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Homomorphisms of direct powers of algebras

by

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"Wieleż lat czekać trzeba, nim się przedmiot świeży
Jak figa ucukruje, jak tytuń uleży?" [13]

0. Abstract. Given algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, a set X and a homomorphism $h: \mathfrak{A}^X \to \mathfrak{B}$ we study the collection of all supports of h, i.e., sets $Y \subseteq X$ such that for all $f, g \in \mathfrak{A}^X$ if $f \restriction Y = g \restriction Y$ then $h(f) = h(g)$.

1. Terminology and generalities. We identify every ordinal number ξ with the set of ordinal numbers smaller than ξ, e.g., $n = \{0, 1, \ldots, n-1\}$ and $\omega = \{0, 1, \ldots\}$. Cardinal numbers are the initial ordinals. α and β denote cardinals. If X is a set then $|X|$ denotes the cardinal of X. α^+ denotes the cardinal successor of α. A filter F of subsets of X is called α-*complete* iff for every $G \subseteq F$ with $|G| < \alpha$ we have $\bigcup G \in F$, and F is called an *ultrafilter* if from any two complementary sets in X at least one is in F. We shall use the following surprising characterisation of α-complete ultrafilters.

1.1 (Galvin and Horn [9]). Let F be a family of subsets of X and α be a cardinal $\geqslant 4$. Then the following two conditions are equivalent

(i) F is an α-complete ultrafilter and $\emptyset \notin F$.

(ii) For every partition P of X with $|P| < \alpha$ we have $|F \cap P| = 1$.

For any cardinal α we denote by $\mu(\alpha)$ the least cardinal such that there exists a nonprincipal α^+-complete ultrafilter of subsets of $\mu(\alpha)$. We recall that $\mu(n) = \omega$ for $2 \leqslant n < \omega$, $\mu(\alpha)$ is a measurable cardinal and

1.2. Every α^+-complete ultrafilter of subsets of $\mu(\alpha)$ is $\mu(\alpha)$-complete.

In fact $\mu(\alpha)$ has many other "closure properties", see [11]. Even the existence of $\mu(\omega)$ does not follow from the Zermelo–Fraenkel axioms of set theory, but the main results of this paper could be easily reformulated so as to avoid the assumptions of the existence of $\mu(\alpha)$ for any infinite α. On the other hand the existence of $\mu(\alpha)$ for every cardinal α is already a well established axiom of set theory, see e.g. [24] p. 47, 48 or [29] p. 675.

For any function $f\colon X\to U$ and any set $Y\subseteq X$ we denote by $f\restriction Y$ the restriction of f to Y, i.e., $f\restriction Y=f\cap(Y\times U)$. German capitals $\mathfrak{A}$ and $\mathfrak{B}$ denote algebras and A and B their respective universes. We write $|\mathfrak{A}|$ for $|A|$ and $a\in\mathfrak{A}$ for $a\in A$. $h\colon \mathfrak{A}\to\mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are algebras of the same similarity type and h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. For any algebra $\mathfrak{A}$ and any set X, $\mathfrak{A}^X$ denotes the direct power of $\mathfrak{A}$ with exponent X, i.e., the algebra of all functions $f\colon X\to A$ with operations defined coordinatewise. $\cong$ denotes isomorphism of algebras.

2. Introduction, main concepts and history.

2.1. Definition. If $h\colon \mathfrak{A}^X\to\mathfrak{B}$ then Y is called a *support of* h iff $Y\subseteq X$ and for every $f,g\in\mathfrak{A}^X$ if $f\restriction Y=g\restriction Y$ then $h(f)=h(g)$.

The following proposition was stated in [23] p. 130.

2.2. Proposition. *The collection of all supports of a homomorphism $h\colon \mathfrak{A}^X\to\mathfrak{B}$ is a filter of subsets of X.*

Proof. It is clear that if $Y\subseteq Z\subseteq X$ and Y is a support of h then Z is a support of h. It remains to show that if Y and Z are supports of h then $Y\cap Z$ also is a support of h. Let $f,g\in\mathfrak{A}^X$ and $f\restriction Y\cap Z=g\restriction Y\cap Z$. We put

$$p(x)=\begin{cases} f(x) & \text{for all } x\in Y,\\ g(x) & \text{for all } x\in X-Y.\end{cases}$$

Hence $f\restriction Y=p\restriction Y$ and $p\restriction Z=g\restriction Z$. Therefore $h(f)=h(p)=h(g)$.

2.3. Definition. $S(\mathfrak{A},\mathfrak{B},\alpha,\beta)$ means that for every set X with $|X|<\alpha$ and every homomorphism $h\colon \mathfrak{A}^X\to\mathfrak{B}$, h has a support of cardinality $<\beta$.

The relation S permits one to formulate briefly some old results on abelian groups which motivated this paper. We shall deal only with $\beta\leqslant\omega$. Several theorems of the form $S(\mathfrak{A},\mathfrak{B},\omega,2)$ are proved in Section 4. Let $\mathfrak{Z}$ denote the infinite cyclic group. In the literature on abelian groups an abelian group $\mathfrak{B}$ is called slender if $S(\mathfrak{Z},\mathfrak{B},\omega_1,\omega)$ holds (we shall not use this terminology here). In 1950 [25] Specker proved $S(\mathfrak{Z},\mathfrak{Z},\omega_1,\omega)$. (His aim was to show that $\mathfrak{Z}^\omega$ is not a free abelian group and this follows from $S(\mathfrak{Z},\mathfrak{Z},\omega_1,\omega)$ in the following way. If $\mathfrak{Z}^\omega$ were free abelian then it would have 2^{2^ω} different homomorphisms into $\mathfrak{Z}$ but $S(\mathfrak{Z},\mathfrak{Z},\omega_1,\omega)$ yields that it has only ω such homomorphisms. The theorem of Specker has been complemented by a more recent result of Nöbeling [17] (for a simpler proof see [1] or [8]) which says that for every set X the group of all functions $f\colon X\to\mathfrak{Z}$ with $|f[X]|<\omega$ is free abelian.) In 1959 [5] (see also [8]) Ehrenfeucht and Łoś proved that for every group $\mathfrak{B}$

$$S(\mathfrak{Z},\mathfrak{B},\omega_1,\omega)\Rightarrow S(\mathfrak{Z},\mathfrak{B},\mu(\omega),\omega)\,. \tag{2.4}$$

Corollary 3.2 of this paper partly extends (2.4) substituting an arbitrary algebra $\mathfrak{A}$ for $\mathfrak{Z}$, although ω_1 had to be replaced by $|\mathfrak{A}^\omega|^+$ (see Problem 3.3).

