model $M \models \lambda^+$ in which $\lambda^+ = \lambda^+$ and $\lambda^+$ is not closed under exponentiation. We do not know whether the G.C.H. can be removed or whether this $\lambda^+$ can be of the form $\lambda^+$ for some non-standard $\alpha \in M$.