Improving a theorem of Sąsiada [19] R. J. Nunke [18, 19] (see also [8]) proved that if $\mathfrak{B}$ is a torsion free abelian group then $\mathfrak{B}$ satisfies $S(\mathfrak{Z},\mathfrak{B},\omega_1,\omega)$ (and hence

also $S(\mathfrak{Z},\mathfrak{B},\mu(\omega),\omega)$) iff $\mathfrak{B}$ has no subgroups isomorphic to $\mathfrak{Z}^\omega$ nor to the additive group of rational numbers nor to the additive group of p-adic integers for any prime p. For related easy facts on rings see [28].

Generalizing the theorem of Ehrenfeucht and Łoś, Mrówka [14] proved that if X is a closed subspace of a product space N^T, where N is a countable discrete space and T is any set, then every homomorphism $h\colon \mathfrak{Z}^X\to\mathfrak{Z}$, where $\mathfrak{Z}^X$ is the group of all continuous functions $f\colon X\to\mathfrak{Z}$, has a compact support. Fajtlowicz and Mrówka [6] gave a partial characterization of which compact subsets of X can be supports of such homomorphisms h.

Galvin and Horn [9] gave another result related to the relation S which we have translated into Theorem 4.12 below.

The problems 3.3, 3.7, 4.3, 4.11 and 4.24 show that our understanding of S is still quite limited.

For simplicity of notation we shall write only about homomorphisms of direct powers $\mathfrak{A}^X$, but some results could be generalized to direct products (e.g. 4.6 and 4.9). Other generalizations are mentioned in 6.12.

We are indebted to Walter Taylor for many helpful remarks and for introducing us to some material which lead to 4.6, . ., 4.10. Also 4.20 and 4.21 are due to him.

3. Results for $\beta=\omega$.

3.1. Theorem. *If $S(\mathfrak{A},\mathfrak{B},|\mathfrak{A}^\omega|^+,\omega)$ then for every set X and every homomorphism $h\colon \mathfrak{A}^X\to\mathfrak{B}$ the collection of all supports of h is a filter of the form $\bigcap_{k<n}F_k$, where $n<\omega$ and all F_k are $|\mathfrak{A}|^+$-complete ultrafilters of subsets of X. Moreover there exists a homomorphism $h_0\colon \mathfrak{A}^n\to\mathfrak{B}$ such that for all $f\in\mathfrak{A}^X$*

$$h(f)=h_0(a_0,\ldots,a_{n-1})$$

where each a_k is defined by the condition $f^{-1}\{a_k\}\in F_k$.

The proof will be given Section 6. For related theorems see Mrówka [15]. This result immediately yields the following corollary which was announced in [4].

3.2. Corollary. $S(\mathfrak{A},\mathfrak{B},|\mathfrak{A}^\omega|^+,\omega)\to S(\mathfrak{A},\mathfrak{B},\mu(|\mathfrak{A}|),\omega)$.

3.3. Problem. Would the assumption $S(\mathfrak{A},\mathfrak{B},|\mathfrak{A}|^+,\omega)$ be sufficient for the conclusions of 3.1 and 3.2? (In the case $\mathfrak{A}=\mathfrak{Z}$ the answer is yes, see (2.4) above. See also Example 6.11 below.)

The number $\mu(|\mathfrak{A}|)$ in the conclusion of 3.2 is the largest possible as the following proposition shows.

3.4. Proposition. *If $|\mathfrak{A}|>1$ then $S\big(\mathfrak{A},\mathfrak{A},(\mu(|\mathfrak{A}|))^+,\mu(|\mathfrak{A}|)\big)$ fails.*

Proof. Let $|X|=\mu(|\mathfrak{A}|)$. Given an $|\mathfrak{A}|^+$-complete non-principal ultrafilter F on X, for any $f\in\mathfrak{A}^X$ we put $h(f)=a$ iff $f^{-1}\{a\}\in F$. Then $h\colon \mathfrak{A}^X\to\mathfrak{A}$ is a homomorphism without any support of cardinality less than $\mu(|\mathfrak{A}|)$. Q.E.D.

The following proposition shows that the two properties "$\mathfrak{A}$ is equationally compact" (see [30], [26] and [27]) and $S(\mathfrak{A},\mathfrak{A},\omega_1,\omega)$ are opposing extremes (most algebras do not satisfy any one of them).

3.5. PROPOSITION. *If $\mathfrak{A}$ is equationally compact and $|\mathfrak{A}|>1$ then $S(\mathfrak{A}, \mathfrak{A}, \omega_1, \omega)$ fails.*

Proof. For any ultrapower $\mathfrak{A}^\omega/F$, where $\mathfrak{A}$ is equationally compact, there exists a homomorphism r: $\mathfrak{A}^\omega/F\to\mathfrak{A}$ extending the natural isomorphism of the diagonal of $\mathfrak{A}^\omega/F$ with $\mathfrak{A}$ (r is a retraction see [30]). Let h: $\mathfrak{A}^\omega\to\mathfrak{A}^\omega/F$ be the natural homomorphism. Then, if F is a nonprincipal ultrafilter, the homomorphism $r\circ h$: $\mathfrak{A}^\omega\to\mathfrak{A}$ has no finite supports. Q.E.D.

For completeness we list the following facts which are obvious.

3.6. PROPOSITION. (i) *$S(\mathfrak{A}, \mathfrak{B}, \alpha, \omega)\to S(\mathfrak{A}^{\alpha_0}, \mathfrak{B}, \alpha, \omega)$ for every $\alpha_0<\alpha$.*

(ii) *$S(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)\Rightarrow S(h(\mathfrak{A}), \mathfrak{B}_0, \alpha, \beta)$ for every homomorphism h of $\mathfrak{A}$ and every subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$.*

(iii) $[S(\mathfrak{A}, \mathfrak{B}_1, \alpha, \omega)\ \&\ S(\mathfrak{A}, \mathfrak{B}_2, \alpha, \omega)]\Rightarrow S(\mathfrak{A}, \mathfrak{B}_1\times\mathfrak{B}_2, \alpha, \omega)$.

(iv) *Let $\mathfrak{A}_0=\langle A, p_s\rangle_{s\in S_0}$, $\mathfrak{B}_0=\langle B, q_s\rangle_{s\in S_0}$, $\mathfrak{A}=\langle A, p_s\rangle_{s\in S}$, $\mathfrak{B}=\langle B, q_s\rangle_{s\in S}$ and $S_0\subseteq S$ then $S(\mathfrak{A}_0, \mathfrak{B}_0, \alpha, \beta)\Rightarrow S(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.*

3.7. PROBLEM. Does the implication

$$[S(\mathfrak{A}_1, \mathfrak{B}, \alpha, \omega)\ \&\ S(\mathfrak{A}_2, \mathfrak{B}, \alpha, \omega)]\to S(\mathfrak{A}_1\times\mathfrak{A}_2, \alpha, \omega)$$

hold? (By 3.6 (i) and (ii) the answer is yes if $\mathfrak{A}_2$ is a homomorphic image of $\mathfrak{A}_1$.)

4. Results for $\beta<\omega$.

4.1. THEOREM. *If $n<\omega$ and $S(\mathfrak{A}, \mathfrak{B}, |\mathfrak{A}|^+ +\omega, n)$ then the conclusion of 3.1 holds and no more than n ultrafilters are needed.*

The proof is similar to the proof of 3.1, see Section 6. This result immediately yields the following corollary.

4.2. COROLLARY. $S(\mathfrak{A}, \mathfrak{B}, |\mathfrak{A}|^+ +\omega, n)\Rightarrow S(\mathfrak{A}, \mathfrak{B}, \mu(|\mathfrak{A}|), n)$.

By 3.4 the number $\mu(|\mathfrak{A}|)$ in the conclusion of 4.2 is the largest possible.

4.3. EXAMPLE. The implication $S(\mathfrak{A}, \mathfrak{A}, n, 2)\Rightarrow S(\mathfrak{A}, \mathfrak{A}, n+1, 2)$ fails in general, as the following example shows. Let $\mathfrak{A}=\langle\{0, 1\}, x+1, x+y\rangle$, where $+$ is modulo 2. Then every homomorphism h: $\mathfrak{A}^2\to\mathfrak{A}$ is a projection but

$$x_0+x_1+x_2:\ \mathfrak{A}^3\to\mathfrak{A}$$

is a homomorphism without one-element supports. We are lacking counterexamples for larger n's.

Now we shall prove a number of concrete theorems of the form $S(\mathfrak{A}, \mathfrak{B}, \omega, 2)$.

An algebra of the form $\langle A, d(\cdot, \cdot, \cdot, \cdot)\rangle$, where

$$d(x, y, u, v)=\begin{cases}u & \text{if}\quad x=y,\\ v & \text{if}\quad x\neq y\end{cases}\tag{4.4}$$

is called a *discriminator algebra*. In a finite field of power p^k, where p is a prime, the function

$$d=u+(x-y)^{p^k-1}(v-u)$$

satisfies (4.4). Hence, cf. 3.6(iv), the following facts will apply to all finite fields.

4.5. PROPOSITION. *Every discriminator algebra is simple.*

This follows immediately from (4.4).

The next three results are close to some theorems of Bergman [2] and Foster and Pixley [7] see also [21].

4.6. THEOREM. *If $\mathfrak{A}$ is a discriminator algebra, X is any set and $\mathfrak{A}'\subseteq\mathfrak{A}^X$ then for every congruence $\equiv$ in $\mathfrak{A}'$ there exists a filter F of subsets of X such that for all $f, g\in\mathfrak{A}'$*

$$f\equiv g\ \Leftrightarrow\ \{x: f(x)=g(x)\}\in F.$$

Proof. For any $f, g\in\mathfrak{A}'$ we put $E_{fg}=\{x: f(x)=g(x)\}$. First we show for all $f, g, p, q\in\mathfrak{A}'$

(4.7) If $f\equiv g$ and $E_{fg}\subseteq E_{pq}$ then $p\equiv q$.

Indeed by the assumptions

$$p=d(f, f, p, q)\equiv d(f, g, p, q)=q.$$

Now we show for all $f, g, p, q, r, s\in\mathfrak{A}'$

(4.8) If $f\equiv g$, $p\equiv q$ and $E_{fg}\cap E_{pq}\subseteq E_{rs}$ then $r\equiv s$.

Let $t=d(f, g, r, s)$. Then $E_{fg}\subseteq E_{rt}$ and $E_{pq}\subseteq E_{ts}$. Hence, by (4.7), $r\equiv t\equiv s$.

4.6 follows of course from (4.8). Q.E.D.

4.9. THEOREM. *If $\mathfrak{A}$ and $\mathfrak{B}$ are discriminator algebras then every homomorphism h: $\mathfrak{A}^X\to\mathfrak{B}$ is of the form $h=i\circ h_F$, where*

$$\mathfrak{A}^X\overset{h_F}{\to}\mathfrak{A}^X/F\overset{i}{\to}\mathfrak{B},$$

F is an ultrafilter of subsets of X, h_F is the natural homomorphism and i is an injection.

Proof. By 4.6 we get a filter F corresponding to the congruence determined by h. By 4.5 $\mathfrak{A}^X/F$ is simple and hence F is an ultrafilter.

4.10. COROLLARY. *If $\mathfrak{A}$ and $\mathfrak{B}$ are discriminator algebras then $S(\mathfrak{A}, \mathfrak{B}, \omega, 2)$.*

This follows immediately from 4.9.

4.11. PROBLEM. By 3.4 and 4.10 the implication $S(\mathfrak{A}, \mathfrak{A}, \omega, 2)\Rightarrow S(\mathfrak{A}, \mathfrak{A}, \omega_1, 2)$ fails in general. Does $S(\mathfrak{A}, \mathfrak{A}, \omega_1, 2)\Rightarrow S(\mathfrak{A}, \mathfrak{A}, \omega_2, 2)$?

An algebra of the form $\langle A, e_a(\cdot)_{a\in A}, F\rangle$ where, for all $a\in A$,

$$e_a(x)=e_a(y)\neq a\quad\text{if}\quad x\neq a\neq y,$$

and

$$e_a(a)=a,$$

is called a *full algebra*. In a finite field of power p^k the functions

$$e_a=a+(x-a)^{p^k-1}$$

satisfy the above requirements. Therefore the following theorem applies to all finite fields with all elements added as constants.

4.12. THEOREM. *If $|\mathfrak{A}|\geqslant 3$ and $\mathfrak{A}$ is a full algebra then for every set X and every homomorphism $h\colon \mathfrak{A}^X\to\mathfrak{A}$ the collection F of all supports of h is an $|\mathfrak{A}|^+$-complete ultrafilter of subsets of X and moreover for all $a\in\mathfrak{A}$ and $f\in\mathfrak{A}^X$*

$$h(f)=a \quad \textit{iff} \quad f^{-1}\{a\}\in F.$$

Proof. Let us put $\mathfrak{A}^*=\langle A, R_a\rangle_{a\in A}$, where R_a is an equivalence relation over A whose equivalence classes are $\{a\}$ and $A-\{a\}$. By the theorem of Galvin and Horn [9] it is enough to prove the following lemma.

If $\mathfrak{A}$ is full and $h\colon \mathfrak{A}^X\to\mathfrak{A}$ then h is also a homomorphism of $(\mathfrak{A}^*)^X$ into $\mathfrak{A}^*$.

Suppose that $fR_a g$ holds in $(\mathfrak{A}^*)^X$, i.e., $f^{-1}\{a\}=g^{-1}\{a\}$. We have to show that $h(f)R_a h(g)$ holds in $\mathfrak{A}^*$. Since h is a homomorphism and $e_a\circ f=e_a\circ g$ we have

$$e_a(h(f))=h(e_a\circ f)=h(e_a\circ g)=e_a(h(g)).$$

Therefore $h(f)=a$ iff $h(g)=a$ and $h(f)R_a h(g)$ follows. Q.E.D.

4.13. COROLLARY. *If $|\mathfrak{A}|\leqslant 3$ and $\mathfrak{A}$ is a full algebra and $|X|<\mu(|\mathfrak{A}|)$ then every homomorphism $h\colon \mathfrak{A}^X\to\mathfrak{A}$ is a projection onto one axis.*

4.14. Remark. The supposition $|\mathfrak{A}|\geqslant 3$ in 4.12 and 4.13 is essential. In fact if $|\mathfrak{A}|=2$ then the e_a's are definable by the term x, but if $\mathfrak{A}$ is the two-element group then the homomorphism $x_1+x_2\colon \mathfrak{A}^2\to\mathfrak{A}$ does not have any one-element support. However if $\mathfrak{A}$ is the two-element Boolean algebra then the conclusions of 4.12 and 4.13 are of course valid.

4.15. THEOREM. *If $\mathfrak{A}$ is a self-simple lattice, i.e., every homomorphism $h\colon \mathfrak{A}\to\mathfrak{A}$ is either an automorphism or a constant, then $S(\mathfrak{A},\mathfrak{A},\omega,2)$.*

Proof. Let $h\colon \mathfrak{A}^n\to\mathfrak{A}$. We shall prove by induction on n that h has a 1-element support. We need some terminology and lemmas. A subset of $\mathfrak{A}^n$ of the form $\{(x_0,\ldots,x_{n-1})\colon (x_{i_0},\ldots,x_{i_{k-1}})=(a_0,\ldots,a_{k-1})\}$, where $(a_0,\ldots,a_{k-1})\in\mathfrak{A}^k$ and $0\leqslant i_0<i_1<\ldots<i_{k-1}<n$, will be called a k-plane or briefly a plane. A subset of $\mathfrak{A}^n$ of the form $A_0\times\ldots\times A_{n-1}$ will be called a product set.

(4.16) If S is a product set and for every 1-plane L if $|L\cap S|\geqslant 2$ then $L\subseteq S$ then S is a plane.

The proof is an obvious induction based on the following fact. If $S=A_0\times\ldots\times A_{n-1}$, $|A_i|\geqslant 2$ and $S_i=A_0\times\ldots\times A_{i-1}\times A\times A_{i+1}\times\ldots\times A_{n-1}$ then $S_i\subseteq S$. This in turn follows from the assumption of (4.16) since if $p\in S_i$ then there exists a 1-plane L whose fixed coordinates are those of p, with $|L\cap S|\geqslant 2$.

(4.17) If $S\subseteq\mathfrak{A}^n$ has the property that whenever $u,v\in S$ then $\{x\colon u\wedge v\leqslant x\leqslant u\vee v\}\subseteq S$ then S is a product set.

Let S_i be the projection of S into the ith axis, we have to show that $S_0\times\ldots\times S_{n-1}\subseteq S$. Let $s_i\in S_i$ and $p_i\in S$ be such that the ith coordinates of p_i is s_i then

$$p_0\wedge\ldots\wedge p_{n-1}\leqslant(s_0,\ldots,s_{n-1})\leqslant p_0\vee\ldots\vee p_{n-1}$$

and $(s_0,\ldots,s_{n-1})\in S$ follows.

(4.18) For every $a\in\mathfrak{A}$ the coset $h^{-1}(a)$ is a plane.

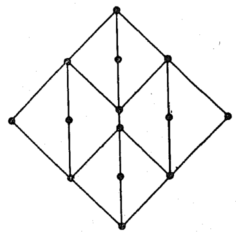
In fact $h^{-1}(a)$ is a sublattice of $\mathfrak{A}^n$. Also, since h preserves $\leqslant$, if $u\leqslant x\leqslant v$ and $u,v\in h^{-1}(a)$ then $x\in h^{-1}(a)$. Hence, by (4.17) $h^{-1}(a)$ is a product set. If L is a 1-plane and $|L\cap h^{-1}(a)|\geqslant 2$ then $h\restriction L$ is not injective. Since L is a sublattice of $\mathfrak{A}^n$ isomorphic to $\mathfrak{A}$ hence by the assumption of 4.15 $h\restriction L$ is a constant. Hence $L\subseteq h^{-1}(a)$. Thus $h^{-1}(a)$ satisfies the assumption of (4.16), and (4.18) follows.

(4.19) If $|\mathfrak{A}|$, $n\geqslant 2$ then there exists an $a_0\in\mathfrak{A}$ such that $|h^{-1}(a_0)|\geqslant 2$.

Take any two disjoint 1-planes L_1 and L_2. Since they are sublattices of $\mathfrak{A}^n$ isomorphic to $\mathfrak{A}$ hence either $h\restriction L_1$ and $h\restriction L_2$ are both constants or one of them is surjective and their ranges intersect. In either case we get an a_0 as desired.

Now we can conclude our inductive proof of 4.15. For $|\mathfrak{A}|=1$ or $n=1$ the assertion is obvious. Thus we can assume $|\mathfrak{A}|$, $n\geqslant 2$. Let $h^{-1}(a_0)$ be given by (4.19). By (4.18) $h^{-1}(a_0)$ is a plane and it includes some 1-plane P_0. Let P be any 1-plane parallel to P_0. Then P is either included in or disjoint with $h^{-1}(a_0)$. Hence $h\restriction P$ cannot be surjective and it is a constant. Thus h does not depend on the coordinate which parametrises P_0. Hence by the inductive assumption h has a 1-element support. Q.E.D.

4.20. EXAMPLE (W. Taylor). The lattice of the figure is self-simple but is not simple.



The next theorem is also due to W. Taylor and is published here with his permission.

4.21. THEOREM. *If $\mathfrak{A}$ is a lattice and $\mathfrak{B}$ is a linearly ordered lattice then $S(\mathfrak{A},\mathfrak{B},\omega,2)$.*

Proof. Let $h\colon \mathfrak{A}^n\to\mathfrak{B}$. Since $\mathfrak{A}$ is a union of a chain of bounded lattice intervals the theorem reduces to the case when $\mathfrak{A}$ has a 0 and a 1. Then we can assume without loss of generality that

$$h(1,0,0,\ldots,0)\geqslant h(0,1,0,\ldots,0)\geqslant h(0,0,1,\ldots,0)\geqslant\ldots\geqslant h(0,\ldots,0,1).$$

Hence

$$h(1,0,\ldots,0)=h(1,0,\ldots,0)\vee\ldots\vee h(0,\ldots,0,1)=h(1,1,\ldots,1).$$

Now for any $(a_0, \ldots, a_{n-1}) \in \mathfrak{A}^n$

$$\begin{aligned} h(a_0, \ldots, a_{n-1}) &= h((1, 1, \ldots, 1) \wedge (a_0, \ldots, a_{n-1})) \\ &= h(1, 1, \ldots, 1) \wedge h(a_0, \ldots, a_{n-1}) \\ &= h(1, 0, \ldots, 0) \wedge h(a_0, \ldots, a_{n-1}) \\ &= h((1, 0, \ldots, 0) \wedge (a_0, \ldots, a_{n-1})) \\ &= h(a_0, 0, 0, \ldots, 0) . \end{aligned}$$

Hence $\{0\}$ is a support for h. Q.E.D.

4.22. THEOREM. *If $\mathfrak{A}$ is a centerless group and $\mathfrak{B}$ is a group such that every homomorphism h: $\mathfrak{A} \to \mathfrak{B}$ is either injective or a constant and there is no injection of $\mathfrak{A}^2$ into $\mathfrak{B}$ then $S(\mathfrak{A}, \mathfrak{B}, \omega, 2)$.*

Proof. Let h: $\mathfrak{A}^n \to \mathfrak{B}$. We use induction with respect to n. For $n = 1$ the assertion is obvious. For $n > 1$ h is not an injection. Let $a = (a_0, \ldots, a_{n-1}) \in \operatorname{Ker}(h)$. We can assume without loss of generality that $a_0 \neq 1$. Since $\mathfrak{A}$ is centerless there is a $b_0 \in \mathfrak{A}$ such that $a_0 b_0 \neq b_0 a_0$. Let

$$c = (b_0, 1, \ldots, 1) a (b_0, 1, \ldots, 1)^{-1} = (b_0 a_0 b_0^{-1}, a_1, a_2, \ldots, a_{n-1}) .$$

Then $c \in \operatorname{Ker}(h)$, and $ac^{-1} = (d_0, 1, \ldots, 1) \in \operatorname{Ker}(h)$, where $d_0 = a_0 b_0 a_0^{-1} b_0^{-1} \neq 1$. Hence $(d, 1, \ldots, 1) \in \operatorname{Ker}(h)$ for all $d \in \mathfrak{A}$. Thus h has a one-element support by the inductive assumption.

4.23. EXAMPLE. If $\mathfrak{A}$ is a finite nonabelian simple group then $S(\mathfrak{A}, \mathfrak{A}, \omega, 2)$ holds. This example is known, see [9, Theorem 9.12(b), p. 51]. Finiteness is essential here, as the group of even permutations of an infinite set shows.

4.24. PROBLEM. An algebra $\mathfrak{A}$ is called weakly functionally complete (w.f.c.) iff every function f: $A^2 \to A$ can be represented by a term with constants, i.e., by a term in the extended algebra $\langle \mathfrak{A}, a \rangle_{a \in A}$. E.g. finite discriminator algebras are w.f.c. and finite nonabelian simple groups are w.f.c., see H. Werner [28, 29]. All w.f.c. algebras are simple. Does $S(\mathfrak{A}, \mathfrak{A}, \omega, 2)$ hold for all w.f.c. algebras $\mathfrak{A}$?

4.25. THEOREM. *If $\mathfrak{A}$ is a ring unity and $\mathfrak{B}$ is a ring such that every homomorphism h: $\mathfrak{A} \to \mathfrak{B}$ is either injective or a constant and there is no injection of $\mathfrak{A}^2$ into $\mathfrak{B}$ then $S(\mathfrak{A}, \mathfrak{B}, \omega, 2)$.*

Proof. Let h: $\mathfrak{A}^n \to \mathfrak{B}$. We use induction with respect to n. For $n = 1$ the assertion is obvious. For $n > 1$ h is not an injection. Let $\bar{a} = (a_0, \ldots, a_{n-1}) \in \operatorname{Ker}(h)$ and $\bar{a} \neq 0$. We can assume without loss of generality that $a_0 \neq 0$. Then

$$(a_0, 0, \ldots, 0) = \bar{a}(1, 0, \ldots, 0) \in \operatorname{Ker}(h) .$$

Hence $(a, 0, \ldots, 0) \in \operatorname{Ker}(h)$ for all $a \in \mathfrak{A}$. Thus h has a one-element support by the inductive assumption.

4.26. Remark. The assumption that the ring $\mathfrak{A}$ has a unity could be replaced by the weaker assumption that for every $a \in \mathfrak{A}$, $a \neq 0$ there exists a $b \in \mathfrak{A}$ with $ab \neq 0$ or $ba \neq 0$.

5. Homomorphisms of subalgebras of $\mathfrak{A}^X$. First we generalize some of the above concepts to the case when X is a topological space.

5.1. DEFINITION. If X is a topological space then $\mathfrak{A}^X$ denotes the algebra of all continuous functions f: $X \to A$, A being endowed with the discrete topology.

5.2. PROPOSITION. *If X is a topological space then the collection of all supports of any homomorphism h: $\mathfrak{A}^X \to \mathfrak{B}$ which are closed and open in X constitutes a filter in the Boolean algebra of all clopen subsets of X.*

Proof. Same as the proof of 2.2.

The following theorem is closely related to some results and conjectures of Mrówka and Shore [16].

5.3. THEOREM. *If $S(\mathfrak{A}, \mathfrak{B}, \omega, n)$ then for every compact Hausdorff space X every homomorphism h: $\mathfrak{A}^X \to \mathfrak{B}$ has an n-element support.*

The proof will be given in Section 6.

5.4. COROLLARY. *If $S(\mathfrak{A}, \mathfrak{B}, \omega, n)$ and, for any set X, $\mathfrak{A}^X_\omega$ denotes the algebra of all the functions f: $X \to A$ with $|f(X)| < \omega$, then every homomorphism h: $\mathfrak{A}^X_\omega \to \mathfrak{B}$ has an n-element support.*

Proof. By 5.3, since every homomorphism h: $\mathfrak{A}^X_\omega \to \mathfrak{B}$ can be extended (uniquely) to a homomorphism h^*: $\mathfrak{A}^{X^*} \to \mathfrak{B}$, where X^* is the Čech compactification of X discrete.

5.5. Remark. Many examples of algebras satisfying the assumptions of 5.3 and 5.4 with $n = 2$ are given in Section 4.

For any sets A and X and any $x \in X$ and $f \in A^X$ we put

$$A^X_f = \{g \in A^X \colon |\{y \colon g(y) \neq f(y)\}| < \omega\}$$

(notice that for every $g \in A^X_f$ we have $A^X_g = A^X_f$) and

$$A^X_{xf} = \{g \in A^X \colon \forall y \in X [g(y) \neq f(y) \Rightarrow y = x]\}$$

(notice that $A^X_{xf} \subseteq A^X_f$). For any $D \subseteq A^X$ and h: $D \to A$ we consider the following property of h

$$\mathscr{P}(h) \Leftrightarrow \forall x \in X \forall f \in D [\forall g \in D \cap A^X_{xf} [h(g) = g(x)] \text{ or } h \restriction D \cap A^X_{xf} \text{ is a constant}] .$$

The following theorem is a generalization of [20, Problem 2, p. 38].

5.6. THEOREM. *If $|A| \geqslant 3$. $f \in A^X$, h: $A^X_f \to A$ and $\mathscr{P}(h)$ then h has a support of cardinality $\leqslant 1$.*

Proof. If for every $x \in X$ and $g \in A^X_f$, $h \restriction A^X_{xg}$ is a constant then it is easy to check that h is a constant. From now on we may assume that for all $p \in A^X_{x_0 g_0}$ we have $h(p) = p(x_0)$, and have to prove that for all $p \in A_f$ $h(p) = p(x_0)$. Take the case $|X| = 2$ first. Then A^X and A^X_f can be regarded as $A \times A$, and h: $A \times A \to A$. Let us show that $\mathscr{P}(h)$ makes it impossible to have

$$\exists a, b \forall x, y [h(a, y) = y \text{ and } h(x, b) = x] . \tag{5.7}$$

Indeed if (5.7) were true then $a = h(a, b) = b$. Since $|A| \geqslant 3$ we choose $c, d \in A$ with $a \neq c \neq d \neq a$. By the same argument which gave $a = b$ we see that if $\forall y$ $h(c, y) = y$ then $c = b$. Hence, by $\mathscr{P}(h)$, $h \restriction \{c\} \times A$ is a constant and, by (5.7), $h(c, x) = c$. In the same way we show that $h(x, d) = d$. Hence $c = h(c, d) = d$, contradiction. Thus (5.7) fails. Then we can assume without loss of generality that $h \restriction \{a\} \times A$ is a constant for all $a \in A$ and $h(x, b_0) = x$ for some $b_0 \in A$. This yields $h(x, b) = x$ for all $b \in A$.

Now we consider an arbitrary X. To show $h(p) = p(x_0)$ we proceed by induction on the number $n(p) = |\{x: p(x) \neq g_0(x) \text{ and } x \neq x_0\}|$. Freezing all but two coordinates of p, namely $p(x_0)$ and $p(x_1)$, where $p(x_1) \neq g_0(x_1)$ and $x_1 \neq x_0$ we see, by the 2-dimensional case, that h turns into a function which depends only on the value at x_0. Then we take

$$p'(x) = \begin{cases} p(x) & \text{if} \quad x \neq x_1, \\ g_0(x_1) & \text{if} \quad x = x_1, \end{cases}$$

and $n(p') < n(p)$. Hence by the inductive assumption and the above property of h we have $h(p) = h(p') = p'(x_0) = p(x_0)$. Q.E.D.

5.8. Theorem. *If $|A| \geqslant 3$, X is a compact Hausdorff space, A^X is as in 5.1, $h: A^X \to A$ and $\mathscr{P}(h)$ then h has a support of cardinality $\leqslant 1$.*

For the proof see Section 6.

5.9. Corollary. *If $|A| \geqslant 3$, X is a set, $A^X_\omega = \{f \in A^X: |f[X]| < \omega\}$, $h: A^X_\omega \to A$ and $\mathscr{P}(h)$ then h has a support of cardinality $\leqslant 1$.*

Proof. 5.9 follows from 5.8 in the same way as 5.4 followed from 5.3.

5.10. Examples. In contrast to 5.6 and 5.9 there exist sets A, X and maps $h: A^X \to A$ with $|A| \geqslant 3$ and $\mathscr{P}(h)$ but without finite supports. E.g. if A is finite and F is a nonprincipal ultrafilter of subsets of X then the ultrapower map $h: A^X \to A$ defined by F satisfies $\mathscr{P}(h)$ but has no finite supports. Also the supposition $|A| \geqslant 3$ in 5.6, 5.8 and 5.9 is essential as the map $h: \{0, 1\}^X \to \{0, 1\}$ defined by $h(f) = \prod_{x \in X} f(x)$ shows.

6. Proofs of 3.1, 4.1, 5.3 **and** 5.8. The main idea is the following. For P any partition of X, we let

(6.1) $\mathfrak{A}_P$ be the subalgebra of $\mathfrak{A}^X$ consisting of all the functions which are constant on each block of P.

Then of course $\mathfrak{A}_P \cong \mathfrak{A}^{|P|}$, and $\mathfrak{A}^X = \bigcup_{|P| \leqslant |\mathfrak{A}|} \mathfrak{A}_P$. We shall look at supports in the partitions and prove that they form ultrafilters with the desired completeness properties.

We shall use the notations

(6.2) $$P(f) = \{f^{-1}\{a\}: a \in A\} \quad \text{for all } f: X \to A.$$

If $P_0, P_1, \ldots$ are partitions of X then

(6.3) $$P_0 \wedge P_1 = \{A \cap B: A \in P_0, B \in P_1\},$$

(6.4) $$\bigwedge_{n<\omega} P_n = \{\bigcap_{n<\omega} A_n: A_n \in P_n\}.$$

Proof of 3.1. First the first part of 3.1. Let $\boldsymbol{P}$ be the set of all partitions of X into no more than $|\mathfrak{A}^\omega|$ blocks.

(6.5) If $P_n \in \boldsymbol{P}$ for all $n < \omega$ then $\bigwedge_{n<\omega} P_n \in \boldsymbol{P}$.

This follows from (6.4) and $|\mathfrak{A}^\omega|^\omega = |\mathfrak{A}^\omega|$.

By the assumption $S(\mathfrak{A}, \mathfrak{B}, |\mathfrak{A}^\omega|^+, \omega)$, for every $P \in \boldsymbol{P}$ there exists a finite set $F(P) \subseteq P$ such that for all $f, g \in \mathfrak{A}_P$ (see (6.1)) with $f \restriction \bigcup F(P) = g \restriction \bigcup F(P)$ we have $h(f) = h(g)$. We assume that $F(P)$ is minimal. By 2.2 a minimal set of this kind is unique. We shall study the function $F(P)$.

(6.6) If $P, Q \in \boldsymbol{P}$ and P is a refinement of Q, i.e., $P \wedge Q = P$, then each block in $F(Q)$ includes some block of $F(P)$.

This follows from the minimality of $F(Q)$ by an obvious argument.

To get the next property we need (6.5) (see Example 6.11).

(6.7) There exists an $n < \omega$ such that $|F(P)| \leqslant n$ for all $P \in \boldsymbol{P}$.

Suppose to the contrary that there exists a sequence $P_0, P_1, \ldots$ with $|F(P_n)| > n$. Let $P = \bigwedge_{n<\omega} P_n$. Then, by (6.5), $P \in \boldsymbol{P}$. By

(6.6) $|F(P)| \geqslant |F(P_n)| > n$ for all n, which is a contradiction.

Let n be the least integer satisfying (6.7) and let P_0 be such that

$$F(P_0) = \{A_0, \ldots, A_{n-1}\},$$

where A_i are disjoint and non-empty. We put $\boldsymbol{P}_0 = \{P \wedge P_0: P \in \boldsymbol{P}\}$ (see (6.3)).

(6.8) For every $P \in \boldsymbol{P}_0$ the set $\bigcup F(P)$ is a support of h.

This follows easily from (6.6) and (6.7).

Now we prove the following key lemma.

(6.9) There exist n $|\mathfrak{A}|^+$-complete ultrafilters $F_0, \ldots, F_{n-1}$ of subsets of X such that

(i) $A_k \in F_k$ for all $k < n$;

(ii) $|F_k \cap P| = 1$ for all $P \in \boldsymbol{P}_0$ and $k < n$;

(iii) $F(P) = P \cap \bigcup_{k<n} F_k$ for all $P \in \boldsymbol{P}_0$.

By (6.6), for all $P \in \boldsymbol{P}_0$, $F(P)$ has exactly one element included in A_k, call it $A_k(P)$. Let us show that for any $P_1, P_2 \in \boldsymbol{P}_0$ we have $A_k(P_1) \cap A_k(P_2) \neq \emptyset$. By the minimality of $F(P_0)$ there are two functions $g_1, g_2 \in \mathfrak{A}_{P_0}$ such that $g_1 \upharpoonright (X-A_k) = g_2 \upharpoonright (X-A_k)$ but $h(g_1) \neq h(g_2)$. If $A_k(P_1) \cap A_k(P_2) = \emptyset$ was true then there would be a function g such that $g \upharpoonright A_k(P_1) = g_1 \upharpoonright A_k(P_1)$ and $g \upharpoonright (X-A_k(P_1)) = g_2 \upharpoonright (X-A_k(P_1))$ and $g \upharpoonright A_k(P_2) = g_2 \upharpoonright A_k(P_2)$. Then $g, g_1 \in \mathfrak{A}_{P_1}$, $g, g_2 \in \mathfrak{A}_{P_2}$, $g_1 \upharpoonright \bigcup F(P_1) = g \upharpoonright \bigcup F(P_1)$ and $g_2 \upharpoonright \bigcup F(P_2) = g \upharpoonright \bigcup F(P_2)$. Hence $h(g_1) = h(g) = h(g_2)$ which is a contradiction.

It follows that for every $P_1, P_2 \in \boldsymbol{P}_0$, if P_1 and P_2 split A_k in the same way then $A_k(P_1) = A_k(P_2)$. Hence, by 1.1, there exists an $|\mathfrak{A}^{\omega}|^+$-complete, and hence $|\mathfrak{A}|^+$-complete, ultrafilter G_k of subsets of A_k such that $\{A_k(P)\} = P \cap G_k$ for all $P \in \boldsymbol{P}_0$. Let F_k be the filter of subsets of X generated by G_k, and (6.9) follows.

(6.10) $\bigcap_{k<n} F_k$ is the collection of all supports of h.

Let $Y \in \bigcap_{k<n} F_k$, $f, g \in \mathfrak{A}^X$ and $f \upharpoonright Y = g \upharpoonright Y$. Let $Y_k = Y \cap A_k$ for all $k<n$, and $P_1 = P_0 \wedge P(f) \wedge P(g) \wedge (\{Y_k\colon k<n\} \cup \{X - \bigcup_{k<n} Y_k\})$ (see (6.2)). Then $f, g \in \mathfrak{A}_{P_1}$ and, by (6.9), $\bigcup F(P_1) \subseteq Y$. Hence $h(f) = h(g)$. Let now Y be a support of h. Then, by 2.2 and (6.8), $\bigcup F(P_0 \wedge \{Y, X-Y\}) \subseteq Y$, and by (6.9), $Y \in F_k$ for all $k<n$.

By (6.9) and (6.10) the first part of 3.1 follows.

The second part of 3.1 is a corollary of the first. Suppose that n is minimal. Then there are disjoint sets $A_k \in F_k$ for all $k<n$. For any $a_0, \ldots, a_{n-1} \in \mathfrak{A}$ let $h_0(a_0, \ldots, a_{n-1}) = h(f)$, where $f(x) = a_k$ for all $x \in A_k$ and $k<n$. It is obvious that h_0 has the required property. Q.E.D.

6.11. Example. The following example shows that our proof would not work under the suppositions $|\mathfrak{A}| = \omega$ and $S(\mathfrak{A}, \mathfrak{B}, \omega_1, \omega)$ and, as mentioned in 3.3, we do not know if 3.1 is valid in this case. Let $X = \omega \cup \{0, 1\}^{\omega}$, and $\boldsymbol{P}$ be the set of all partitions of X into at most ω parts. Then there exists a function F with domain $\boldsymbol{P}$ such that $F(P) \subseteq P$ and $|F(P)| < \omega$ for all $P \in \boldsymbol{P}$ and F satisfies (6.6) but it violates (6.7). To define such an F let a set $I \subseteq \{0, 1\}^{\omega}$ be called an interval of diameter $1/3^n$ iff $I = \{(x_0, x_1, \ldots) \in \{0, 1\}^{\omega}\colon (x_0, \ldots, x_{n-1}) = (a_0, \ldots, a_{n-1})\}$ for some $(a_0, \ldots, a_{n-1}) \in \{0, 1\}^n$. For every $P \in \boldsymbol{P}$ let $n(P) = \min\{n\colon$ there is an $A \in P$ which intersects two disjoint intervals of diameter $1/3^n\}$. Since $|P| < 2^{\omega}$ it follows that $n(P) < \omega$. We put $F(P) = \{A \in P\colon A \cap n(P) \neq \emptyset\}$. It is easy to check that our F satisfies (6.6) and violates (6.7).

Proof of 4.1. This proof is identical to the proof of 3.1 except that now we let $\boldsymbol{P}$ be the set of all partitions of X into less than $|\mathfrak{A}|^+ + \omega$ parts and we get (6.7) directly from the assumption $S(\mathfrak{A}, \mathfrak{B}, |\mathfrak{A}|^+ + \omega, n)$.

Proof of 5.3. Again this proof is quite similar to the proof of 4.1. Now $\boldsymbol{P}$ is the set of all finite partitions of X into clopen sets, and (6.7) follows from the assumption $S(\mathfrak{A}, \mathfrak{B}, \omega, n)$.

6.12. Generalizations. Some possibilities of generalizations were mentioned at the end of Section 2. Let us remark here that the proofs of 3.1, 4.1 and 5.3 (unlike the proofs given in Section 4) do not use explicitly the algebraic structure of the algebras involved. Hence they generalize (without any changes) to relational structures, algebras with relations, topological spaces, topological algebras, etc.

Proof of 5.8. Again the proof is quite similar to the proof of 5.3 except that now we get (6.7) with $n = 1$ from 5.6.

7. Adjacent facts on topological spaces.

7.1. Proposition. *If A is a Hausdorff topological space and F is a finite discrete space then for every set X every continuous map $h\colon A^X \to F$ has a finite support.*

Proof. For every $p \in F$ the inverse $h^{-1}\{p\}$ is clopen. Every clopen set in A^X is a cylinder over a clopen subset of a finite power of A. Hence 7.1 follows.

In particular if A is a finite discrete space then

(7.2) Every continuous map $h\colon A^X \to A$ has a finite support.

It was asked if there are any infinite spaces A with this property. In 1967 Cook [3] constructed a continuum $A \subseteq \boldsymbol{R}^3$ such that

(7.3) Every continuous function $f\colon A \to A$ is either the identity or a constant.

The following theorem generalizes a result of [12] and answers the above question.

7.4. Theorem. *If A is a Hausdorff space with Property (7.3) then for every set X every continuous map $h\colon A^X \to A$ has a support of cardinality ≤ 1.*

Proof. Pick any point $p \in A$. For any finite set $Y \subseteq X$ and any $f \in A^X$ we let

$$f_Y(x) = \begin{cases} f(x) & \text{for } x \in Y, \\ p & \text{for } x \in X - Y, \end{cases}$$

and $S = \{f_Y\colon f \in A^X, Y \subseteq X, |Y| < \omega\}$. By 5.6 $h \upharpoonright S$ is either a constant or there exists an $x_0 \in X$ such that $h(g) = g(x_0)$ for all $g \in S$. In the first case, since S is dense in A^X and h is continuous, h is a constant and $\emptyset$ is a support for h. In the second case, since f_Y converges to f over the net of finite sets Y and since $h(f_Y) = f_Y(x_0) = f(x_0)$ whenever $x_0 \in Y$, we get $h(f) = f(x_0)$ for all $f \in A^X$. Thus $\{x_0\}$ is a support. Q.E.D.

Addenda

1. R. Quackenbush gave the following simple solution of Problem 3.7. If $\mathfrak{A}_1 = \langle\{0, 1\}, +, 0, 1\rangle$, $\mathfrak{A}_2 = \langle\{0, 1\}, +, 1, 0\rangle$ and $\mathfrak{B} = \langle\{0, 1\}, +, 1, 1\rangle$, where $+$ is mod 2, then the implication in 3.7 fails.

2. R. Quackenbush and H. Werner gave the following positive solution of Problem 2.24. First $\mathfrak{A}^* = \langle \mathfrak{A}, a\rangle_{a \in A}$ has the same congruences as $\mathfrak{A}$, and the same is true for $(\mathfrak{A}^*)^n$ and $\mathfrak{A}^n$ for any n. Then, if $\mathfrak{A}$ is w.f.c. and $|\mathfrak{A}| > 1$, by the results of G. A. Fraser and A. Horn (Congruence relations in direct products, Proc. Amer.

Math. Soc. 26 (1970), pp. 390–394) it follows that $(\mathfrak{A}^*)^n$, and hence $\mathfrak{A}^n$, has exactly 2^n congruences. Therefore $S(\mathfrak{A}, \mathfrak{A}, \omega, 2)$ holds.

3. Additional reference related to our work: B. Poizat, *Une relation particulierement rigide*, C. R. Acad. Sci. Paris 282 (1976), Série A, pp. 671–673.

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